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STABILITY OF AN l-VARIABLE CUBIC FUNCTIONAL EQUATION

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Abstract. Using the direct and fixed point methods, we obtain the solution and prove the Hyers-Ulam stability of the l-variable cubic functional equation

$$\begin{aligned} & f\left(\sum_{i=1}^{l} x_i\right)+\sum_{j=1}^{l} f\left(-l x_j+\sum_{i=1, i \neq j}^{l} x_i\right) \\ = & -2(l+1) \sum_{i=1, i \neq j \neq k}^{l} f\left(x_i+x_j+x_k\right)+\left(3 l^2-2 l-5\right) \sum_{i=1, i \neq j}^{l} f\left(x_i+x_j\right) \\ & -3\left(l^3-l^2-l+1\right) \sum_{i=1}^{l} f\left(x_i\right), \end{aligned}$$

$l \in \mathbb{N}$, $l \geq 3$, in random normed spaces.

1. Introduction

The theory of random normed space (briefly, RN-space) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. It is a practical tool for handling situations where classical theories fail to explain. Random theory has much application in several fields, for example, population dynamics, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence and so forth. The Cauchy additive equation

$$f(x+y)=f(x)+f(y)$$

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has been studied by many authors [7, 9, 13, 15, 19]. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping (see [5, 8, 11, 12, 20]). A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [23] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof [23] is still true if the relevant domain X is replaced by an Abelian group. Jun and Kim [14] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.1}$$

and they established the solution and the Hyers-Ulam stability for the functional equation. The function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation (see [3, 6, 10, 16, 21, 22]). Czerwik [5] proved the Hyers-Ulam stability of the additive, quadratic and cubic functional equation.

Using the direct and fixed point methods, we obtain the solution and prove the Hyers-Ulam stability of the l-variable cubic functional equation

$$\begin{aligned} & f\left(\sum_{i=1}^{l} x_i\right) + \sum_{j=1}^{l} f\left(-lx_j + \sum_{i=1, i\neq j}^{l} x_i\right) \\ &= -2(l+1)\sum_{i=1, i\neq j\neq k}^{l} f(x_i+x_j+x_k) + (3l^2-2l-5)\sum_{i=1, i\neq j}^{l} f(x_i+x_j) \\ &\quad -3(l^3-l^2-l+1)\sum_{i=1}^{l} f(x_i), \end{aligned} \tag{1.2}$$

$l \in \mathbb{N}$, $l \geq 3$, in random normed spaces.

2. Preliminaries

In this section, we present some notations and basic definitions used in this article.

Definition 2.1. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm if T satisfies the following condition:

a) T is commutative and associative;
b) T is continuous;
c) $T(a, 1) = a$ for all $a \in [0, 1]$;
d) $T(a, b) \leq T(c, d)$ when $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_p(a, b) = ab$, $T_m(a, b) = \min\{a, b\}$ and $T_L(a, b) = \max\{a+b-1, 0\}$ (The Lukasiewicz t-norm). Recall [7] that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^{n} x_{n+i}$ is defined recurrently by $T_{i=1}^{1}x_i = x_i$ and $T_{i=1}^{n}x_i = T\left(T_{i=1}^{n-1}x_i, x_n\right)$ for $n \geq 2$, $T_{i=1}^{\infty}x_i$ is defined as $T_{i=1}^{\infty}x_{n+i}$. It is known that, for the Lukasiewicz t-norm, the following holds:

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

Definition 2.2. A random normed space (briefly, RN-space) is a triple, where X is a vector space. T is a continuous t-norm and μ is a mapping from X into D^+ satisfying the following conditions:

(RN1) $\mu_x(t) = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{ax}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, μ, T) be an RN-space.

1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ for all $n > N$.
2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
3) The RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X. For more details we can go through [1, 2, 4, 13, 18].

Throughout this paper, assume that X is a vector space and (Y, μ, T) is a complete random normed space. All over this paper we use the following notation for a given mapping $f : X \to Y$

$$\begin{aligned} Df(x_1, \ldots, x_l) =& f\left(\sum_{i=1}^{l} x_i\right) + \sum_{j=1}^{l} f\left(-lx_j + \sum_{i=1, i\neq j}^{l} x_i\right) \\ &+ 2(l+1)\sum_{i=1, i\neq j\neq k}^{l} f(x_i + x_j + x_k) - (3l^2 - 2l - 5)\sum_{i=1, i\neq j}^{l} f(x_i + x_j) \\ &+ 3(l^3 - l^2 - l + 1)\sum_{i=1}^{l} f(x_i), \end{aligned}$$

for all $x_1, x_2, x_3, \ldots, x_l \in X$.

3. Solution of the l-Variable Cubic Functional Equation in (1.2)

In this section, we investigate the solution of the l-variable cubic functional equation (1.2).

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies (1.2), then the mapping $f : X \to Y$ is cubic.*

Proof. Letting $x_1 = x_2 = \cdots = x_l = 0$ in (1.2), we get

$$\begin{aligned}2f(0) = &-2(l+1)\left(1+3(l-3)+\frac{3(l-3)(l-4)}{2}+\frac{(l-3)(l-4)(l-5)}{6}\right)f(0)\\&+(3l^2-2l-5)\left(3+3(l-3)+\frac{(l-3)(l-4)}{2}\right)f(0)\\&-3l(l^3-l^2-l+2)f(0).\end{aligned}\tag{3.1}$$

It follows from (3.1) that $f(0) = 0$. Setting $x_1 = x_3 = \cdots = x_l = 0$ and $x_2 = x$ in (1.2), we have

$$\begin{aligned}lf(x)+f(-lx) = &-2(l+1)\left(1+2(l-3)+\frac{(l-3)(l-4)}{2}\right)f(x)\\&+(3l^2-2l-5)\left(1+2(l-3)+\frac{(l-3)(l-4)}{2}\right)f(x)\\&-3(l-1)(l^3-l^2-l+1)f(x),\end{aligned}\tag{3.2}$$

for all $x \in X$. It follows from (3.2) that

$$f(-x) = -f(x),\tag{3.3}$$

for all $x \in X$. Letting $x_2 = x_3 = \cdots = x_l = 0$ and $x_1 = x$ in (1.2), we get

$$\begin{aligned}lf(x)+f(-lx) =&(3l^2-2l-5)\left(1+2(l-3)+\frac{l^2-7l+12}{2}\right)f(x)\\&-3(l-1)(l^3-l^2-l+1)f(x)\\&-2(l+1)\left(1+2(l-3)+\frac{l^2-7l+12}{2}\right)f(x),\end{aligned}\tag{3.4}$$

for all $x \in X$. It follows from (3.4) and the oddness of f that

$$f(lx) = l^3 f(x),\tag{3.5}$$

for all $x \in X$. Letting $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_l = 0$, we get

$$\begin{aligned}&(l-1)f(x)+2f((-l+1)x)\\=&-2(l+1)(l-2)f(2x)-2(l+1)\left(2(n-3)+\frac{2(l-3)(l-4)}{2}\right)f(x)\\&+(3l^2-2l-5)f(2x)+2(3l^2-2l-5)(n-2)f(x)-6(l^3-l^2-l+1)f(x),\end{aligned}\tag{3.6}$$

for all $x \in X$. It follows from (3.6), (3.5) and the oddness of f that

$$f(2x) = 8f(x),\tag{3.7}$$

for all $x \in X$. Setting $x_1 = x_2 = x_3 = x$ and $x_4 = x_5 = \cdots = x_l = 0$ in (1.2), we have

$$\begin{aligned}&(l-2)f(3x) + 3f((-l+2)x)\\ =& -2(l+1)f(3x) - 6(l+1)f(2x) - 3(l+1)(l-3)(l-4)f(x)\\ &+ 3(3l^2-2l-5)f(2x) + 3(3l^2-2l-5)(n-3)f(x) - 9(l^3-l^2-l+1)f(x),\end{aligned} \tag{3.8}$$

for all $x \in X$. It follows from (3.8) that

$$f(3x) = 27f(x), \tag{3.9}$$

for all $x, y \in X$. Setting $x_1 = x_3 = x_4 = x$ and $x_2 = x_5 = \cdots = x_l = 0$ in (1.2), we get

$$\begin{aligned}&(l-2)f(2x+y) + 2f(-2x+y) + f(2x-3y)\\ =& -2(l+1)\Big(f(2x+y) + (l-3)f(2x) + 2(l-3)f(x+y)\Big)\\ &- 2(l+1)\Big((l-3)(l-4)\Big)f(x) + (l-3)(l-4)f(y)\\ &+ (3l^2-2l-5)\Big(f(2x) + 2(l-3)f(x) + (n-3)f(y) + 2f(x+y)\Big)\\ &- 3(l^3-l^2-l+1)(2f(x)+f(y)),\end{aligned} \tag{3.10}$$

for all $x, y \in X$. It follows from (3.10) and the oddness of f that

$$\begin{aligned}&f(2x+y) - 2f(2x-y) + f(2x-3y)\\ =& -8f(2x+y) + 128f(x) + 32f(x+y) - 96f(x) - 48f(y),\end{aligned} \tag{3.11}$$

for all $x, y \in X$. Replacing y by $-y$ in (3.11), we get

$$\begin{aligned}&f(2x-y) - 2f(2x+y) + f(2x+3y)\\ =& -8f(2x-y) + 128f(x) + 32f(x-y) - 96f(x) + 48f(y),\end{aligned} \tag{3.12}$$

for all $x, y \in X$. Adding (3.11) and (3.12), we have

$$\begin{aligned}&f(2x+3y) + f(2x+3y) - f(2x-y) - f(2x+y)\\ =& -8f(2x+y) - 8f(2x+y) + 32f(x+y) + 32f(x-y) + 64f(x),\end{aligned} \tag{3.13}$$

for all $x, y \in X$. It follows from (3.13) and (1.1) that

$$7f(2x+y) + 7f(2x-y) = 14(f(x+y) + f(x-y)) + 84f(x), \tag{3.14}$$

for all $x, y \in X$. It follows from (3.14) that

$$f(2x+y) + f(2x-y) = 2(f(x+y) + f(x-y)) + 12f(x),$$

for all $x, y \in X$. Therefore, the mapping $f : X \to Y$ is cubic. $\square$

4. Hyers-Ulam Stability of the l-Variablel Cubic Functional Equation (1.2): Direct Approach

In this setion, we prove the Hyers-Ulam stability of the l-variablel cubic functional equation (1.2) in RN-spaces by using the direct method.

Theorem 4.1. *Let $j = \pm 1$ and $f : X \to Y$ be a mapping for which there exists a function $\eta : X^l \to D^+$ with the condition*

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{l^{(k+i)}x_1, l^{(k+i)}x_2, l^{(k+i)}x_3, \ldots, l^{(k+i)}x_l}\left(l^{(k+i+1)j}t\right)\right) = \lim_{k\to\infty} \eta_{l^{(kj)}x_1, l^{(kj)}x_2, l^{(kj)}x_3, \ldots, l^{(kj)}x_l}\left(l^{kj}t\right) = 1, \tag{4.1}$$

such that $f(0) = 0$ and

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \eta_{(x_1,x_2,\ldots,x_l)}(t), \tag{4.2}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.2) and

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{m^{(i+1)j}x, \underbrace{0, \ldots, 0}_{(l-1)-times}}\left(l^{(i+1)j}t\right)\right), \tag{4.3}$$

for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(l^{kj}x)}{l^{3kj}}}(t), \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_1, x_2, \ldots, x_n) = (x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}})$ in (4.3), we have

$$\mu_{f(lx)-l^3 f(x)}(t) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}(t), \tag{4.5}$$

for all $x \in X$ and all $t > 0$. It follows from (4.4) and (RN2) that

$$\mu_{\frac{f(lx)}{l^3} - f(x)}(t) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(l^3 t\right),$$

for all $x \in X$ and all $t > 0$. Replacing x by $l^k x$ in (4.5), we catch

$$\mu_{\frac{f(l^{k+1}x)}{l^{3(k+1)}} - \frac{f(l^k x)}{l^{3k}}}(t) \geq \eta_{l^k x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(l^{3k} l^3 t\right) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(\frac{l^{3k} l}{\alpha^k} t\right), \tag{4.6}$$

for all $x \in X$ and all $t > 0$. It follows from

$$\frac{f\left(l^n x\right)}{l^{3n}} - f(x) = \sum_{k=0}^{n-1} \frac{f\left(l^{k+1}x\right)}{l^{3(k+1)}} - \frac{f\left(l^k x\right)}{l^{3k}}$$

and (4.6) that

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}\right) \geq T_{k=0}^{n-1}\left(\eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t)\right) = \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t),$$

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right), \tag{4.7}$$

for all $x \in X$ and all $t > 0$. Replacing x by $l^m x$ in (4.7), we get

$$\mu_{\frac{f(l^{n+m}x)}{l^{n+m}}-\frac{f(l^m x)}{l^{3m}}}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{l^{3k}l^3}}\right). \tag{4.8}$$

Since $\eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{l^{3k}l^3}}\right) \to 1$ as $m, n \to \infty$, $\left\{\frac{f(l^n x)}{l^{3n}}\right\}$ is a Cauchy sequence in (Y, μ, T). Since (Y, μ, T) is complete, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (4.8). Then we have

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right)$$

and so, for every $\delta > 0$, we have

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\delta) &\geq T\left(\mu_{C(x)-\frac{f(l^n x)}{l^{3n}}}(\delta), \mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t)\right)\\ &\geq T\left(\mu_{C(x)-\frac{f(l^n x)}{l^{3n}}}(\delta), \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right)\right).\end{aligned} \tag{4.9}$$

Taking limit as $n \to \infty$ and using (4.9), we have

$$\mu_{C(x)-f(x)}(t+\delta) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left((l^3-\alpha)t\right). \tag{4.10}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (4.10), we have

$$\mu_{C(x)-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left((l^3-\alpha)t\right). \tag{4.11}$$

Replacing $(x_1, x_2, \ldots, x_l)$ by $(2^n x_1, 2^n x_2, \ldots, 2^n x_l)$ in (4.2), we have

$$\mu_{Df(l^n x_1, l^n x_2, \ldots, l^n x_l)}(t) \geq \eta_{l^n x_1, l^n x_2, \ldots, l^n x_l}\left(l^{3n}t\right),$$

for all $x_1, x_2, \ldots, x_l \in X$ and all $t > 0$. Since

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{l^{(k+i)}x_1, l^{(k+i)}x_2, \ldots, l^{(k+i)}x_l}\left(l^{3(k+i+1)j}t\right)\right) = 1,$$

we conclude that C fulfills (1.2).

To prove the uniqueness of the cubic mapping C, assume that there exists another cubic mapping D from X to Y, which satisfies (4.11). Fix $x \in X$. Clearly, $C(l^n x) = l^{3n} C(x)$ and $D(l^n x) = l^{3n} D(x)$ for all $x \in X$. It follows from (4.11) that

$$\begin{aligned}
\mu_{C(x)-D(x)}(t) &= \lim_{n\to\infty} \mu_{\frac{C(l^n x)}{l^{3n}} - \frac{D(l^n x)}{l^{3n}}}(t), \\
\mu_{C(x)-D(x)}(t) &\geq \min\left\{\mu_{\frac{C(l^n x)}{l^{3n}} - \frac{f(l^n x)}{l^{3n}}}\left(\frac{t}{2}\right), \mu_{\frac{D(l^n x)}{l^{3n}} - \frac{f(l^n x)}{l^{3n}}}\left(\frac{t}{2}\right)\right\} \\
&\geq \eta_{l^n x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(l^{3n}\left(l^3-\alpha\right)t\right) \\
&\geq \eta_{x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{l^{3n}\left(l^3-\alpha\right)t}{\alpha^n}\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{l^{3n}(l^3-\alpha)t}{\alpha^n}\right) = \infty$, we get

$$\lim_{n\to\infty} \eta_{x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\left(\frac{l^{3n}(l^3-\alpha)t}{\alpha^n}\right)\right) = 1.$$

Therefore, it follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $C(x) = D(x)$.

For $j = -1$, we can prove the theorem by a similar way. This completes the proof. □

The following corollary is an immediate consequence of Theorem 4.1, concerning the stability of (1.2).

Corollary 4.1. *Let ξ and ρ be nonnegative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \begin{cases} \eta_\xi(t), & \\ \eta_{\xi\sum_{i=1}^n \|x_i\|^\rho}(t), & s \neq 3, \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho + \sum_{i=1}^n \|x_i\|^{n\rho}\right)}(t), & p \neq \frac{3}{n}, \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\xi}{|l^3-1|}}(t), \\ \eta_{\frac{\xi\|x\|^\rho}{|l^3-1^\rho|}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{|l^3-1^{n\rho}|}}(t), \end{cases}$$

for all $x \in X$ and all $t > 0$.

5. Hyers-Ulam Stability of the l-Variablel Cubic Functional Equation (1.2): Fixed Point Approach

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2) in random normed spaces by using the fixed point approach.

Theorem 5.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^l \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \ldots, \delta_i^k x_l}(\delta_i^k t) = 1,$$

for all $x_1, x_2, \ldots, x_l \in X$, $t > 0$ and $\delta_i = \begin{cases} l, & i = 0, \\ \frac{1}{l}, & i = 1, \end{cases}$ satisfying the functional inequality

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \eta_{x_1,x_2,\ldots,x_l}(t),$$

for all $x_1, x_2, \ldots, x_l \in X$ and $t > 0$. If there exists $L = L(i)$ such that the function

$$x \mapsto \beta(x,t) = \eta_{\frac{x}{l}, \underbrace{0,\ldots,0}_{(l-1)-times}}(t)$$

has the property that

$$\beta(x,t) \leq L \frac{1}{\delta_i^3} \beta(\delta_i x, t), \tag{5.1}$$

for all $x \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.2) and

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L} t\right) \geq \beta(x,t),$$

for all $x \in X$ and $t > 0$.

Proof. Let $\Omega := \{f : X \to Y : f \text{ is a function}\}$ and d be a generalized metric on Ω such that

$$d(g,h) = \inf\left\{k \in (0,\infty) / \mu_{(g(x)-h(x))}(kt) \geq \beta(x,t) : x \in X, t > 0\right\}.$$

It is easy to see that (Ω, d) is complete (see [17]). Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x)$ for all $x \in X$. Now, for $g, h \in \Omega$ we have $d(g,h) \leq K$, which implies

$$\begin{aligned} \mu_{(g(x)-h(x))}(Kt) &\geq \beta(x,t), \\ \mu_{(Tg(x)-Th(x))}\left(\frac{Kt}{\delta_i}\right) &\geq \beta(x,t), \\ d(Tg(x), Th(x)) &\leq KL, \\ d(Tg, Th) &\leq L d(g,h), \end{aligned}$$

for all $g, h \in \Omega$. Therefore, T is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (4.5) that

$$\mu_{f(lx)-l^3 f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t),$$

for all $x \in X$. It follows from (4.5) that

$$\mu_{\frac{f(lx)}{l^3}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(l^3 t),$$

for all $x \in X$. Using (5.1) for the case $i = 0$, we get

$$\mu_{\frac{f(lx)}{l^3}-f(x)}(t) \geq L\beta(x,t),$$

for all $x \in X$. Hence, we obtain

$$d(\mu_{Tf,f}) \leq L = L^{1-i} < \infty, \tag{5.2}$$

for all $x \in X$. Replacing x by $\frac{x}{l}$ in (4), we get

$$\mu_{\frac{f(x)}{l}-f\left(\frac{x}{l}\right)}(t) \geq \eta_{\frac{x}{l},\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(l^3 t\right),$$

for all $x \in X$. By using (5.1) for the case $i = 1$, it reduce to

$$\mu_{l^3 f\left(\frac{x}{l}\right)-f(x)}(t) \geq \beta(x,t) \Rightarrow \mu_{Tf(x)-f(x)}(t) \geq \beta(x,t),$$

for all $x \in X$. Hence, we get

$$d\left(\mu_{Tf,f}\right) \leq L = L^{1-i} < \infty, \tag{5.3}$$

for all $x \in X$. From (5.2) and (5.3), we can conclude

$$d\left(\mu_{Tf,f}\right) \leq L = L^{1-i} < \infty,$$

for all $x \in X$.

The remaining proof is similar to the proof of Theorem 4.1. Since C is a unique fixed point of T in the set $\Delta = \{f \in \Omega \mid d(f,C) < \infty\}$, C is a unique mapping such that

$$\mu_{f(x)-C(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t),$$

for all $x \in X$ and $t > 0$. This completes the proof. $\square$

From Theorem 5.1, we obtain the following corollary concerning the stability for the functional equation (1.2).

Corollary 5.1. *Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \begin{cases} \eta_\xi(t), & \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho}(t), & \rho \neq 3, \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho + \sum_{i=1}^n \|x_i\|^{n\rho}\right)}(t), & p \neq \frac{3}{n}, \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$, where ρ, ξ are constants with $\xi > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\xi}{|l^3-1|}}(t), \\ \eta_{\frac{\xi\|x\|^\rho}{|l^3-1^\rho|}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{|l^3-1^{n\rho}|}}(t), \end{cases}$$

for all $x \in X$ and $t > 0$.

Proof. Set

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \begin{cases} \eta_\xi(t), \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho}(t), \\ \eta_{\xi(\prod_{i=1}^n \|x_i\|^\rho + \sum_{i=1}^n \|x_i\|^{n\rho})}(t), \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. Then

$$\eta_{(\delta_i^k x_1, \delta_i^k x_2, \ldots, \delta_i^k x_l)}(\delta_i^k t) = \begin{cases} \eta_\xi \delta_i^{3k}(t) \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho \delta_i^{(3-\rho)k}}(t) \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho \delta_i^{(3-\rho)k} + \sum_{i=1}^n \|x_i\|^{n\rho} \delta_i^{(3-n\rho)k}\right)}(t) \end{cases}$$
$$\to \begin{cases} 1 \quad \text{as} \quad k \to \infty, \\ 1 \quad \text{as} \quad k \to \infty, \\ 1 \quad \text{as} \quad k \to \infty. \end{cases}$$

But we have that $\beta(x,t) = \eta_{\frac{x}{l}, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}(t)$ has the property $L \frac{1}{\delta_i^3} \beta(\delta_i x, t)$ for all $x \in X$ and $t > 0$.

Now,

$$\beta(x,t) = \begin{cases} \eta_\xi(t), \\ \eta_{\frac{\xi\|x\|^\rho}{l^{3s}}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{l^{3ns}}}(t), \end{cases}$$

$$L \frac{1}{\delta_i^3} \beta(\delta_i x, t) = \begin{cases} \eta_{\delta_i^{-3}\beta(x)}(t), \\ \eta_{\delta_i^{\rho-3}\beta(x)}(t), \\ \eta_{\delta_i^{n\rho-3}\beta(x)}(t). \end{cases}$$

Using Theorem 4.1, we prove the following six cases.

$L = l^{-3}$ if $i = 0$; $L = l$ if $i = 1$; $L = l^{\rho-3}$ for $\rho < 1$ if $i = 0$; $L = l^{3-\rho}$ for $s > 1$ if $i = 1$; $L = l^{n\rho-3}$ for $\rho < \frac{1}{n}$ if $i = 0$; $L = l^{3-n\rho}$ for $\rho > \frac{1}{n}$ if $i = 1$.

Case 1. $L = l^{-3}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L \frac{1}{\delta_i^3} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi}{l^3 - l}\right)}(t).$$

Case 2. $L = l^3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi}{1-l^3}\right)}(t).$$

Case 3. $L = l^{\rho-3}$ for $\rho < 1$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^\rho}{(l^3-l^{3\rho})}\right)}(t).$$

Case 4. $L = l^{3-\rho}$ for $\rho > 1$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^\rho}{(l^{3\rho}-l^3)}\right)}(t).$$

Case 5. $L = l^{n\rho-3}$ for $\rho < \frac{1}{n}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^{n\rho}}{(l^3-l^{3n\rho})}\right)}(t).$$

Case 6. $L = l^{3-n\rho}$ for $\rho > \frac{1}{n}$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^{n\rho}}{(l^{3n\rho}-l^3)}\right)}(t).$$

Hence, the proof is complete. □

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