Kragujevac Journal of Mathematics
Volume 51(6) (2027), Pages 909–933.

A FINITE DIFFERENCE SCHEME FOR THE ONE-DIMENSIONAL FRACTIONAL TRANSMISSION PROBLEM

SANDRA ŽIVANOVIĆ[1], VUKAŠIN BRKOVIĆ[2], ALEKSANDRA DELIĆ[1], AND ZORICA MILOVANOVIĆ JEKNIĆ[3]

Abstract. In this paper, we study a time-fractional diffusion transmission problem on two disjoint intervals separated by an interface, and formulated with the Riemann-Liouville fractional derivative. First, we develop a rigorous analytical framework for this interface problem with zero initial data: we prove existence and uniqueness of a weak solution and derive an a priori estimate within a suitable Sobolev-like function space. Second, we derive an interface-consistent finite difference scheme that is unconditionally stable under minimal regularity assumptions. We obtain a convergence rate estimate that depends on the regularity on the input data. The theoretical results are confirmed by numerical examples.

1. Introduction

Fractional partial differential equations (FPDEs) have attracted growing attention in recent years due to their broad applicability across science and engineering. Often, fractional-order models prove to be more suitable than their integer-order counterparts, as fractional derivatives and integrals facilitate the description of memory properties inherent in various materials and processes. Recently, considerable effort has been devoted to developing efficient numerical schemes for fractional differential models. In [23], hybrid explicit group methods were proposed for a two-dimensional advection-diffusion problem with a Caputo time derivative, where stability and convergence were established. The study [22] introduced fractional explicit group methods for the time-fractional mobile/immobile equation, achieving convergence of order $O(\tau^{2-\alpha}+h^2)$.

Key words and phrases. fractional derivative, transmission problem, non-local boundary condition, finite differences, Sobolev spaces, convergence rate estimates.

2020 *Mathematics Subject Classification.* Primary: 35R11. Secondary: 65N06, 65N12, 65N15.

DOI

Received: August 18, 2025.

Accepted: January 14, 2026.

In [7], a modified implicit algorithm was designed for the Rayleigh-Stokes problem involving a Riemann-Liouville fractional derivative, confirming accuracy and stability through numerical tests. Collectively, these works demonstrate efficient strategies for fractional-in-time models with improved computational performance. A space-time finite element formulation combining the discontinuous Galerkin method in time with the Nitsche-XFEM approach in space has also been developed for time-fractional diffusion interface problems, yielding sharp error estimates validated by numerical experiments [27].

Transmission problems describe the transfer of physical quantities through different media or across interfaces separating them. The effect of intermediate regions can be modeled by nonlocal jump conditions, which capture discontinuities or flux exchanges across boundaries. Such problems appear in heat transfer, wave propagation, electromagnetism, and related physical processes. Mathematically, they are represented by systems of partial differential equations supplemented with boundary, initial, and conjugation conditions. Radiative heat transfer becomes significant when surface temperatures are high or when conduction and convection are negligible. In complex configurations, radiation exchange between visible surfaces and the influence of shadow zones must be considered, leading to nonlocal integro-differential formulations such as the radiosity equation for diffuse-gray surfaces in transparent cavities. Numerous studies have addressed heat transfer problems with nonlocal boundary conditions [9]. In [8], a transmission spectral problem in two disjoint intervals was analyzed, establishing the existence of countable generalized solutions, describing the spectral structure, and constructing a suitable finite difference scheme. Further investigations [10, 11, 14, 18–20] examined elliptic and parabolic transmission problems in disjoint domains, proving existence and uniqueness of weak solutions in Sobolev-type spaces and convergence of the corresponding finite difference approximations. Hyperbolic transmission problems were analyzed in [5, 13]. In this study, we focus on a fractional-in-time transmission problem defined on two disjoint intervals.

We introduce some basic definitions and properties of fractional derivatives that we will use throughout the paper. The left and the right Riemann-Liouville fractional time-derivative $\partial^{\alpha}_{t,0+}u$ are defined by [15, 21]:

$$\partial^{\alpha}_{t,0+}u(x,t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^n}{\partial t^n} \int_0^t u(x,s)(t-s)^{n-\alpha-1}ds, \quad t > 0,$$

and

$$\partial^{\alpha}_{t,T-}u(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \cdot \frac{\partial^n}{\partial t^n} \int_t^T u(x,s)(s-t)^{n-\alpha-1}ds, \quad t < T,$$

respectively, where $n-1 \le \alpha < n$, $n \in \mathbb{N}$ and $\Gamma(\cdot)$ denotes the Gamma function. Analogously, one defines the left Caputo fractional time-derivative:

$${}^{C}\partial^{\alpha}_{t,0+}u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\partial^n u(x,s)}{\partial t^n}(t-s)^{n-\alpha-1}ds, \quad t > 0.$$

The following relation between Riemann-Liouville and Caputo fractional derivatives is valid:

$$^{C}\partial^{\alpha}_{t,0+}u(x,t) = \partial^{\alpha}_{t,0+}u(x,t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \cdot \frac{\partial^k u(x,0)}{\partial t^k}.$$

Thus, for the zero initial conditions, the Riemann-Liouville and Caputo fractional derivatives coincide.

Under the assumption that u has a sufficient number of continuous partial derivatives of integer order and

$$\frac{\partial^j u}{\partial t^j}(x,0) = 0, \quad j = 0, 1, \dots, n-1,$$

where $n = \max\{n_1, n_2\}$, $n_i - 1 \le \alpha_i < n_i$, $n_i \in \mathbb{N}$, $i = 1, 2$, the fractional derivatives of the left Riemann-Liouville satisfy the semigroup property (see [21]):

$$\partial^{\alpha_1}_{t,0+}\partial^{\alpha_2}_{t,0+}u = \partial^{\alpha_2}_{t,0+}\partial^{\alpha_1}_{t,0+}u = \partial^{\alpha_1+\alpha_2}_{t,0+}u.$$

Let Ω be a domain in $\mathbb{R}^n$. As usual, we denote by $C^k(\Omega)$ and $C^k(\bar{\Omega})$, $k \in \mathbb{N}$, the space of k-fold differentiable functions. By $C_0^{\infty}(\Omega)$ we denote the space of infinitely differentiable functions with compact support in Ω. By $L^p(\Omega)$, $p \ge 1$, we denote Lebesgue spaces. The inner product and norm in the space of real-valued measurable functions whose square is integrable in Ω, are defined in the usual way. We also use $W_p^{\alpha}(\Omega)$ and $W_{p,0}^{\alpha}(\Omega)$ to denote Sobolev spaces [1]. In particular, for $p = 2$ we set $H^{\alpha}(\Omega) = W_2^{\alpha}(\Omega)$ and $H_0^{\alpha}(\Omega) = W_{2,0}^{\alpha}(\Omega)$.

For $0 < \alpha < 1$, and $\Omega = (a, b)$ we define seminorms

$$|u|_{C_+^{\alpha}(\Omega)} = \|\partial^{\alpha}_{a+}u\|_{C(\bar{\Omega})}, \quad |u|_{C_-^{\alpha}(\Omega)} = \|\partial^{\alpha}_{b-}u\|_{C(\bar{\Omega})},$$
$$|u|_{H_+^{\alpha}(\Omega)} = \|\partial^{\alpha}_{a+}u\|_{L^2(\Omega)}, \quad |u|_{H_-^{\alpha}(\Omega)} = \|\partial^{\alpha}_{b-}u\|_{L^2(\Omega)},$$

and norms

$$\|u\|^2_{C_\pm^{\alpha}(\bar{\Omega})} = \left(\|u\|^2_{C(\bar{\Omega})} + |u|^2_{C_\pm^{\alpha}(\bar{\Omega})}\right)^{1/2},$$
$$\|u\|^2_{H_\pm^{\alpha}(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{H_\pm^{\alpha}(\Omega)}\right)^{1/2}.$$

Moreover, we define the spaces $C_\pm^{\alpha}(\bar{\Omega})$ and $H_\pm^{\alpha}(\Omega)$ as spaces of functions with finite norms $\|u\|_{C_\pm^{\alpha}(\bar{\Omega})}$ and $\|u\|_{H_\pm^{\alpha}(\Omega)}$, respectively.

For functions of more variables, which have a different smoothness in different settings, we can define, so called, anisotropic Sobolev spaces. Let, for example, $Q = (0, T) \times \Omega$. Then, we define

$$H^{\alpha,\beta}(Q) = L^2((0,T), H^{\alpha}(\Omega)) \cap H^{\beta}((0,T), L^2(\Omega)),$$

endowed with norm

$$\|u\|_{H^{\alpha,\beta}(Q)} = \left(\|u\|^2_{L^2((0,T),H^{\alpha}(\Omega))} + \|u\|^2_{H^{\beta}((0,T),L^2(\Omega))}\right)^{1/2}.$$

Similarly, we define anisotropic spaces $H_\pm^{\alpha,\beta}(Q)$.

Here, we state two important lemmas that we use thereafter.

Lemma 1.1 ([6,16]). *Let* $\alpha > 0$, $u \in C^\infty(\mathbb{R})$ *and* $\operatorname{supp} u \subset (0, T]$. *Then,*

$$\left(\partial^\alpha_{t,0+}u, \partial^\alpha_{t,T-}u\right)_{L^2(0,T)} = \cos\pi\alpha\, \|\partial^\alpha_{t,0+}u\|^2_{L^2(0,+\infty)}. \tag{1.1}$$

Lemma 1.2 ([16,17]). *Let* $0 < \alpha < 1$, $u \in H^\alpha_+(0, T)$, $v \in \dot{C}^\infty(0, T)$. *Then,*

$$\left(\partial^\alpha_{t,0+}u, v\right)_{L^2(0,T)} = \left(u, \partial^\alpha_{t,T-}v\right)_{L^2(0,T)}. \tag{1.2}$$

The rest of the paper is organized as follows. Section 2 presents the problem formulation along with its weak formulation and the analysis of well-posedness in appropriate function spaces. Section 3 focuses on the numerical approximation of the problem based on the theory of finite differences. Section 4 discusses the convergence of the proposed difference scheme and provides error estimates. Finally, Section 5 is devoted to numerical examples that confirm our expectations regarding the order of convergence.

2. Problem Formulation

Let $\Omega_i = (a_i, b_i)$ and $Q_i = \Omega_i \times (0, T)$, $i = 1, 2$, and $0 < \alpha < 1$. We consider a fractional transmission problem

$$\partial^\alpha_{t,0_+}u_i - \frac{\partial}{\partial x}\left(p_i(x)\frac{\partial u_i}{\partial x}\right) + q_i(x)u_i = f_i(x,t), \quad (x,t) \in Q_i, \quad i = 1, 2, \tag{2.1}$$

with homogeneous Dirichlet boundary condition

$$u_1(a_1, t) = 0, \quad u_2(b_2, t) = 0, \quad t \in [0, T], \tag{2.2}$$

and non-local Robin–Dirichlet boundary conditions

$$\begin{aligned} p_1(b_1)\frac{\partial u_1}{\partial x}(b_1, t) + \sigma_1 u_1(b_1, t) &= \beta_1 u_2(a_2, t), \\ -p_2(a_2)\frac{\partial u_2}{\partial x}(a_2, t) + \sigma_2 u_2(a_2, t) &= \beta_2 u_1(b_1, t), \end{aligned} \tag{2.3}$$

for $t \in [0, T]$, where σ_i and β_i, $i = 1, 2$ are given coefficients.

Also, we introduce initial conditions:

$$u_i(x, 0) = 0, \quad x \in \bar{\Omega}_i, \quad i = 1, 2. \tag{2.4}$$

For the input data and the coefficients of the proposed problem some conditions must be satisfied. Thus we have the regularity and ellipticity conditions

$$p_i(x),\, q_i(x) \in L^\infty(\Omega_i), \quad i = 1, 2, \tag{2.5}$$

$$0 < p_{0i} \le p_i(x), \quad 0 \le q_i(x), \quad i = 1, 2, \tag{2.6}$$

and

$$\sigma_i > 0, \quad \beta_i > 0, \quad i = 1, 2, \quad \text{and} \quad \beta_1\beta_2 \le \sigma_1\sigma_2. \tag{2.7}$$

We also define the product space

$$L^2 = L^2(Q_1) \times L^2(Q_2) = \{v = (v_1, v_2) : \, v_i \in L^2(Q_i), \, i = 1, 2\},$$

endowed with the inner product

$$(v, w)_{L^2} = \beta_2 (v_1, w_1)_{L^2(Q_1)} + \beta_1 (v_2, w_2)_{L^2(Q_2)}.$$

For $\alpha \geq 0$, we introduce the anisotropic Sobolev type space

$$H_0^{1,\alpha/2} = \{v = (v_1, v_2) : v_i \in \; H^{1,\alpha/2}(Q_i), \, i = 1, 2, \, v_1(a_1, t) = v_2(b_2, t) = 0\},$$

with zero boundary conditions involved in the definition of the space.

The weak formulation of the problem (2.1)–(2.4) has the form: find $u \in H_0^{1,\alpha/2}$ such that

$$a(u, v) = l(v), \quad \text{for all } v \in H_0^{1,\alpha/2}, \tag{2.8}$$

where

$$\begin{aligned} a(u, v) =& \sum_{i=1}^{2} \beta_{3-i} \int_0^T \int_{a_i}^{b_i} \partial_{t,0+}^{\alpha/2} u_i \, \partial_{t,T-}^{\alpha/2} v_i \, dxdt \\ &+ \sum_{i=1}^{2} \beta_{3-i} \int_0^T \int_{a_i}^{b_i} \left(p_i(x) \frac{\partial u_i}{\partial x} \frac{\partial v_i}{\partial x} + q_i(x) u_i v_i \right) dxdt \\ &+ \sigma_1 \beta_2 \int_0^T u_1(b_1, t) v_1(b_1, t) dt + \sigma_2 \beta_1 \int_0^T u_2(a_2, t) v_2(a_2, t) dt \\ &- \beta_1 \beta_2 \int_0^T u_2(a_2, t) v_1(b_1, t) dt - \beta_2 \beta_1 \int_0^T u_1(b_1, t) v_2(a_2, t) dt, \end{aligned} \tag{2.9}$$

is a bilinear functional and

$$l(v) = \beta_2 \int_0^T \int_{a_1}^{b_1} f_1 v_1 dxdt + \beta_1 \int_0^T \int_{a_2}^{b_2} f_2 v_2 dxdt, \tag{2.10}$$

is a linear functional.

Lemma 2.1. *Under the conditions* (2.5)*, the bilinear form defined by* (2.9) *is bounded on* $H_0^{1,\alpha/2}$*. If the conditions* (2.6) *and* (2.7) *are also satisfied, this form is coercive:*

$$a(u, u) \geq c_0 \|u\|_{H^{1,\alpha/2}}, \quad c_0 > 0. \tag{2.11}$$

Moreover, there is a unique solution u *of a weak problem* (2.8).

Proof. With the aid of the Cauchy-Schwarz inequality, conditions (2.5) and the embedding $H^1(\Omega_i) \subset C(\bar{\Omega}_i)$, $i = 1, 2$, we obtain the boundedness of the bilinear form (2.9).

In order to prove that the bilinear form is coercive, first we use the scalar product property (1.1) and the conditions (2.6) that coefficients satisfy, to obtain inequality

$$\begin{aligned} a(u, u) \geq& \sum_{i=1}^{2} \beta_{3-i} \left[\cos \frac{\pi \alpha}{2} \|\partial_{t,0+}^{\alpha/2} u_i\|_{L^2(Q_i)}^2 + p_{0i} \int_0^T \left\| \frac{\partial u_i}{\partial x}(\cdot, t) \right\|_{L^2(\Omega_i)}^2 dt \right] \\ &+ \int_0^T \Big(\sigma_1 \beta_2 u_1^2(b_1, t) + \sigma_2 \beta_1 u_2^2(a_2, t) - 2 \beta_1 \beta_2 u_1(b_1, t) u_2(a_2, t) \Big) \, dt. \end{aligned} \tag{2.12}$$

By virtue of conditions (2.7), it follows that

$$\sigma_1\beta_2 u_1^2(b_1,t) + \sigma_2\beta_1 u_2^2(a_2,t) - 2\beta_1\beta_2 u_1(b_1,t)u_2(a_2,t) \geq 0,$$

so we can omit it in the previous inequality. In the end, using the Poincaré-Friedrichs inequality adjusted to the function space $H_0^{1,\alpha/2}$ we obtain the coercivity of the bilinear form.

The boundedness of the right-hand side can be easily proved using the Cauchy-Schwarz inequality, and thus, all conditions for the application of the Lax-Milgram lemma, which guarantees the existence and uniqueness of the solution, are satisfied. □

Using the previous result, it is straightforward to derive the following a priori estimate.

Theorem 2.1. *Let* $\alpha \in (0,1)$, $f = (f_1, f_2) \in L^2$, *and let conditions* (2.5)-(2.7) *be satisfied. Then, the problem* (2.8) *is well posed and its solution satisfies the following a priori estimate*

$$\|u\|_{H^{1,\alpha/2}} \leq C\|f\|_{L^2}. \tag{2.13}$$

3. Approximation

Let $N_l > 1$, $l = 1,2$ be positive integers and set $h_l = (b_l - a_l)/N_l$. We introduce uniform spatial meshes

$$\bar{\omega}_{h_l} = \{x_{li} = a_l + ih_l,\ i = 0,1,\ldots,N_l\},$$

in the intervals $[a_l, b_l]$, $l = 1,2$ and denote $\omega_{h_l} := \bar{\omega}_{h_l} \cap (a_l, b_l)$, $\omega_{h_l}^- := \omega_{h_l} \cup \{a_l\}$ and $\omega_{h_l}^+ := \omega_{h_l} \cup \{b_l\}$.

Let M be a positive integer and set $\tau = T/M$. Also, we define a uniform temporal mesh

$$\bar{\omega}_\tau = \{t_j = j\tau,\ j = 0,1,\ldots,M\},$$

in $[0,T]$ and denote $\omega_\tau := \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$, $\bar{Q}_{h_l\tau} = \bar{\omega}_{h_l} \times \bar{\omega}_\tau$, $l = 1,2$, and $\bar{Q}_{h\tau} = \bar{Q}_{h_1\tau} \times \bar{Q}_{h_2\tau}$.

Then, for the grid functions v_l, $l = 1,2$, defined on $\bar{Q}_{h_l\tau}$, we introduce the standard notations of the finite differences (see [24])

$$v_l = v_l(x,t), \quad v_{l,x} = \frac{v_l(x+h_l,t) - v_l(x,t)}{h_l} = v_{l,\bar{x}}(x+h_l,t),$$

$$v_{l,t} = \frac{v_l(x,t+\tau) - v_l(x,t)}{\tau} = v_{l,\bar{t}}(x,t+\tau),$$

$$v_{l,\bar{x}x} = \frac{v_{l,x} - v_{l,\bar{x}}}{h_l}, \quad v_{l,\bar{t}t} = \frac{v_{l,t} - v_{l,\bar{t}}}{\tau},$$

$$v_l^j = v_l(x,t_j), \qquad \bar{v}_l^j = \frac{v_l(x,t_j) + v_l(x,t_{j-1})}{2}.$$

We approximate the Caputo fractional derivative of order $\alpha \in (0,1)$ by a standard $L1$ scheme. Namely, on every subinterval $[t_{k-1}, t_k]$ we approximate the function $u(x,t)$ by a linear interpolant

$$L_{1,k}(t) = \frac{t - t_k}{t_{k-1} - t_k} u^{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} u^k,$$

and use $L'_{1,k}(t) = u_{\bar{t}}^k$. Then, the approximation of the Caputo fractional derivative of order α for $t = t_j$ is given by

$$ {}^C\partial_{t,0_+}^{\alpha} u(x, t_j) \approx \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{j} a_{j-k} u_{\bar{t}}^k := \left(\Delta_{t,0_+}^{\alpha} u \right)^j, \tag{3.1}$$

where $a_{j-k} = a_{j-k}^{[1-\alpha]} = (j-k+1)^{1-\alpha} - (j-k)^{1-\alpha} > 0$.

Using basic results from finite difference theory we obtain

$$p_2\left(a_2 + \frac{h_2}{2}\right) \frac{\partial u_2}{\partial x}\left(a_2 + \frac{h_2}{2}, t\right) = p_2\left(a_2 + \frac{h}{2}\right) u_{2,x}(a_2, t) + O(h_2^2). \tag{3.2}$$

Furthermore, by means of Taylor expansion and using (2.2) and (2.3), we have

$$\begin{aligned} & p_2\left(a_2 + \frac{h}{2}\right) \frac{\partial u_2}{\partial x}\left(a_2 + \frac{h}{2}, t\right) \\ =& \sigma_2 u_2(a_2,t) - \beta_2 u_1(b_1,t) + \frac{h}{2}\left(\partial_{t,0_+}^{\alpha} u_2(a_2,t) + q_2(a_2) u_2(a_2,t) - f_2(a_2,t)\right) + O(h^2). \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we get the following:

$$\begin{aligned} & \partial_{t,0_+}^{\alpha} u_2(a_2,t) + q_2(a_2) u_2(a_2,t) \\ & + \frac{2}{h}\left(-p_2\left(a_2 + \frac{h}{2}\right) u_{2,x}(a_2,t) + \sigma_2 u_2(a_2,t) - \beta_2 u_1(b_1,t)\right) + O(h^2) = f_2(a_2,t). \end{aligned}$$

After similar reasoning for point $x = b_1$, we are inspired to approximate the problem (2.1)–(2.4) with the following finite difference scheme, expressed in operator form:

$$\begin{gathered} \Delta_{t,0_+}^{\alpha} v_l + L_{h_l} v_l = \tilde{f}_l, \quad (x,t) \in \bar{\omega}_{h_l} \times \omega_\tau^+, \\ v_l^0 = 0, \quad x \in \bar{\omega}_{h_l}, \end{gathered} \tag{3.4}$$

where, for $t \in \omega_\tau$, we define

$$L_{h_1} v = \begin{cases} -(\bar{p}_1 v_{1,\bar{x}})_x + \tilde{q}_1 v_1, & x \in \omega_{h_1}, \\ \tilde{q}_1(b_1) v_1(b_1,t) + \frac{2}{h_1}\left[\bar{p}_1(b_1) v_{1,\bar{x}}(b_1,t) + \sigma_1 v_1(b_1,t) - \beta_1 v_2(a_2,t)\right], & x = b_1, \\ 0, & x = a_1, \end{cases}$$

$$L_{h_2} v = \begin{cases} -(\bar{p}_2 v_{2,\bar{x}})_x + \tilde{q}_2 v_2, & x \in \omega_{h_2}, \\ \tilde{q}_2(a_2) v_2(a_2,t) + \frac{2}{h_2}\left[-\bar{p}_2(a_2+h_2) v_{2,x}(a_2,t) + \sigma_2 v_2(a_2,t) - \beta_2 v_1(b_1,t)\right], & x = a_2, \\ 0, & x = b_2, \end{cases}$$

$$\bar{p}_l(x) = p_l(x - h_l/2), \quad l = 1,2,$$

and we average the right-hand side f in the following way:

$$\tilde{f}_1 := \begin{cases} T_x^{2-} f_1, & x = b_1, \\ T_x^2 f_1, & x \in \omega_{h_1}, \end{cases} \qquad \tilde{f}_2 := \begin{cases} T_x^2 f_2, & x \in \omega_{h_2}, \\ T_x^{2+} f_2, & x = a_2. \end{cases}$$

Here T_x is the Steklov smoothing operator [25]:

$$T_x f(x,t) = \int_{-1/2}^{1/2} f(x + hx', t)\,\mathrm{d}x',$$

$$T_x^2 f(x,t) = T_x\left(T_x f(x,t)\right) = \int_{-1}^{1} (1 - |x'|) f(x + hx', t)\,\mathrm{d}x'.$$

Asymmetric Steklov smoothing operators are defined as follows:

$$\begin{aligned} T_x^+ v(x,t) =& \frac{1}{h}\int_x^{x+h} v(x',t)dx' = T_x^- v(x+h,t), \\ T_x^{2+} f(x,t) =& \frac{2}{h}\int_0^h \left(1 - \frac{x'}{h}\right) f(x',t)dx', \quad x = 0, \\ T_x^{2-} f(x,t) =& \frac{2}{h}\int_{1-h}^1 \left(1 + \frac{x'-1}{h}\right) f(x',t)dx', \quad x = 1. \end{aligned}$$

We use the same averaging for coefficients q_l. We also denote $L_h v = (L_{h_1} v_1, L_{h_2} v_2)$, $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ in order to compactly express equation (3.4) with

$$\Delta_{t,0_+}^{\alpha} v + L_h v = \tilde{f}. \tag{3.5}$$

In order to prove the stability of the proposed finite difference scheme, for $l = 1, 2$, we define discrete inner products and norms:

$$\begin{aligned} |[v_l]|_{C,h_l} =& \max_{x \in \bar{\omega}_{h_l}} |v_l(x)|, \\ (u_l, v_l)_{h_l} =& (u_l, v_l)_{L^2(\omega_{h_l})} = h_l \sum_{x \in \omega_{h_l}} u_l v_l, \\ [u_l, v_l]_{h_l} =& [u_l, v_l]_{L^2(\bar{\omega}_{h_l})} = h_l \sum_{x \in \omega_{h_l}} u_l(x) v_l(x) + \frac{h_l}{2}\left[u_l(b_l)v_l(b_l) + u_l(a_l)v_l(a_l)\right], \\ (u, v)_h =& \beta_2 (u_1, v_1)_{h_1} + \beta_1 (u_2, v_2)_{h_2}, \quad \|v\|_h = (v, v)_h^{1/2}, \\ [u, v]_h =& \beta_2 [u_1, v_1]_{h_1} + \beta_1 [u_2, v_2]_{h_2}, \quad |[v]|_h = [v, v]_h^{1/2}, \\ (u_1, v_1]_{h_1} =& (u_1, v_1]_{L^2(\omega_{h_1}^+)} = h_1 \sum_{x \in \omega_{h_1}^+} u_1(x) v_1(x), \\ ||v_1]|_{h_1} =& ||v_1]|_{L^2(\omega_{h_1}^+)} = (v_1, v_1]_{h_1}^{1/2}, \\ [u_2, v_2)_{h_2} =& [u_2, v_2)_{L^2(\omega_{h_2}^-)} = h_2 \sum_{x \in \omega_{h_2}^-} u_2(x) v_2(x), \\ |[v_2\|_{h_2} =& |[v_2\|_{L^2(\omega_{h_2}^-)} = [v_2, v_2)_{h_2}^{1/2}, \end{aligned}$$

$$\begin{aligned}|v_1|_{H^1(\omega_{h_1})} =&\|v_{1,\bar{x}}]|_{h_1}, \quad |v_2|_{H^1(\omega_{h_2})} = |[v_{2,x}\|_{h_2},\\ |v|^2_{H^1} =&\beta_2|v_1|^2_{H^1(\omega_{h_1})} + \beta_1|v_2|^2_{H^1(\omega_{h_2})},\\ \|v\|_{L^2(Q_{h\tau})} =& \left(\tau \sum_{t\in\omega_\tau^+} |[v(\cdot,t)]|^2_h\right)^{1/2},\\ \|v\|_{B^{1,\alpha/2}(Q_{h\tau})} =& \left(\tau \sum_{t\in\omega_\tau^+} |v(\cdot,t)|^2_{H^1} + \tau \sum_{t\in\omega_\tau^+} \Delta^\alpha_{t,0_+}|[v(\cdot,t)]|^2_h\right)^{1/2}.\end{aligned}$$

Lemma 3.1 ([24])**.** *For every function $v(x)$ defined on the mesh $\bar{\omega}_h = \{x_i = a+ih,\ i = 0,1,\dots,N,\ h = (b-a)/N\}$ and equal to zero at one of the endpoints of the interval $[a,b]$ the following inequality is valid*

$$|[v]|_h \le \sqrt{b-a}|[v]|_{C,h} \le (b-a)||v_{\bar{x}}]|_h. \tag{3.6}$$

Lemma 3.2 ([2])**.** *Let v be a function defined on the mesh $\bar{\omega}_\tau$ which satisfies the condition $v(0) = 0$ and $\alpha \in (0,1)$. Then, the following inequality is valid*

$$v^j\left(\Delta^\alpha_{t,0_+} v\right)^j \ge \frac{1}{2}\left(\Delta^\alpha_{t,0_+} v^2\right)^j.$$

Lemma 3.3 ([3])**.** *For every function $v(t)$ defined on the mesh $\bar{\omega}_\tau$ which satisfies $v(0) = 0$ and $\alpha \in (0,1)$, the following equality is valid*

$$\tau\sum_{j=1}^{M}\left(\Delta^\alpha_{t,0_+} v^2\right)^j = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{M} a_{M-j}(v^j)^2, \quad a_{M-j} = a^{[1-\alpha]}_{M-j}.$$

Theorem 3.1. *Under the assumptions of Theorem* 2.1*, the finite difference scheme* (3.4) *is absolutely stable and its solution satisfies the a priori estimate:*

$$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})} \le C|[\tilde{f}]|_{L^2(Q_{h\tau})}. \tag{3.7}$$

Proof. Taking the inner product $[\Delta^\alpha_{t,0_+} v + L_h v, v]_h = [\tilde{f}, v]_h$ and applying partial summation, we get

$$\begin{aligned}&[\Delta^\alpha_{t,0+} v, v]_h + \beta_2(\bar{p}_1 v_{1,\bar{x}}, v_{1,\bar{x}}]_{h_1} + \beta_1[\bar{p}_2(\cdot + h_2)v_{2,x}, v_{2,x})_{h_2} + [\tilde{q}v, v]_h\\ &+ \beta_2\sigma_1 v_1^2(b_1,t) + \beta_1\sigma_2 v_2^2(a_2,t) - 2\beta_1\beta_2 v_1(b_1,t)v_2(a_2,t) = [\tilde{f}, v]_h.\end{aligned} \tag{3.8}$$

Applying Lemma 3.2 to the first term in the previous equation and using properties (2.5)–(2.7) for the rest of the terms on the left side of the previous equation we obtain the inequality

$$\frac{1}{2}\Delta^\alpha_{t,0_+}|[v]|^2_h + C_0|v|^2_{H^1} \le [\tilde{f}, v]_h, \quad t \in \omega_\tau^+, \tag{3.9}$$

where $C_0 = \min\{p_{01}, p_{02}\}$. For the estimate of the right hand side, one can use the ε-inequality

$$[\tilde{f}, v]_h \le \varepsilon|[v]|^2_h + \frac{1}{4\varepsilon}|[\tilde{f}]|^2_h.$$

Finally, using Lemma 3.1, multiplying by τ and summing over the mesh ω_τ^+ we obtain the result (3.7). $\square$

4. Convergence of the Finite Difference Scheme

Let u be the solution of the initial-boundary value problem (2.1)-(2.4) and v the solution of the difference problem (3.4). The error $z = u - v$ satisfies

$$\begin{gathered}\Delta_{t,0_+}^{\alpha} z_l + L_{h_l} z_l = \psi_l, \quad (x,t) \in \bar{\omega}_{h_l} \times \omega_\tau^+, \\ z_l^0 = 0, \quad x \in \bar{\omega}_h,\end{gathered} \tag{4.1}$$

where

$$\begin{aligned}
&\psi_1(x,t) = \xi_1(x,t) + \rho_{1,x}(x,t) + \mu_1(x,t), \quad x \in \omega_{h_1},\, t \in \omega_\tau^+,\\
&\psi_2(x,t) = \xi_2(x,t) + \rho_{2,\bar{x}}(x,t) + \mu_2(x,t), \quad x \in \omega_{h_2},\, t \in \omega_\tau^+,\\
&\psi_1(b_1,t) = \xi_1(b_1,t) - \frac{2}{h_1}\rho_1(b_1,t) + \mu_1(b_1,t), \quad t \in \omega_\tau^+,\\
&\psi_2(a_2,t) = \xi_2(a_2,t) + \frac{2}{h_2}\rho_2(a_2,t) + \mu_2(a_2,t), \quad t \in \omega_\tau^+,\\
&\xi_i(x,t) = \Delta_{t,0_+}^{\alpha} u_i(x,t) - T_x^2 \partial_{t,0_+}^{\alpha} u_i(x,t), \quad x \in \omega_{h_i},\, t \in \omega_\tau^+, \quad i = 1,2,\\
&\xi_1(b_1,t) = \Delta_{t,0_+}^{\alpha} u_1(b_1,t) - T_x^{2-} \partial_{t,0_+}^{\alpha} u_1(b_1,t), \quad t \in \omega_\tau^+,\\
&\xi_2(a_2,t) = \Delta_{t,0_+}^{\alpha} u_2(a_2,t) - T_x^{2+} \partial_{t,0_+}^{\alpha} u_2(a_2,t), \quad t \in \omega_\tau^+,\\
&\rho_i = -\bar{p}_i u_{i,\bar{x}} + T_x^-\left(p_i \frac{\partial u_i}{\partial x}\right), \quad x \in \omega_{h_i},\, t \in \omega_\tau^+, \quad i = 1,2,\\
&\rho_1 = \bar{p}_1 u_{1,\bar{x}} - T_x^-\left(p_1 \frac{\partial u_1}{\partial x}\right), \quad x = b_1,\, t \in \omega_\tau^+,\\
&\rho_2 = \bar{p}_2(\cdot + h_2) u_{2,\bar{x}} - T_x^+\left(p_2 \frac{\partial u_2}{\partial x}\right), \quad x = a_2,\, t \in \omega_\tau^+,\\
&\mu_i = (T_x^2 q_i) u_i - T_x^2(q_i u_i), \quad x \in \omega_{h_i},\, t \in \omega_\tau^+, \quad i = 1,2,\\
&\mu_1 = (T_x^{2-} q_1) u_1 - T_x^{2-}(q_1 u_1), \quad x = b_1,\, t \in \omega_\tau^+,\\
&\mu_2 = (T_x^{2+} q_2) u_2 - T_x^{2+}(q_2 u_2), \quad x = a_2,\, t \in \omega_\tau^+.
\end{aligned}$$

Lemma 4.1 ([26]). *Suppose that $\alpha \in (0,1)$, $u \in C^2[0,t]$, $t \in \omega_\tau^+$. Then,*

$$|\Delta_{t,0_+}^{\alpha} u - {}^C\partial_{t,0_+}^{\alpha} u| \le C\tau^{2-\alpha} \max_{0 \le s \le t} |u''(s)|. \tag{4.2}$$

Theorem 4.1 ([25]). *Let $\hat{\Omega}_h$ be the boundary strip of width h of the domain Ω with piecewise smooth boundary $\partial\Omega$. Then, for $u \in H^k(\Omega)$, $\frac{1}{2} < k \le 1$, the following holds:*

$$\|u\|_{L^2(\hat{\Omega}_h)} \le C\sqrt{h}\|u\|_{H^k(\Omega_h)}. \tag{4.3}$$

Theorem 4.2. *For the difference scheme* (4.1), *the following a priori estimate holds*

$$\begin{aligned}\|z\|^2_{B^{1,\alpha/2}(Q_{h\tau})} \leq & C\tau \sum_{t\in\omega_\tau^+} \bigg(\frac{h_1^2}{4}\beta_2|\xi_1(b_1,t)|^2 + \|\xi(\cdot,t)\|_h^2 + \frac{h_2^2}{4}\beta_1|\xi_2(a_2,t)|^2 \\ & + h_1\beta_2|\rho_1(b_1,t)|^2 + \|\rho(\cdot,t)\|_h^2 + h_2\beta_1|\rho_2(a_2,t)|^2 \\ & + \frac{h_1^2}{4}\beta_2|\mu_1(b_1,t)|^2 + \|\mu(\cdot,t)\|_h^2 + \frac{h_2^2}{4}\beta_1|\mu_2(a_2,t)|^2\bigg).\end{aligned} \tag{4.4}$$

Proof. Following the proof of Theorem 3.1, we have

$$\frac{1}{2}\Delta^\alpha_{t,0_+}\|[z]\|_h^2 + C_0|z|^2_{H^1} \leq [\Delta^\alpha_{t,0_+}z + L_hz, z]_h = [\psi, z]_h, \quad t\in\omega_\tau^+. \tag{4.5}$$

It follows from Lemma 3.1 that

$$\begin{aligned}\|z\|_h^2 &\leq \beta_2(b_1-a_1)^2\|z_{1,\bar{x}}]|^2_{h_1} + \beta_1(b_2-a_2)^2|[z_{2,x}\|^2_{h_2} \\ &\leq \max\Big\{(b_1-a_1)^2,(b_2-a_2)^2\Big\}|z|^2_{H^1}.\end{aligned} \tag{4.6}$$

Also, we have $z_1(b_1,t) = h_1\sum_{x\in\omega^+_{h_1}} z_{1,\bar{x}}$, and thus, using the Cauchy-Schwarz inequality, we get

$$z_1^2(b_1,t) = \left(h_1\sum_{x\in\omega^+_{h_1}} z_{1,\bar{x}}(x,t)\right)^2 \leq (b_1-a_1)\|z_{1,\bar{x}}(\cdot,t)]|^2_{h_1}.$$

Analogously, we have $z_2^2(a_2,t)\leq (b_2-a_2)|[z_{2,x}(\cdot,t)\|^2_{h_2}$ and therefore

$$\beta_2z_1^2(b_1,t) + \beta_1z_2^2(a_2,t) \leq \max\{b_1-a_1, b_2-a_2\}|z(\cdot,t)|^2_{H^1}, \quad t\in\omega_\tau^+. \tag{4.7}$$

After performing partial summation, we easily get

$$\begin{aligned}[\psi,z]_h = & \frac{h_1}{2}\beta_2\xi_1(b_1,t)z_1(b_1,t) + (\xi,z)_h + \frac{h_2}{2}\beta_1\xi_2(a_2,t)z_2(a_2,t) \\ & - \beta_2(\rho_1, z_{1,\bar{x}}]_{h_1} - \beta_1[\rho_2, z_{2,x})_{h_2} \\ & + \frac{h_1}{2}\beta_2\mu_1(b_1,t)z_1(b_1,t) + (\mu,z)_h + \frac{h_2}{2}\beta_1\mu_2(a_2,t)z_2(a_2,t),\end{aligned}$$

and thus, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}[\psi,z]_h \leq & \Big(2\beta_2z_1^2(b_1,t) + 2\|z\|_h^2 + 2\beta_1z_2^2(a_2,t) + \beta_2\|z_{1,\bar{x}}]|^2_{h_1} + \beta_1|[z_{2,x}\|^2_{h_2}\Big)^{1/2} \\ & \times\bigg(\frac{h_1^2}{4}\beta_2|\xi_1(b_1,t)|^2 + \|\xi(\cdot,t)\|_h^2 + \frac{h_2^2}{4}\beta_1|\xi_2(a_2,t)|^2 \\ & + h_1\beta_2|\rho_1(b_1,t)|^2 + \|\rho(\cdot,t)\|_h^2 + h_2\beta_1|\rho_2(a_2,t)|^2 \\ & + \frac{h_1^2}{4}\beta_2|\mu_1(b_1,t)|^2 + \|\mu(\cdot,t)\|_h^2 + \frac{h_2^2}{4}\beta_1|\mu_2(a_2,t)|^2\bigg)^{1/2}.\end{aligned} \tag{4.8}$$

As a result of (4.5), (4.6) and (4.7) one finds that

$$\begin{aligned}
&2z_1^2(b_1,t)+2\|z\|_h^2+2z_2^2(a_2,t)+\beta_2\|z_{1,\bar{x}}]|_{h_1}^2+\beta_1|[z_{2,x}\|_{h_2}^2\\
&\le C_1|z(\cdot,t)|_{H^1}^2\le\frac{C_1}{C_0}[\psi,z]_h=:C[\psi,z]_h,\quad t\in\omega_\tau^+,
\end{aligned}\tag{4.9}$$

where $C_1=1+2\max\{b_1-a_1,b_2-a_2\}+2\max\left\{(b_1-a_1)^2,(b_2-a_2)^2\right\}$.

Finally, using (4.8) and (4.9) we have

$$\begin{aligned}
[\psi,z]_h\le&C\left(\frac{h_1^2}{4}\beta_2|\xi_1(b_1,t)|^2+\|\xi(\cdot,t)\|_h^2+\frac{h_2^2}{4}\beta_1|\xi_2(a_2,t)|^2\right.\\
&+h_1\beta_2|\rho_1(b_1,t)|^2+\|\rho(\cdot,t)\|_h^2+h_2\beta_1|\rho_2(a_2,t)|^2\\
&\left.+\frac{h_1^2}{4}\beta_2|\mu_1(b_1,t)|^2+\|\mu(\cdot,t)\|_h^2+\frac{h_2^2}{4}\beta_1|\mu_2(a_2,t)|^2\right).
\end{aligned}$$

Using the last result, we obtain (4.4) after multiplying (4.5) by τ and summing over the mesh ω_τ^+.

□

Theorem 4.3. *Let the assumptions of Theorem* 2.1 *hold,* $p_i,q_i\in H^2(\Omega_i)$, $i=1,2$, *and let* $u=(u_1,u_2)$ *be the solution of the initial-boundary value problem* (2.1)-(2.4), *where* u_i *belongs to the space* $C^2([0,T],C(\bar\Omega_i))\cap C([0,T],H^3(\Omega_i))\cap C_+^\alpha([0,T],H^2(\Omega_i))$. *Then, the solution* v *of the finite difference scheme* (3.4) *converges to* u *and the following convergence rate estimate holds:*

$$\|u-v\|_{B^{1,\alpha/2}(Q_{h\tau})}=O(h^2+\tau^{2-\alpha}),$$

where $h=\max\{h_1,h_2\}$.

Proof. Directly from Theorem 4.2, an a priori estimate for the problem (4.1) follows

$$\begin{aligned}
\|z\|_{B^{1,\alpha/2}(Q_{h\tau})}^2\le&C\tau\sum_{t\in\omega_\tau^+}\left(h_1^2|\xi_1(b_1,t)|^2+\|\xi(\cdot,t)\|_h^2+h_2^2|\xi_2(a_2,t)|^2\right.\\
&+h_1|\rho_1(b_1,t)|^2+\|\rho(\cdot,t)\|_h^2+h_2|\rho_2(a_2,t)|^2\\
&\left.+h_1^2|\mu_1(b_1,t)|^2+\|\mu(\cdot,t)\|_h^2+h_2^2|\mu_2(a_2,t)|^2\right).
\end{aligned}\tag{4.10}$$

Now it remains to determine the rate of convergence of the implicit scheme (3.4) by estimating the terms on the right-hand side of the inequality (4.10).

For $t\in\omega_\tau^+$, we decompose term ξ_i in the following manner $\xi_i=\xi_{i,1}+\xi_{i,2}$, where

$$\xi_{i,1}=\Delta_{t,0+}^\alpha u_i-\partial_{t,0+}^\alpha u_i,\quad \xi_{i,2}=\partial_{t,0+}^\alpha(u_i-T_x^2u_i),$$

for $x\in\omega_{h_i}$, $i=1,2$, and for $x=b_1$ and $x=a_2$ are

$$\begin{aligned}
\xi_{1,1}(b_1,t)&=\Delta_{t,0+}^\alpha u_1(b_1,t)-\partial_{t,0+}^\alpha u_1(b_1,t),\\
\xi_{1,2}(b_1,t)&=\partial_{t,0+}^\alpha(u_1-T_x^{2-}u_1)(b_1,t),\\
\xi_{2,1}(a_2,t)&=\Delta_{t,0+}^\alpha u_2(a_2,t)-\partial_{t,0+}^\alpha u_2(a_2,t),\\
\xi_{2,2}(a_2,t)&=\partial_{t,0+}^\alpha(u_2-T_x^{2+}u_2)(a_2,t).
\end{aligned}$$

Using inequality (4.2) we have

$$\tau \sum_{t\in\omega_\tau^+} \left(h_1|\xi_{1,1}(b_1,t)| + \|\xi_{1,1}(\cdot,t)\|_{h_1}\right)^2 \le C\tau^{2(2-\alpha)}\|u_1\|^2_{C^2([0,T],C(\bar\Omega_1))}. \tag{4.11}$$

Analogously, we evaluate $\xi_{2,1}$.

For $x \in \omega_{h_i}$, $i = 1, 2$, the following integral representation is valid

$$u_i - T_x^2 u_i = -\frac{1}{h_i}\int_{x-h_i}^{x+h_i}\int_{x'}^{x}\int_{x''}^{x}\left(1-\frac{|x'-x|}{h_i}\right)\frac{\partial^2 u_i}{\partial x^2}(x''',t)\,dx'''dx''dx',$$

while for $x = b_1$,

$$\begin{aligned} u_1 - T_x^{2-}u_1 =& \frac{2}{h_1}\int_{b_1-h_1}^{b_1}\int_{x'}^{b_1}\int_{b_1}^{x''}\left(1+\frac{x'-b_1}{h_1}\right)\frac{\partial^2 u_1}{\partial x^2}(x''',t)dx'''dx''dx' \\ &+ \frac{h_1}{3}\cdot\frac{\partial u_1}{\partial x}(b_1,t). \end{aligned}$$

Applying Cauchy-Schwarz inequality to the above integral representations, one can easily derive the following bounds

$$|u_i(x,t) - T_x^2u_i(x,t)| \le h_i^{3/2}\left\|\frac{\partial^2 u_i(\cdot,t)}{\partial x^2}\right\|_{L^2(x-h_i,x)}, \quad x\in\omega_{h_i}, \quad i = 1,\ 2,$$

and

$$|u_1(b_1,t) - T_x^{2-}u_1(b_1,t)| \le h_1^{1/2}\left\|\frac{\partial u_1(\cdot,t)}{\partial x}\right\|_{L^2(x-h_1,x)}.$$

At this point, we use boundary-strip estimate (4.3) on $\left\|\frac{\partial u_1(\cdot,t)}{\partial x}\right\|_{L^2(x-h_1,x)}$, which gives

$$\left\|\frac{\partial u_1(\cdot,t)}{\partial x}\right\|_{L^2(x-h_1,x)} \le C\sqrt{h_1}\|u_1(\cdot,t)\|_{H^2(\Omega_1)}.$$

Using these integral representations, we obtain

$$\begin{aligned} \tau\sum_{t\in\omega_\tau^+}\left(h_1^2|\xi_{1,2}(b_1,t)|^2 + \|\xi_{1,2}(\cdot,t)\|^2_{h_1}\right) &\le Ch_i^4\tau\sum_{t\in\omega_\tau^+}\|\partial^\alpha_{t,0+}u_1(\cdot,t)\|^2_{H^2(\Omega_1)} \\ &\le Ch_1^4\|u_1\|^2_{C_+^\alpha([0,T],H^2(\Omega_1))}. \end{aligned} \tag{4.12}$$

Analogously, we evaluate $\xi_{2,2}$. Let $\rho_i = \rho_{i,1} + \rho_{i,2} + \rho_{i,3}$, where

$$\begin{aligned} \rho_{i,1} =& T_x^-\left(p_i\frac{\partial u_i}{\partial x}\right) - \left(T_x^-p_i\right)\left(T_x^-\frac{\partial u_i}{\partial x}\right), \\ \rho_{i,2} =& \left(T_x^-p_i - \bar p_i\right)\left(T_x^-\frac{\partial u_i}{\partial x}\right), \\ \rho_{i,3} =& \bar p_i\left(T_x^-\frac{\partial u_i}{\partial x} - u_{\bar x}\right). \end{aligned}$$

These three terms are estimated by applying arguments analogous to those used in [4] and [12], so we obtain

$$\tau \sum_{t\in\omega_\tau^+} \|\rho_i(\cdot,t)\|_{h_i}^2 \le Ch_i^4 \|p\|_{H^2(\Omega_i)}^2 \|u_i\|_{C([0,T],H^3(\Omega_i))}^2.$$

For $x = b_1$, $t \in \omega_\tau^+$, we use decomposition $\rho_1 = \rho_{1,1} + \rho_{1,2}$, with

$$\rho_{1,1} = \left(T_x^- p_1\right)\left(T_x^- \frac{\partial u_1}{\partial x}\right) - T_x^-\left(p_1 \frac{\partial u_1}{\partial x}\right)$$

and

$$\rho_{1,2} = \left(\bar{p}_1 - T_x^- p_1\right)\left(T_x^- \frac{\partial u_1}{\partial x}\right).$$

Using the inequality (4.3), the term $\rho_{1,1}$ can be bounded by

$$|\rho_{1,1}(b_1,t)| \le Ch_1^2 \|p_1\|_{C^1(\bar\Omega_1)} \|u_1(\cdot,t)\|_{H^3(\Omega_1)},$$

while for the term $\rho_{1,2}$ we use the integral representation and the Cauchy-Schwarz inequality

$$|\rho_{1,2}(b_1,t)| \le Ch_1^{3/2} \|p_1\|_{H^2(b_1-h_1,b_1)} \|u_1(\cdot,t)\|_{H^3(\Omega_1)}.$$

For $x \in \omega_{h_i}$, $i = 1,2$, we can decompose $\mu_i = \mu_{i,1} + \mu_{i,2}$, with

$$\begin{aligned}\mu_{i,1} =& (T_x^2 q_i)(u_i - T_x^2 u_i),\\ \mu_{i,2} =& (T_x^2 q_i)(T_x^2 u_i) - T_x^2(q_i u_i),\end{aligned}$$

while for $x = b_1$, $t \in \omega_\tau^+$, we set $\mu_1 = \mu_{1,1} + \mu_{1,2} + \mu_{1,3}$ where

$$\begin{aligned}\mu_{1,1} =& \left(T_x^{2-} q_1\right)\left(T_x^{2-} u_1\right) - T_x^{2-}(q_1 u_1),\\ \mu_{1,2} =& T_x^{2-} u_1 \left(q_1 - T_x^{2-} q_1\right),\\ \mu_{1,3} =& q_1 \left(u_1 - T_x^{2-} u_1\right).\end{aligned}$$

Using similar techniques we bound these terms by

$$\begin{aligned}&\tau \sum_{t\in\omega_{\tau+}} \|\mu_{i,1}(\cdot,t)\|_{L^2(\omega_{h_i})}^2 \le Ch_i^4 \|q_i\|_{C(\bar\Omega_i)}^2 \|u_i\|_{C([0,T],H^2(\Omega_i))}^2,\\ &\tau \sum_{t\in\omega_{\tau+}} \|\mu_{i,2}(\cdot,t)\|_{L^2(\omega_{h_i})}^2 \le Ch_i^4 \|q_i\|_{C^1(\bar\Omega_i)}^2 \|u_i\|_{C([0,T],H^2(\Omega_i))}^2,\\ &|\mu_{1,1}(b_1,t)| \le Ch_1 \|q_1'\|_{L^2(b_1-h_1,b_1)} \|u_1(\cdot,t)\|_{H^1(b_1-h_1,b_1)},\\ &|\mu_{1,2}(b_1,t)| \le Ch_1 \|u_1(\cdot,t)\|_{C(\bar\Omega_1)} \|q_1\|_{H^2(\Omega_1)}\end{aligned}$$

and

$$|\mu_{1,3}(b_1,t)| \le Ch_1 \|u_1(\cdot,t)\|_{H^2(\Omega_1)} \|q_1\|_{C(\bar\Omega_1)}.$$

Analogously, we can estimate terms ρ_2 and μ_2, for $x = a_2$, $t \in \omega_\tau^+$. Combining previous results, we obtain the result of the theorem. □

5. Numerical Experiment

Example 5.1. For the first numerical experiment, we consider an example with $\Omega_1 = (0,1)$ and $\Omega_2 = (2,3)$, $T = 1$, $p_1(x) = p_2(x) = 1$, $q_1(x) = q_2(x) = 1$, $\beta_1 = \beta_2 = 1$, $\sigma_1 = e^3$, $\sigma_2 = e^{-3}$,

$$f_1(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} x e^{-x} + 2t^2 e^{-x}$$

and

$$f_2(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} (3-x) e^{x} + 2t^2 e^{x}.$$

Exact solution to the appropriate problem is given by

$$u_1(x,t) = t^2 x e^{-x}, \quad u_2(x,t) = t^2 (3-x) e^{x}.$$

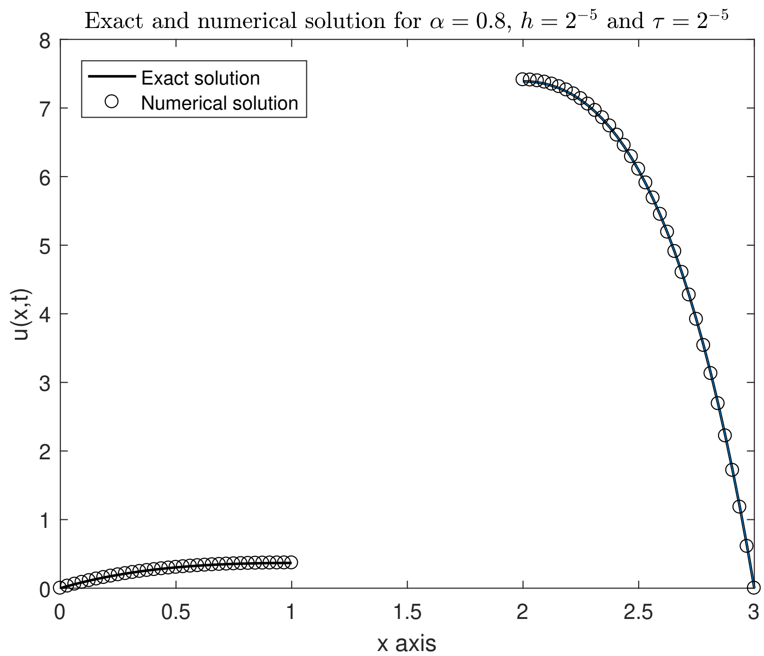

Figure 1. Exact and numerical solution for Example 5.1 with $\alpha = 0.8$, $h = 2^{-5}$ and $\tau = 2^{-5}$

The solutions u_1 and u_2 exhibit a jump or discontinuity at the interface and cannot be smoothly matched.

Figure 1 illustrates both the exact and numerical solutions. Tables 1 and 2 present the computational results for scheme (3.4). The errors have been computed in the discrete maximum norm (denoted by $\|\cdot\|_{C(Q_{h\tau})}$) and discrete $\|\cdot\|_{B^{1,\alpha/2}(Q_{h\tau})}$- norm (denoted in Tables by $\|\cdot\|_{B(Q_{h\tau})}$).

TABLE 1. Errors and convergence rates in the spatial direction with $\tau = 2^{-14}$ of scheme (3.4) for Example 5.1

α	h	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h/2,\tau})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h/2,\tau})}}$
0.9	2^{-2}	$1.1585e-01$	1.98	$8.0452e-02$	1.99
	2^{-3}	$2.9455e-02$	2.00	$2.0299e-02$	2.00
	2^{-4}	$7.3651e-03$	2.02	$5.0576e-03$	2.03
	2^{-5}	$1.8109e-03$		$1.2346e-03$	
0.7	2^{-2}	$1.2294e-01$	1.98	$7.8713e-02$	1.98
	2^{-3}	$3.1244e-02$	1.99	$1.9895e-02$	2.00
	2^{-4}	$7.8405e-03$	2.00	$4.9845e-03$	2.00
	2^{-5}	$1.9589e-03$		$1.2439e-03$	
0.5	2^{-2}	$1.2932e-01$	1.98	$7.7423e-02$	1.98
	2^{-3}	$3.2831e-02$	1.99	$1.9577e-02$	2.00
	2^{-4}	$8.2394e-03$	2.00	$4.9082e-03$	2.00
	2^{-5}	$2.0616e-03$		$1.2276e-03$	

TABLE 2. Errors and convergence rates in the time direction with $h = 2^{-13}$ of scheme (3.4) for Example 5.1

α	τ	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h\tau/2})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h\tau/2})}}$
0.9	2^{-5}	$4.1210e-02$	1.10	$4.8016e-02$	1.10
	2^{-6}	$1.9276e-02$	1.10	$2.2456e-02$	1.10
	2^{-7}	$9.0041e-03$	1.10	$1.0490e-02$	1.10
	2^{-8}	$4.2032e-03$		$4.8972e-03$	
0.7	2^{-5}	$1.4416e-02$	1.29	$1.7240e-02$	1.29
	2^{-6}	$5.8826e-03$	1.30	$7.0453e-03$	1.29
	2^{-7}	$2.3957e-03$	1.30	$2.8727e-03$	1.30
	2^{-8}	$9.7457e-04$		$1.1696e-03$	
0.5	2^{-5}	$4.7843e-03$	1.48	$5.8173e-03$	1.48
	2^{-6}	$1.7130e-03$	1.49	$2.0916e-03$	1.48
	2^{-7}	$6.1096e-04$	1.49	$7.4863e-04$	1.49
	2^{-8}	$2.1732e-04$		$2.6704e-04$	

Example 5.2. For our second example, we consider $\Omega_1 = (0, 1/3)$ and $\Omega_2 = (2/3, 1)$, $T = 1$, $p_1(x) = p_2(x) = 1$, $q_1(x) = q_2(x) = 0$, $\beta_1 = 2\sqrt{2}$, $\beta_2 = \frac{\sqrt{2}}{2}$, $\sigma_1 = 2$, $\sigma_2 = 1$ and

$$f_i(x,t) = \frac{\Gamma(3.4)}{\Gamma(3.4-\alpha)} t^{2.4-\alpha} x^{\frac{5}{2}}(1-x) + \frac{5}{4} t^{2.4} x^{\frac{1}{2}}\left(7x - 3\right), \quad i = 1, 2.$$

Exact solution for this problem is

$$u_i(x,t) = t^{2.4} x^{\frac{5}{2}}(1-x), \quad i = 1, 2.$$

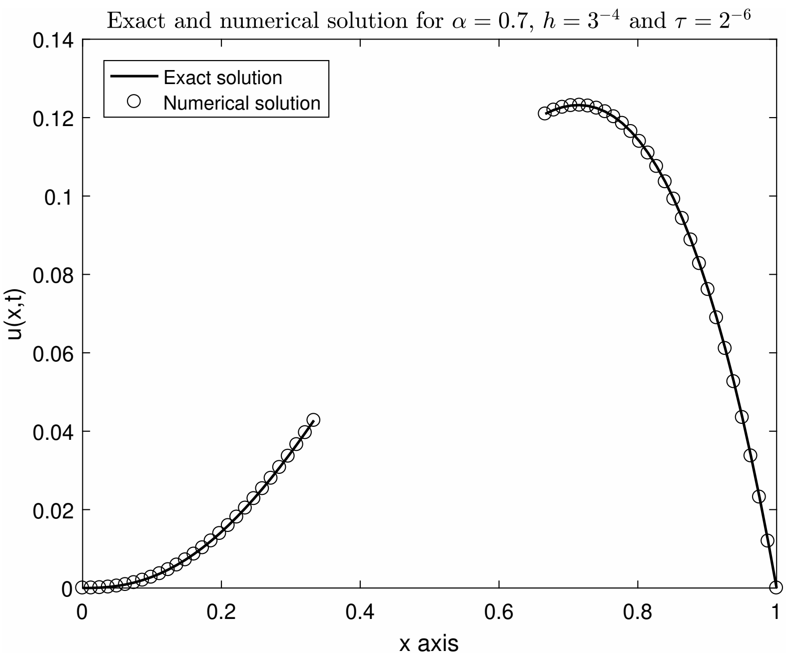

FIGURE 2. Exact and numerical solution for Example 5.2 with $\alpha = 0.7$, $h = 3^{-4}$ and $\tau = 2^{-6}$.

Exact and numerical solutions for this example are shown in Figure 2, while the computational results for the scheme (3.4) are given in Tables 3 and 4.

TABLE 3. Errors and convergence rates in the spatial direction with $\tau = 2^{-14}$ of scheme (3.4) for Example 5.2.

α	h	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h\tau/2})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h\tau/2})}}$
0.5	3^{-2}	$4.7944e-03$	1.99	$6.1940e-03$	1.99
	3^{-3}	$5.3600e-04$	2.00	$6.9688e-04$	1.99
	3^{-4}	$5.9698e-05$	2.00	$7.8146e-05$	1.99
	3^{-5}	$6.6397e-06$		$8.7471e-06$	
0.7	3^{-2}	$4.7451e-03$	1.99	$6.2193e-03$	1.99
	3^{-3}	$5.3069e-04$	2.00	$6.9950e-04$	1.99
	3^{-4}	$5.9093e-05$	2.00	$7.8403e-05$	2.00
	3^{-5}	$6.5576e-06$		$8.7460e-06$	
0.9	3^{-2}	$4.6918e-03$	1.99	$6.2718e-03$	1.99
	3^{-3}	$5.2482e-04$	2.00	$7.0480e-04$	2.00
	3^{-4}	$5.8303e-05$	2.02	$7.8709e-05$	2.02
	3^{-5}	$6.3314e-06$		$8.5089e-06$	

TABLE 4. Errors and convergence rates in the time direction with $h = 3^{-9}$ of scheme (3.4) for Example 5.2.

α	τ	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h\tau/2})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h\tau/2})}}$
0.9	2^{-5}	$1.6604e-04$	1.10	$4.4695e-04$	1.10
	2^{-6}	$7.7709e-05$	1.10	$2.0844e-04$	1.10
	2^{-7}	$3.6312e-05$	1.10	$9.7239e-05$	1.10
	2^{-8}	$1.6955e-05$		$4.5365e-05$	
0.7	2^{-5}	$6.0823e-05$	1.29	$1.6308e-04$	1.29
	2^{-6}	$2.4880e-05$	1.29	$6.6560e-05$	1.29
	2^{-7}	$1.0147e-05$	1.30	$2.7128e-05$	1.30
	2^{-8}	$4.1310e-06$		$1.1044e-05$	
0.5	2^{-5}	$2.0858e-05$	1.48	$5.5449e-05$	1.48
	2^{-6}	$7.5010e-06$	1.48	$1.9936e-05$	1.48
	2^{-7}	$2.6827e-06$	1.49	$7.1384e-06$	1.49
	2^{-8}	$9.5561e-07$		$2.5472e-06$	

Example 5.3. For our last example, we consider $\Omega_1 = (0, 1/4)$ and $\Omega_2 = (3/4, 1)$, $T = 1$, $p_1(x) = p_2(x) = x$, $q_1(x) = q_2(x) = 0$, $\beta_1 = \beta_2 = \sigma_1 = \sigma_2 = 1$, and

$$f_i(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin^2(2\pi x)) - t^2(8\pi^2 x \cos(4\pi x) + 2\pi \sin(4\pi x)), \quad i = 1, 2.$$

Unlike the previous examples, in this case we have functional coefficients. The exact solution is known to be $u_i(x,t) = t^2 \sin^2(2\pi x)$, $i = 1, 2$.

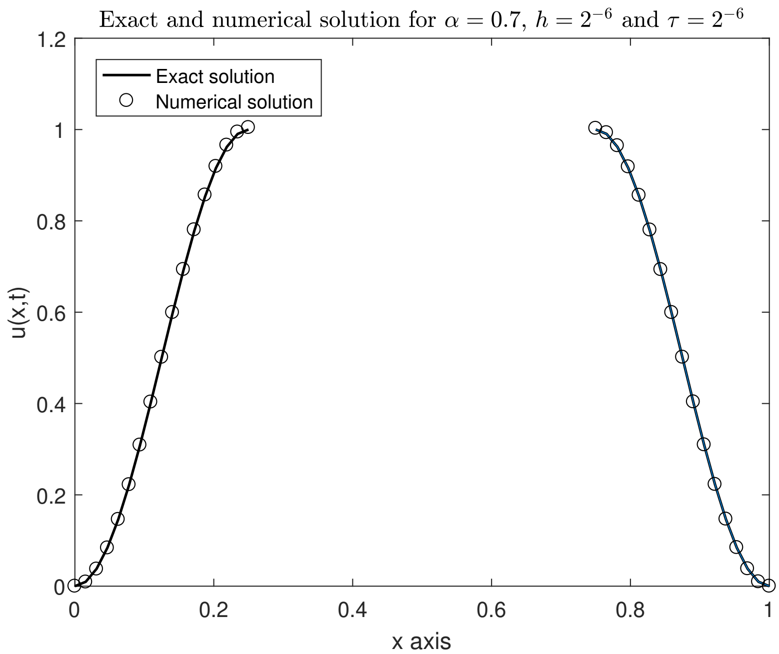

FIGURE 3. Exact and numerical solution for Example 5.3 with $\alpha = 0.7$, $h = 2^{-6}$ and $\tau = 2^{-6}$

Exact and numerical solutions for this example are shown in Figure 3, while the computational results for the scheme (3.4) are given in Tables 6 and 5.

The numerical results in all of these examples indicate that the convergence order is $2 - \alpha$ in the time direction and 2 in the spatial direction.

Table 5. Errors and convergence rates in the spatial direction with $\tau = 2^{-14}$ of scheme (3.4) for Example 5.3

α	h	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h\tau/2})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h\tau/2})}}$
0.9	2^{-4}	$6.6497e-02$	2.05	$9.1579e-02$	1.93
	2^{-5}	$1.6074e-02$	2.02	$2.4079e-02$	1.92
	2^{-6}	$3.9610e-03$	2.01	$6.3786e-03$	1.90
	2^{-7}	$9.8386e-04$		$1.7078e-03$	
0.7	2^{-4}	$6.7811e-02$	2.05	$9.1475e-02$	1.93
	2^{-5}	$1.6391e-02$	2.02	$2.3961e-02$	1.92
	2^{-6}	$4.0372e-03$	2.01	$6.3287e-03$	1.90
	2^{-7}	$1.0008e-03$		$1.6931e-03$	
0.5	2^{-4}	$6.9036e-02$	2.05	$9.1550e-02$	1.94
	2^{-5}	$1.6687e-02$	2.02	$2.3896e-02$	1.93
	2^{-6}	$4.1099e-03$	2.01	$6.2923e-03$	1.91
	2^{-7}	$1.0185e-03$		$1.6787e-03$	

Table 6. Errors and convergence rates in the time direction with $h = 2^{-14}$ of scheme (3.4) for Example 5.3

α	τ	$\|z\|_{C(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{C(Q_{h\tau})}}{\|z\|_{C(Q_{h\tau/2})}}$	$\|z\|_{B(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{B(Q_{h\tau})}}{\|z\|_{B(Q_{h\tau/2})}}$
0.9	2^{-5}	$3.3153e-03$	1.10	$7.4862e-02$	1.10
	2^{-6}	$1.5514e-03$	1.10	$3.5031e-02$	1.10
	2^{-7}	$7.2488e-04$	1.10	$1.6370e-02$	1.10
	2^{-8}	$3.3843e-04$		$7.6438e-03$	
0.7	2^{-5}	$1.1588e-03$	1.29	$2.7340e-02$	1.29
	2^{-6}	$4.7301e-04$	1.30	$1.1184e-02$	1.29
	2^{-7}	$1.9267e-04$	1.30	$4.5633e-03$	1.30
	2^{-8}	$7.8394e-05$		$1.8586e-03$	
0.5	2^{-5}	$3.8409e-04$	1.48	$9.3671e-03$	1.47
	2^{-6}	$1.3756e-04$	1.49	$3.3715e-03$	1.48
	2^{-7}	$4.9079e-05$	1.49	$1.2076e-03$	1.49
	2^{-8}	$1.7470e-05$		$4.3084e-04$	

6. Conclusion

In this work, we studied a fractional-in-time transmission problem with a Caputo derivative of order $\alpha \in (0,1)$, posed on two disjoint spatial subdomains with appropriate boundary and interface conditions. We first established the well-posedness and stability of the weak formulation in suitable Sobolev-type spaces and derived a priori estimates for the exact solution. Building on this analysis, we developed a finite-difference scheme that employs the L1 time discretization for the fractional derivative and a second-order spatial approximation.

Our convergence analysis shows that the proposed scheme achieves a temporal accuracy of order $2-\alpha$ for solutions that are C^2 in time, while C^2 spatial regularity yields second-order accuracy in space. The error estimates are explicit with respect to the smoothness of the input data, clarifying how regularity impacts the accuracy of the numerical approximation.

Future research directions include extending the analysis to diffusion–wave equations with Caputo derivatives of order $\alpha \in (1,2)$, as well as to higher-dimensional transmission problems and more complex interface conditions.

Appendix A. Different conditions

In this appendix we are going to consider a case where Conditions (2.7) are replaced with

$$\sigma_i > 0, \quad \beta_i > 0, \quad i = 1,2, \quad \text{and} \quad \beta_1 \le \sigma_1, \quad \beta_2 \le \sigma_2. \tag{A.1}$$

For these conditions we can prove another stability result for the solution of (3.4).

Theorem A.1. *Let $\alpha \in (0,1)$, $f = (f_1, f_2) \in L^2$, and let conditions* (2.5), (2.6) *and* (A.1) *be satisfied. Then, the initial-boundary value problem* (2.8) *is well posed in $H_+^{2,\alpha} \cap H_0^{1,\alpha/2}$ and its solution satisfies the following a priori estimate*

$$\|u\|_{H_+^{2,\alpha}} \le C\|f\|_{L^2}. \tag{A.2}$$

Proof. Let us define $\partial^\alpha_{t,0_+} u = (\partial^\alpha_{t,0_+} u_1, \partial^\alpha_{t,0_+} u_2)$ and $\mathcal{L}u = (\mathcal{L}_1 u_1, \mathcal{L}_2 u_2)$, where

$$\mathcal{L}_i u_i = -\frac{\partial}{\partial x}\left(p_i(x)\frac{\partial u_i}{\partial x}\right) + q_i(x) u_i, \quad i = 1,2. \tag{A.3}$$

Then, (2.1) is equivalent to

$$\partial^\alpha_{t,0_+} u + \mathcal{L}u = f. \tag{A.4}$$

It can be easily shown that $(\mathcal{L}u, v)_{L^2} = (u, \mathcal{L}v)_{L^2}$. From (A.4) we have

$$\left(\partial^\alpha_{t,0_+} u, \mathcal{L}u\right)_{L^2} + (\mathcal{L}u, \mathcal{L}u)_{L^2} = (f, \mathcal{L}u)_{L^2}. \tag{A.5}$$

Using integration by parts we have

$$\left(\partial^\alpha_{t,0_+} u, \mathcal{L}u\right)_{L^2} = \beta_1\beta_2 \left(\partial^\alpha_{t,0_+}(u_1(b_1,\cdot) - u_2(a_2,\cdot)), u_1(b_1,\cdot) - u_2(a_2,\cdot)\right)_{L^2(0,T)}$$

$$
\begin{aligned}
&+ \beta_2(\sigma_1 - \beta_1)\Big(\partial^\alpha_{t,0_+} u_1(b_1,\cdot), u_1(b_1,\cdot)\Big)_{L^2(0,T)} \\
&+\beta_1(\sigma_2 - \beta_2)\Big(\partial^\alpha_{t,0_+} u_2(a_2,\cdot), u_2(a_2,\cdot)\Big)_{L^2(0,T)} \\
&+ \sum_{i=1,2} \beta_{3-i} \left(\int_0^T \int_{a_i}^{b_i} \frac{\partial}{\partial x}\left(\partial^\alpha_{t,0_+} u_i\right) p_i(x) \frac{\partial u_i}{\partial x} dx dt \right.\\
&\left. + \int_0^T \int_{a_i}^{b_i} \left(\partial^\alpha_{t,0_+} u_i\right) q_i(x) u_i dx dt \right).
\end{aligned}
$$

From Lemma 1.1, Lemma 1.2, conditions (2.6) and (A.1) we conclude that

$$(\partial^\alpha_{t,0_+} u, \mathcal{L} u)_{L^2} \geq 0. \tag{A.6}$$

From (A.5) and (A.6), using the Cauchy-Schwarz inequality for the L^2 norm, we get

$$\|\mathcal{L}u\|_{L^2} \leq \|f\|_{L^2}. \tag{A.7}$$

Next, from (A.3) we can obtain the following

$$p_{0i} \left\| \frac{\partial^2 u_i}{\partial x^2} \right\|_{L^2(\Omega_i)} \leq \|\mathcal{L}_i u_i\|_{L^2(\Omega_i)} + \|p_i\|_{W^{1,\infty}(\Omega_i)} \left\| \frac{\partial u_i}{\partial x} \right\|_{L^2(\Omega_i)} + \|q_i\|_{L^\infty(\Omega_i)} \|u_i\|_{L^2(\Omega_i)}.$$

From this, (A.7), and Theorem 2.1 it is clear that

$$\|u\|_{H^2} \leq C\, \|f\|_{L^2}. \tag{A.8}$$

Lastly, from $\left(\partial^\alpha_{t,0_+} u, \partial^\alpha_{t,0_+} u\right)_{L^2} + \left(\mathcal{L}u, \partial^\alpha_{t,0_+} u\right)_{L^2} = \left(f, \partial^\alpha_{t,0_+} u\right)_{L^2}$ and (A.6) we easily get $\left\|\partial^\alpha_{t,0_+} u\right\|_{L^2} \leq \|f\|_{L^2}$, from which, together with (A.8), the a priori estimate (A.2) immediately follows. □

Next, we present and prove the discrete version of the Theorem A.1. First, we define the following discrete norms:

$$
\begin{aligned}
&|v|^2_{H^2} = \beta_2 \|v_{x\bar{x}}\|^2_{h_1} + \beta_1 \|v_{x\bar{x}}\|^2_{h_2}, \\
&\|v\|^2_{H^2} = |v|^2_{H^2} + \|v\|^2_{H^1}, \\
&\|v\|_{H^{2,\alpha}_+(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \left(\|v(\cdot,t)\|^2_{H^2} + \Delta^\alpha_{t,0_+} |[v(\cdot,t)]|^2_h \right) \right)^{1/2}.
\end{aligned}
$$

Theorem A.2. *Under the assumptions of Theorem* A.1 *the finite difference scheme* (3.4) *is absolutely stable and its solution satisfies the a priori estimate:*

$$\|v\|_{H^{2,\alpha}_+(Q_{h\tau})} \leq C |[\tilde{f}]|_{L^2(Q_{h\tau})}. \tag{A.9}$$

Proof. First, let us take the inner product $[\Delta^\alpha_{t,0_+} v + L_h v, L_h v]_h = [\tilde{f}, L_h v]_h$. After applying partial summation we get

$$
\begin{aligned}
[\Delta^\alpha_{t,0_+} v, L_h v]_h =& \beta_2 (\Delta^\alpha_{t,0_+} v_{1,\bar{x}}, \bar{p}_1 v_{1,\bar{x}}]_{h_1} + \beta_1 [\Delta^\alpha_{t,0_+} v_{2,x}, \bar{p}_2(\cdot + h_2) v_{2,x})_{h_2} + [\Delta^\alpha_{t,0_+} v, \tilde{q} v]_h \\
&+ \beta_1\beta_2 (v_1(b_1,t) - v_2(a_2,t))(\Delta^\alpha_{t,0_+}(v_1(b_1,t) - v_2(a_2,t)))
\end{aligned}
$$

$$+ \beta_2(\sigma_1 - \beta_1)v_1(b_1,t)\Delta^{\alpha}_{t,0_+}v_1(b_1,t)$$
$$+ \beta_1(\sigma_2 - \beta_2)v_2(a_2,t)\Delta^{\alpha}_{t,0_+}v_2(a_2,t).$$

Using Lemma 3.2, conditions (2.6) and (A.1) we get

$$[\Delta^{\alpha}_{t,0_+}v, L_h v]_h \geq \frac{1}{2}\min\{p_{01},p_{02}\}\,\Delta^{\alpha}_{t,0_+}|v|^2_{H^1} \geq 0.$$

Thus, using the Cauchy-Schwarz inequality we get $|[L_h v]|_h \leq |[\tilde{f}]|_h$.

Analogously to the proof of the Theorem A.1 we obtain $\|v\|_{H^2} \leq C|[\tilde{f}]|_h$ and $|[\Delta^{\alpha}_{t,0_+}v]|_h \leq |[\tilde{f}]|_h$. The result (A.9) now follows in a straightforward way. □

Example A.1. We consider an example with $\Omega_1 = (0,1/4)$ and $\Omega_2 = (3/4,1)$, $T = 1$, $p_1(x) = p_2(x) = 1$, $q_1(x) = q_2(x) = 0$, $\beta_1 = \beta_2 = 1 = \sigma_1 = \sigma_2 = 1$ and

$$f_i(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin^2(2\pi x) - 8\pi^2 t^2\cos(4\pi x), \quad i = 1,2.$$

The exact solution is $u_i = t^2\sin^2(2\pi x)$, $i = 1,2$.

Computational results for scheme (3.4) are given in Table 7. The values in the table indicate a convergence order of 2 in the temporal direction and $2-\alpha$ in the spatial direction.

Table 7. Errors and convergence orders (CO) in the time direction with $h = 2^{-14}$ and the spatial direction with $\tau = 2^{-14}$, for Example A.1

α	h	$\|z\|_{H^{2,\alpha}(Q_{h\tau})}$	CO(h)	τ	$\|z\|_{H^{2,\alpha}(Q_{h\tau})}$	CO(τ)
0.9	2^{-6}	$4.4127e-02$	1.95	2^{-5}	$1.4239e-02$	1.10
	2^{-7}	$1.1433e-02$	1.97	2^{-6}	$6.6299e-03$	1.10
	2^{-8}	$2.9116e-03$	1.98	2^{-7}	$3.0906e-03$	1.10
	2^{-9}	$7.3955e-04$		2^{-8}	$1.4416e-03$	
0.7	2^{-6}	$4.4286e-02$	1.95	2^{-5}	$5.2174e-03$	1.29
	2^{-7}	$1.1469e-02$	1.98	2^{-6}	$2.1290e-03$	1.29
	2^{-8}	$2.9157e-03$	1.99	2^{-7}	$8.6812e-04$	1.30
	2^{-9}	$7.3540e-04$		2^{-8}	$3.5376e-04$	
0.5	2^{-6}	$4.4425e-02$	1.95	2^{-5}	$1.7749e-03$	1.47
	2^{-7}	$1.1505e-02$	1.98	2^{-6}	$6.3912e-04$	1.48
	2^{-8}	$2.9242e-03$	1.99	2^{-7}	$2.2942e-04$	1.48
	2^{-9}	$7.3698e-04$		2^{-8}	$8.2234e-05$	

Acknowledgements. The research of the authors was partially funded by Faculty of Mathematics University of Belgrade through the grant by the Ministry of Science, Technological Development, and Innovation of the Republic of Serbia (the contract no 451-03-136/2025-03/200104). The work of V. Brkovi' c is supported by the Ministry of Science, Technological Development and Innovations of the Republic of Serbia (the contract no 451-03-136/2025-03/ 200151).

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. A. Alikhanov, *Boundary value problems for the diffusion equation of the variable order in differential and difference settings*, Appl. Math. Comput. **219** (2012), 3938–3946. https://doi.org/10.1016/j.amc.2012.10.029

[3] A. Delić and B. S. Jovanović, *Numerical approximation of an interface problem for fractional in time diffusion equation*, Appl. Math. Comput. **229** (2014), 467–479. https://doi.org/10.1016/j.amc.2013.12.060

[4] A. Delić, S. Živanović and Z. Milovanović Jeknić, *A finite difference scheme for a linear multi-term fractional in time differential equation with concentrated capacities*, Int. J. Numer. Anal. Model. **18** (2021), 265–286.

[5] A. M. Delić, B. S. Jovanović and Z. Milovanović, *On the transmission eigenvalue problem in disjoint domains*, Comput. Methods Appl. Math. **11** (2011), 407–417. https://doi.org/10.2478/cmam-2011-0023

[6] V. J. Ervin and J. P. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Partial Differential Equations **22** (2006), 558–576. https://doi.org/10.1002/num.20112

[7] A. H. Ganie, A. M. Saeed, S. Saeed and U. Ali, *The Rayleigh-Stokes problem for a heated generalized second-grade fluid with fractional derivative: An implicit scheme via Riemann-Liouville integral*, Mathematical Problems in Engineering (2022), Article ID 6948461. https://doi.org/10.1155/2022/6948461

[8] S. Gegovska-Zajkova, B. S. Jovanović and I. M. Jovanović, *On the numerical solution of a transmission eigenvalue problem*, Lecture Notes in Comput. Sci. **5434** (2009), 289–296. https://doi.org/10.1007/978-3-642-00464-3_31

[9] P. Grisvard, *Elliptic Problems in Non-smooth Domains*, Pitman, London, 1985.

[10] B. S. Jovanović and Z. Milovanović, *Finite difference approximation of a parabolic problem with variable coefficients*, Publ. Inst. Math. (Beograd) **95**(109) (2014), 49–62. https://doi.org/10.2298/PIM1409049J

[11] B. S. Jovanović and Z. Milovanović, *Numerical approximation of 2d parabolic transmission problem in disjoint domains*, Appl. Math. Comput. **228** (2014), 508–519. https://doi.org/10.1016/j.amc.2013.12.002

[12] B. S. Jovanović and E. Süli, *Analysis of Finite Difference Schemes: For Linear Partial Differential Equations with Generalized Solutions*, Springer-Verlag, London, 2014.

[13] B. S. Jovanović and L. G. Vulkov, *Numerical solution of a hyperbolic transmission problem*, Comput. Methods Appl. Math. **8** (2008), 374–385. https://doi.org/10.2478/cmam-2008-0027

[14] B. S. Jovanović and L. G. Vulkov, *Numerical solution of a parabolic transmission problem*, IMA J. Numer. Anal. **31** (2011), 233–253. https://doi.org/10.1093/imanum/drn077

[15] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science and Technology, Boston, 2006.

[16] X. Li and C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal. **47** (2009), 2108–2131. https://doi.org/10.1137/080718942

[17] X. Li and C. Xu, *Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation*, Commun. Comput. Phys. **8** (2010), 1016–1051. https://doi.org/10.4208/cicp.020709.221209a

[18] Z. Milovanović, *Convergence of a finite difference scheme for a parabolic transmission problem in disjoint domains*, in: *Proceedings in Applied Mathematics and Mechanics* (2013), 433–434. https://doi.org/10.1002/pamm.201310211
[19] Z. Milovanović, *Finite difference scheme for a parabolic transmission problem in disjoint domains*, in: *Numerical Analysis and Its Applications* 2012, Springer, 2013, 403–410. https://doi.org/10.1007/978-3-642-41515-9_45
[20] Z. Milovanović, *Elliptic transmission problem in disjoint domains*, Mat. Vesnik **66** (2014), 418–429.
[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Boston, New York, London, Tokyo, Toronto, 1998.
[22] F. M. Salama, U. Ali and A. Ali, *Numerical solution of two-dimensional time fractional mobile/immobile equation using explicit group methods*, Int. J. Appl. Comput. Math. **8** (2022), Article ID 188. https://doi.org/10.1007/s40819-022-01408-z
[23] F. M. Salama, N. N. A. Hamid, U. Ali and N. H. M. Ali, *Fast hybrid explicit group methods for solving* 2*d fractional advection-diffusion equation*, AIMS Mathematics **7** (2022), 15854–15880. https://doi.org/10.3934/math.2022868
[24] A. A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker Inc., New York, 2001.
[25] A. A. Samarskii, R. D. Lazarov and V. L. Makarov, *Difference Schemes for Differential Equations with Generalized Solutions*, Vysshaya Shkola, Moscow, 1987.
[26] Z.-Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math. **56** (2006), 193–209. https://doi.org/10.1016/j.apnum.2005.03.003
[27] T. Wang and Y. Chen, *Nitsche-XFEM for a time fractional diffusion interface problem*, Sci. China Math. **67** (2024), 665–682. https://doi.org/10.1007/s11425-021-2062-6

[1]Faculty of Mathematics,
University of Belgrade,
Belgrade, Serbia
Email address: sandra.zivanovic@matf.bg.ac.rs
ORCID iD: https://orcid.org/0000-0002-0576-6916
Email address: aleksandra.delic@matf.bg.ac.rs
ORCID iD: https://orcid.org/0000-0002-7728-4342

[2]Faculty of Organizational Sciences,
University of Belgrade,
Belgrade, Serbia
Email address: vukasin.brkovic@fon.bg.ac.rs
ORCID iD: https://orcid.org/0000-0002-7141-832X

[3]Faculty of Construction Management,
University Union-Nikola Tesla,
Belgrade, Serbia
Email address: zmilovanovic@unionnikolatesla.edu.rs
ORCID iD: https://orcid.org/0000-0002-7867-5281