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MULTIPLICITY OF SOLUTIONS FOR A FRACTIONAL DOUBLE-PHASE PROBLEM WITH $p(t)$-LAPLACE OPERATOR INVOLVING THE ϕ-HILFER FRACTIONAL DERIVATIVE

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Abstract. In this work, we investigate a class of ϕ-Hilfer fractional double-phase problem involving a $p(t)$-Laplacian operator with Dirichlet boundary conditions. More precisely, we will use a variational method with the critical theorem of Bonanno and Marano, to prove the existence of at least three nontrivial solutions for such a problem. An illustrative example is provided at the conclusion of this work to strengthen the validity of our main findings.

1. Introduction

Nowadays, there is a great interest in the theory of fractional differential equations due to its applications in various fields such as science, engineering, finance and quantum mechanics. For further reading, we recommend the references [16, 19–21, 24, 26, 28, 29, 40]. Given their importance, many authors have focused on the study of fractional problems involving Riemann-Liouville, Caputo and Grunwald-Letnikov derivatives. For more details, we refer the reader to [1,7,9,14,35,37] and the references therein. Notable contributions include the work of Padhi et al. [27], who study various fixed point theorems; Ghanmi and Horrigue [13], who apply the Schauder fixed point theorem; and Ghanmi et al. [14, 15], who apply the Nehari manifold method and variational approaches. In addition, Nouf et al. [25] use the Mountain Pass theorem in their analysis. Specifically, Ghanmi and Zhang [14] applied the Nehari manifold

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method to prove the multiplicity of solutions for the following problem

$$\begin{cases} -_t D_T^{\alpha} \left(K_p(\psi(t))\right) = \nabla W(t, \psi(t)) + \lambda g(t)|\psi(t)|^{q-2}\psi(t), & t \in (0,T), \\ \psi(0) = \psi(T) = 0, \end{cases}$$

where $K_p(\psi(t)) = |_0D_t^{\alpha}(\psi(t))|^{p-2}\ {}_0D_t^{\alpha}\psi(t)$, $\lambda > 0$, $2 < q < p$, $\frac{1}{2} < \alpha \leq 1$, $g \in C([0,1])$ and $W \in C([0,1] \times \mathbb{R}^n, \mathbb{R})$, ${}_tD_T^{\alpha}$ and ${}_0D_t^{\alpha}$ are the left and right fractional derivatives in the sense of Riemann-Liouville.

Kefi et al. [18] applied the critical theorem developed by Bonanno and Marano, originally introduced in their seminal work [8], to study the following double-phase biharmonic problem:

$$\begin{cases} \Delta^2_{p(x)}v(x) + \Delta^2_{q(x)}v(x) + \varsigma(x)\dfrac{|v(x)|^{s-2}v(x)}{|x|^{2s}} = \lambda g(x)|v(x)|^{r-2}v(x), & \text{in } \Omega, \\ v = \Delta v = 0, \quad \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

Under suitable conditions for the functions p, q, s, r, ς and g, the authors proved that Problem (1.1) admits three nontrivial solutions in Ω a bounded domain in $\mathbb{R}^n$, $n > 2$, with a boundary of class C^1.

Recently, many studies have focused on problems involving the ϕ-Hilfer fractional derivative, first introduced by Hilfer [17]. Hilfer proposed a fractional computational model that integrates concepts of Riemann-Liouville and Caputo fractional derivatives and includes parameters that enhance flexibility in modeling real-world phenomena. We find that the ϕ-Hilfer derivative provides a flexible fractional framework for two-phase problems. It captures heterogeneous behavior, connects different types of derivatives, and accounts for memory effects, making it particularly useful for models with nonlocal and variable growth. Many authors have studied the existence and multiplicity of solutions to fractional differential equations involving the Hilfer fractional derivative; see e.g. [2–6, 11, 23, 30, 34, 38].

In this study, we extend the concepts introduced in [18] and, inspired by [14], we prove the existence of multiple nontrivial weak solutions to a double-phase problem involving the ϕ-Hilfer fractional derivatives ${}_tD_T^{\alpha,\beta,\phi}$ and ${}_0D_t^{\alpha,\beta,\phi}$, as defined in Section 2. Specifically, we investigate the following problem:

$$\begin{cases} {}_tD_T^{\alpha,\beta,\phi}\left(K_{p(\cdot)}(u) + LK_{q(\cdot)}(u)\right)(t) = \lambda f(t, u(t)) + \mu g(t, u(t)), & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where

$$K_a(u) = \left|{}_0D_t^{\alpha,\beta,\phi}u\right|^{a-2}\ {}_0D_t^{\alpha,\beta,\phi}u, \tag{1.3}$$

functions p, q and L are in $C([0,T], \mathbb{R})$, $\lambda > 0$ and the functions f and g are in $C([0,T] \times \mathbb{R}, \mathbb{R})$ such that $f + \frac{\mu}{\lambda}g$ is positively homogeneous of a variable degree $s(x) - 1$. That is

$$\left(f + \frac{\mu}{\lambda}g\right)(x, tu) = t^{s(x)-1}\left(f + \frac{\mu}{\lambda}g\right)(x, u)$$

holds for all $t > 0$ and $(x, u) \in [0, T] \times \mathbb{R}$.

${}_tD_T^{\alpha,\beta,\phi}$ and ${}_0D_t^{\alpha,\beta,\phi}$ are, respectively, the left and right-sided ϕ-Hilfer fractional derivatives of order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$. The function ϕ is positive and increasing on $[0, T]$ with a continuously differentiable derivative $\phi'(s) \neq 0$ for all $s \in [0, T]$.

While the methods employed in this paper are standard, their configuration within problem (1.2) is novel. To the best of our knowledge, this work presents the first application of Bonanno-Marano's theorem to a problem involving the ϕ-Hilfer fractional derivative. Moreover, the double-phase problem-a central concept in variational calculus-concerns functionals exhibiting two distinct growth behaviors, typically governed by different power terms. Such problems are characterized by terms whose growth rates vary across different regions of the domain, thereby rendering the analysis of minimizers and solutions substantially more challenging. The double-phase problem has significant applications in various fields of mathematics and science.

Before stating our main hypotheses, we introduce some notations and fundamental results.

Let $C_0^\infty([0, T], \mathbb{R})$ be the set of all smooth functions $u \in C^\infty([0, T], \mathbb{R})$ that satisfy $u(0) = u(T) = 0$. For $\eta \in C([0, T], \mathbb{R})$, we set

$$\eta^- := \inf_{x \in [0,T]} \eta(x), \quad \eta^+ := \sup_{x \in [0,T]} \eta(x).$$

We define the Lebesgue space with variable exponents as follows:

$$L^{a(\cdot)}([0, T]) := \left\{ u \mid u : [0, T] \to \mathbb{R} \text{ is measurable, } \int_0^T |u(t)|^{a(t)} dt < +\infty \right\},$$

which is equipped with the so-called Luxemburg norm

$$\|u\|_{a(\cdot)} := \inf \left\{ \nu > 0 \mid \int_0^T \left| \frac{u(t)}{\nu} \right|^{a(t)} dt \leq 1 \right\}.$$

The inclusion between Lebesgue spaces is generalized as follows.

If s_1 and s_2 are such that $s_1(x) \leq s_2(x)$, a.e. $x \in [0, T]$, then there exists a continuous embedding

$$L^{s_2(\cdot)}([0, T]) \hookrightarrow L^{s_1(\cdot)}([0, T]).$$

We denote $L^{s'(\cdot)}([0, T])$ as the conjugate space of $L^{s(\cdot)}([0, T])$, where $\frac{1}{s(x)} + \frac{1}{s'(x)} = 1$. For any $u \in L^{s(\cdot)}([0, T])$ and $v \in L^{s'(\cdot)}([0, T])$, the following Hölder-type inequality holds:

$$\left| \int_0^T u(x)v(x)dx \right| \leq \left(\frac{1}{s^-} + \frac{1}{s'^-} \right) \|u\|_{s(\cdot)} \|v\|_{s'(\cdot)}.$$

To establish the variational structure for our problem (1.2), we introduce the fractional derivative space $E_{p(.)}^{\alpha,\beta,\phi}$ with variable exponents as follows:

$$E_{p(.)}^{\alpha,\beta,\phi} := \left\{ u \in L^{p(\cdot)}([0, T]) \mid {}_0D_t^{\alpha,\beta,\phi} u \in L^{p(\cdot)}([0, T]) \right\},$$

endowed with the norm

$$\|u\|_{\alpha,\beta,\phi,p(\cdot)} = \|u\|_{p(\cdot)} + \|{}_0D_t^{\alpha,\beta,\phi}u\|_{p(\cdot)}.$$

Remark 1.1 ([22, 33]). (i) Consider $E := E_{p(\cdot),0}^{\alpha,\beta,\phi}$ as the closure of $C_0^\infty([0,T],\mathbb{R})$ in $E_{p(\cdot)}^{\alpha,\beta,\phi}$, which can be renormed by the equivalent norm

$$\|u\| := \|{}_0D_t^{\alpha,\beta,\phi}u\|_{p(\cdot)}.$$

(ii) The fractional space E is a separable and reflexive Banach space.

(iii) If $s \in C([0,T],\mathbb{R})$ such that $s(x) > 1$, for all $x \in [0,T]$, then the embedding $E \hookrightarrow L^{s(\cdot)}([0,T])$ is compact and continuous and there exists a positive constant $C > 0$ such that

$$\|u\|_{s(\cdot)} \leq C\,\|u\|, \quad \text{for all } u \in E.$$

Throughout this paper, we put $h = f + \frac{\mu}{\lambda}g$ and $H = F + \frac{\mu}{\lambda}G$, where

$$F(t,\tau) := \int_0^\tau f(t,\xi)d\xi \quad \text{and} \quad G(t,\tau) := \int_0^\tau g(t,\xi)d\xi,$$

for all $(t,\tau) \in [0,T] \times \mathbb{R}$.

In order to study problem (1.2), we shall need the following hypotheses.

(H$_1$) The function $L \in C([0,T],\mathbb{R})$ is such that

$$0 < L^- := L_0 \leq L^+ := L_\infty.$$

(H$_2$) The functions $p, q, s \in C([0,T],\mathbb{R})$ satisfying $1 < q^- \leq q(x) \leq q^+ < s^- < s(x) < s^+ < p^- \leq p(x) \leq p^+ < +\infty$.

(H$_3$) $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ is homogeneous of degree $s(x)$ that is

$$H(x,\xi u) = \xi^{s(x)}H(x,u), \quad \xi > 0, \text{ for all } x \in [0,T] \text{ and } u \in \mathbb{R}.$$

(H$_4$) There exits $s_0 > 0$ and $u_0 \in E$ such that

$$\int_0^T H(t,u_0(t))dt \geq s_0.$$

Remark 1.2. Note that, from **(H$_3$)**, h leads to the so-called Euler identity

$$vh(x,v) = s(x)H(x,v).$$

Moreover, there exists $C_0 > 0$, such that

$$|H(x,v)| \leq C_0|v|^{s(x)}. \tag{1.4}$$

Let $\|u_0\| = b$ and

$$\sigma = \left(\frac{1}{p^-} + \frac{L^+}{q^-}\right)\max\left\{b^{p^+}, b^{q^-}\right\}. \tag{1.5}$$

Our main result is the following.

Theorem 1.1. *Assume that* $(\mathbf{H}_1)$-$(\mathbf{H}_4)$ *are fulfilled and there exists* $r > 0$ *such that*

$$r < \frac{1}{p^+} \min\left\{b^{p^-}, b^{p^+}\right\} \tag{1.6}$$

and

$$\frac{1}{r} \max\left\{\nu^{s^+}, \nu^{s^-}\right\} < \frac{s_0}{\sigma C_0 \max\left\{C^{s^+}, C^{s^-}\right\}}, \tag{1.7}$$

where $\nu = \max\left\{(rp^+)^{\frac{1}{p^-}}, (rp^+)^{\frac{1}{p^+}}\right\}$ *and* C *is given in Remark* 1.1.

Then, Problem (1.2) *admits at least three weak solutions.*

2. Preliminaries

In this section, we provide an overview of key concepts in fractional calculus, focusing on fundamental results related to the ϕ-Hilfer fractional derivative. For further details, we refer the reader to [19, 31, 33, 36].

Let $[a, b]$ be a finite or infinite interval on the real line, and let ϕ be an increasing positive function on $[a, b]$, with a continuously differentiable derivative satisfying $\phi'(y) \neq 0$ for all $y \in [a, b]$.

We begin by introducing the definition of the ϕ-Hilfer fractional integral.

Definition 2.1 ([19, 31])**.** Let $\alpha > 0$ and $h : (a, b) \to \mathbb{R}$ be a measurable function defined a.e. on (a, b). The right (resp. left) fractional integral with respect to ϕ with superior limit b (resp. inferior limit a) of order α of h is given by

$${}_tI_b^{\alpha,\phi}h(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \phi'(s)(\phi(s) - \phi(t))^{\alpha-1}h(s)ds$$

and

$${}_aI_t^{\alpha,\phi}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}h(s)ds,$$

respectively. Here Γ is the well-known Euler's Gamma function.

Definition 2.2 ([33, 36])**.** Let $m \in \mathbb{N}$, $\alpha \in (m-1, m]$, $\beta \in [0, 1]$ and consider h an integrable function over the interval $[a, b]$. The right (resp. left) ϕ-Hilfer fractional derivative of order α of h and of type β, is given by

$${}_tD_b^{\alpha,\beta,\phi}h(t) = {}_tI_b^{\beta(m-\alpha),\phi}\left(-\frac{1}{\phi'(t)} \cdot \frac{d}{dt}\right)^m {}_tI_b^{(1-\beta)(m-\alpha),\phi}h(t), \quad \text{for all } t \in [a, b),$$

and

$${}_aD_t^{\alpha,\beta,\phi}h(t) = {}_aI_t^{\beta(m-\alpha),\phi}\left(\frac{1}{\phi'(t)} \cdot \frac{d}{dt}\right)^m {}_aI_t^{(1-\beta)(m-\alpha),\phi}h(t), \quad \text{for all } t \in (a, b],$$

respectively.

It is essential to emphasize that the ϕ-Hilfer fractional derivatives extend previous concepts, including the ϕ-Riemann-Liouville and ϕ-Caputo fractional derivatives. In particular, the following remark holds.

Remark 2.1. (i) From the ϕ-Hilfer fractional derivatives, as β tends to zero, we obtain the ϕ-Riemann-Liouville fractional derivatives:

$$ {}_tD_b^{\alpha,\phi}h(t) = \left(-\frac{1}{\phi'(t)}\cdot\frac{d}{dt}\right)^m {}_tI_b^{m-\alpha,\phi}h(t) $$

and

$$ {}_aD_t^{\alpha,\phi}h(t) = \left(\frac{1}{\phi'(t)}\cdot\frac{d}{dt}\right)^m {}_aI_t^{m-\alpha,\phi}h(t). $$

(ii) As β tends to 1, the ϕ-Hilfer fractional derivatives become equivalent to the ϕ-Caputo fractional derivatives, given by:

$$ {}_t^cD_b^{\alpha,\phi}h(t) ={}_t I_b^{m-\alpha,\phi}\left(-\frac{1}{\phi'(t)}\cdot\frac{d}{dt}\right)^m h(t) $$

and

$$ {}_a^cD_t^{\alpha,\phi}h(t) ={}_a I_t^{m-\alpha,\phi}\left(\frac{1}{\phi'(t)}\cdot\frac{d}{dt}\right)^m h(t). $$

(iii) The ϕ-Hilfer fractional derivatives are directly related to the ϕ-Riemann-Liouville fractional derivatives via the following relations:

$$ {}_tD_b^{\alpha,\beta,\phi}h(t) ={}_t I_b^{\xi-\alpha,\phi}\,{}_tD_b^{\xi,\phi}h(t) $$

and

$$ {}_aD_t^{\alpha,\beta,\phi}h(t) ={}_a I_t^{\xi-\alpha,\phi}\,{}_aD_t^{\xi,\phi}h(t), $$

where $\xi = \alpha + \beta(m - \alpha)$.

3. A Variational Setting and Proof of the Main Result

In this section, we begin by introducing fundamental findings that will be instrumental in proving our main result stated in Theorem 1.1.

Recall that for $u \in E$, the norm is given by

$$ \|u\| := \|{}_0D_t^{\alpha,\beta,\phi}u\|_{p(\cdot)}. $$

The following key properties will play a crucial role in our analysis.

The next lemma is fundamental to our approach and follows similarly from [10, Proposition 3.2].

Lemma 3.1. *For all $u \in L^{p(\cdot)}([0,T])$, we have the following.*

(i) *If $\|u\| > 1$, then*

$$ \|u\|^{p^-} \le \int_0^T \left|{}_0D_t^{\alpha,\beta,\phi}u(t)\right|^{p(t)} dt \le \|u\|^{p^+}. $$

(ii) *If $\|u\| < 1$, then*

$$ \|u\|^{p^+} \le \int_0^T \left|{}_0D_t^{\alpha,\beta,\phi}u(t)\right|^{p(t)} dt \le \|u\|^{p^-}. $$

By simple calculus, the following lemma is proved in [32].

Lemma 3.2. *For $a > 1$, there exits a positive constant C_a, such that*

$$\left\langle |x|^{a-2}x - |y|^{a-2}y, x - y \right\rangle \geq C_a\, |x-y|^a, \quad \text{for } a \geq 2,$$

and

$$\left\langle |x|^{a-2}x - |y|^{a-2}y, x - y \right\rangle \geq C_a \frac{|x-y|^2}{(|x|+|y|)^{2-a}}, \quad \text{for } 1 < a < 2,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

We present the following proposition (see [12]), which will be essential in Section 4.

Proposition 3.1 ([12])**.** *If X is a reflexive Banach space, Y is a Banach space, $Z \subset X$ is nonempty, closed and convex subset, and $J : Z \to Y$ is completely continuous, then J is compact.*

Our main tool will be the following theorem from [8], which we restate in more convenient form.

Theorem 3.1 (Bonanno-Marano theorem, Theorem 3.6 in [8])**.** *Let X be a reflexive real Banach space and $\psi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X. Let $\theta : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that the following hold.*

(A_0)

$$\inf_{x\in X} \psi(x) = \psi(0) = \theta(0) = 0.$$

(A_1) *Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \psi(\bar{x})$, such that:*

$$\frac{\sup_{\psi(x)\leq r} \theta(x)}{r} < \frac{\theta(\bar{x})}{\psi(\bar{x})}.$$

(A_2) *For each $\lambda \in \Lambda_r := \left(\frac{\psi(\bar{x})}{\theta(\bar{x})}, \frac{r}{\sup_{\psi(x)\leq r} \theta(x)}\right)$, the functional $\psi - \lambda\theta$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\psi - \lambda\theta$ has at least three distinct critical points in X.

Now we are ready to prove the main result of this paper, stated in Theorem 1.1. To this end, we begin by introducing the variational setting for problem (1.2). To do this, we define the functional I_λ associated with problem (1.2) by $I_\lambda(u) = \psi(u) - \lambda\theta(u)$, where

$$\psi(u) = \int_0^T \left(\frac{1}{p(t)} \left| {}_0D_t^{\alpha,\beta,\phi} u(t) \right|^{p(t)} + \frac{L(t)}{q(t)} \left| {}_0D_t^{\alpha,\beta,\phi} u(t) \right|^{q(t)} \right) dt \tag{3.1}$$

and

$$\theta(u) = \int_0^T F(t, u(t))dt + \frac{\mu}{\lambda} \int_0^T G(t, u(t))dt.$$

Note that, a function $\varphi \in E$ is said to be a weak solution of Problem (1.2), if it satisfies the associated weak formulation of the differential equation and boundary conditions. That is for any $v \in E$, we have:

$$\begin{aligned}&\int_0^T \left(K_{p(t)}(\varphi(t))\, {}_0D_t^{\alpha,\beta,\phi}v(t) + L(t)K_{q(t)}(\varphi(t))\, {}_0D_t^{\alpha,\beta,\phi}v(t)\right) dt\\ =&\lambda \int_0^T f(t,\varphi(t))v(t)dt + \mu \int_0^T g(t,\varphi(t))v(t)dt,\end{aligned}$$

where $K_{p(\cdot)}$ is given by (1.3).

We observe that, in order to prove that problem (1.2) admits at least three weak solutions, it is necessary to verify that the functional I_λ satisfies all the conditions (A_0)-(A_2) of Theorem 3.1.

First, based on hypotheses (H_1)-(H_2) and Remark 1.2, it is easy to see that condition (A_0) in Theorem 3.1 is fulfilled. Additionally, $\theta(u)$ is well defined for all $u \in E$. Indeed, for all $u \in E$, we have

$$\theta(u) = \int_0^T H(t,u(t))dt.$$

By using (1.4), it follows from Remark 1.2 that

$$|\theta(u)| \leq C_0 \int_0^T |u(t)|^{s(t)}dt < +\infty.$$

In order to ensure that all conditions of Theorem 3.1 are met, we start by satisfying the assertions in the following proposition.

Proposition 3.2. (i) *The functional ψ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on E.*

(ii) *The functional θ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.*

Proof. The proof consists of two main parts: the first establishes (i), and the second proves (ii).

(i) First, we show that ψ is coercive. Specifically, we need to demonstrate that $\lim_{\|u\|\to+\infty} \psi(u) = +\infty$. Let $\varphi \in E$ be such that $\|\varphi\| > 1$.

From (3.1), we have

$$\psi(\varphi) \geq \int_0^T \frac{1}{p(t)} \left|{}_0D_t^{\alpha,\beta,\phi}\varphi(t)\right|^{p(t)} dt \geq \frac{1}{p^+}\int_0^T \left|{}_0D_t^{\alpha,\beta,\phi}\varphi(t)\right|^{p(t)} dt.$$

So, it follows from Lemma 3.1 that

$$\psi(\varphi) \geq \frac{1}{p^+} \min\left\{\|\varphi\|^{p^-}, \|\varphi\|^{p^+}\right\}.$$

Since $p^- \leq p^+$ and $\|\varphi\| > 1$, we get

$$\psi(\varphi) \geq \frac{1}{p^+}\|\varphi\|^{p^-}.$$

Hence, ψ is coercive.

Next, we show that ψ is a weakly lower semi continuous functional on E.

Let $\{u_k\}$ be a weakly convergent sequence in E and let u be the weak limit. Set $M = \liminf_{k\to+\infty} \psi(u_k)$. Now, we have

$$M \geq \int_0^T \left(\frac{1}{p(t)} \left| {}_0D_t^{\alpha,\beta,\phi} u(t) \right|^{p(t)} dt + \int_0^T \frac{L(t)}{q(t)} \left| {}_0D_t^{\alpha,\beta,\phi} u(t) \right|^{q(t)} \right) dt.$$

Thus, $\liminf_{k\to+\infty} \psi(u_k) \geq \psi(u)$ for every $\{u_k\}$ weakly convergent to u in E. So, ψ is lower semi continuous on E.

Due to hypotheses (H_1)-(H_2), ψ is well defined and continuously Gâteaux differentiable. Moreover, for any $u, v \in E$, we have from (1.3),

$$\langle \psi'(u), v \rangle = \int_0^T \left(K_{p(t)}(u(t)) \; {}_0D_t^{\alpha,\beta,\phi} v(t) + L(t) K_{q(t)} u(t) \; {}_0D_t^{\alpha,\beta,\phi} v(t) \right) dt.$$

Now, we claim that ψ' is strictly monotone in E. For any $u, v \in E$, we have from (1.3),

$$\begin{aligned} \langle \psi'(u) - \psi'(v), u - v \rangle = & \int_0^T \left(K_{p(t)}(u(t)) - K_{p(t)}(v(t)) \right) \\ & \times \left({}_0D_t^{\alpha,\beta,\phi} u(t) -_0 D_t^{\alpha,\beta,\phi} v(t) \right) dt \\ & + \int_0^T L(t) \left(K_{q(t)}(u(t)) - K_{q(t)}(v(t)) \right) \\ & \times \left({}_0D_t^{\alpha,\beta,\phi} u(t) -_0 D_t^{\alpha,\beta,\phi} v(t) \right) dt. \end{aligned}$$

For $q^- \geq 2$, and by using Lemma 3.2, there exists a positive constant C_p such that

$$\langle \psi'(u) - \psi'(v), u - v \rangle \geq \int_0^T C_p \left| {}_0D_t^{\alpha,\beta,\phi} u(t) - \; {}_0D_t^{\alpha,\beta,\phi} v(t) \right|^{p(t)} dt.$$

Similarly, if $1 < q^- < 2$, then there exists a positive constant C_p' such that

$$\langle \psi'(u) - \psi'(v), u - v \rangle \geq \int_I C_p' \frac{\left| {}_0D_t^{\alpha,\beta,\phi} u(t) - {}_0D_t^{\alpha,\beta,\phi} v(t) \right|^2}{\left(\left| {}_0D_t^{\alpha,\beta,\phi} u(t) \right| + \left| {}_0D_t^{\alpha,\beta,\phi} v(t) \right| \right)^{2-p(t)}} dt \geq 0,$$

where $I = [0, T] \cap \{x/1 < p(x) < 2\}$.

So ψ' is strictly monotone. This yields ψ' is an injection.

On the other hand, for any $\varphi \in E$ with $\|\varphi\| > 1$, one has

$$\begin{aligned} \langle \psi'(\varphi), \varphi \rangle &= \int_0^T \left(\left| {}_0D_t^{\alpha,\beta,\phi} \varphi(t) \right|^{p(t)} + L(t) \left| {}_0D_t^{\alpha,\beta,\phi} \varphi(t) \right|^{q(t)} \right) dt \\ &\geq \int_0^T \left| {}_0D_t^{\alpha,\beta,\phi} \varphi(t) \right|^{p(t)} dt + L_0 \int_0^T \left| {}_0D_t^{\alpha,\beta,\phi} \varphi(t) \right|^{q(t)} dt. \end{aligned}$$

Since $\|\varphi\| > 1$, it follows from Lemma 3.1 that

$$\frac{\langle \psi'(\varphi), \varphi \rangle}{\|\varphi\|} \geq \frac{\|\varphi\|^{p^-}}{\|\varphi\|} = \|\varphi\|^{p^- - 1}.$$

Thus, ψ' is coercive, and by the Minty-Browder Theorem [39] for reflexive Banach spaces, it follows that ψ' is surjective. So, the mapping ψ' has a bounded inverse $(\psi')^{-1} : E^* \to E$.

Eventually, we prove the continuity of $(\psi')^{-1}$. Let $\varphi_n \to \varphi$ as $n \to +\infty$ in E^* and define $u_n = (\psi')^{-1}(\varphi_n)$, $u = (\psi')^{-1}(\varphi)$.

Since $(\psi')^{-1}$ is bounded and (φ_n) is also bounded, it follows that $\{u_n\}$ is a bounded sequence in E. Hence, without loss of generality, we may assume that there exists a subsequence of u_n, (still denoted by u_n) and v such that $\{u_n\}$ converges weakly to v in E.

This implies that $|\langle \varphi_n - \varphi, u_n - v\rangle| \leq |\varphi_n - \varphi|_{E^*}\|u_n - v\|$. Thus, we can deduce that

$$\begin{aligned}\lim_{n\to+\infty} \langle \psi'(u_n) - \psi'(v), u_n - v\rangle &= \lim_{n\to+\infty} \langle \varphi_n, u_n - v\rangle \\ &= \lim_{n\to+\infty} \langle \varphi_n - \varphi, u_n - v\rangle = 0,\end{aligned}$$

which implies from [18, Lemma 3.2] that $u_n \to v$ strongly as $n \to +\infty$ in E.

Hence, $(\psi')^{-1}(\varphi_n) \to v$ as $n \to +\infty$, and consequently, $\psi'(v) = \varphi$. By the injectivity of ψ', it follows that $u = v$ and thus $(\psi')^{-1}(\varphi_n) \to (\psi')^{-1}(\varphi)$ as $n \to +\infty$. So, the proof of (i) is completed.

(ii) Now, we aim at proving that θ' is compact.

Let $u \in E$ and $\{\omega_n\}$ be a sequence that converges weakly to w in the space E, that is $\omega_n \rightharpoonup \omega$ as $n \to +\infty$ in E. We have

$$\begin{aligned}\Big|\langle \theta'(u), \omega_n\rangle - \langle \theta'(u), \omega\rangle\Big| &= \left|\int_0^T \Big(f(t, u(t)) + \frac{\mu}{\lambda} g(t, u(t))\Big)(\omega_n - \omega)(t)dt\right| \\ &\leq \int_0^T |h(t, u(t))| \cdot |\omega_n(t) - \omega(t)|dt.\end{aligned}$$

Using Remark 1.2, inequality (1.4) and a Hölder-type inequality, we obtain

$$\begin{aligned}\Big|\langle \theta'(u), \omega_n\rangle - \langle \theta'(u), \omega\rangle\Big| &\leq C_0 s^+ \int_0^T |u(t)|^{s(t)-1} |\omega_n(t) - \omega(t)|dt \\ &\leq C_0 s^+ \Big||u|^{s(t)-1}\Big|_{\frac{s(\cdot)}{s(\cdot)-1}} \|\omega_n - \omega\|_{s(\cdot)}.\end{aligned}$$

Combined with Remark 1.1 (iii), this leads to

$$\langle \theta'(u), \omega_n\rangle \to \langle \theta'(u), \omega\rangle, \quad \text{as } n \to +\infty.$$

This implies that $\theta'(u)$ is completely continuous. Therefore, by Proposition 3.1, we deduce that θ' is compact. □

Proof of Theorem 1.1. From hypothesis (H_2) and Lemma 3.1, we have the following estimates for $\psi(u_0)$:

$$\psi(u_0) \geq \frac{1}{p^+} \min\left\{\|u_0\|^{p^-}, \|u_0\|^{p^+}\right\} \tag{3.2}$$

and

$$\psi(u_0) \leq \left(\frac{1}{p^-} + \frac{L^+}{q^-}\right) \max\left\{\|u_0\|^{p^+}, \|u_0\|^{q^-}\right\}.$$

Thus, from hypothesis (H_4) and (1.5), we obtain the following inequality

$$\frac{\theta(u_0)}{\psi(u_0)} \geq \frac{s_0}{\sigma}. \tag{3.3}$$

Using (3.2), it follows from (1.6) that $\psi(u_0) > r$. Next, consider $u \in \psi^{-1}((-\infty, r])$. From Lemma 3.1, we have

$$\frac{1}{p^+} \min\left\{\|u\|^{p^-}, \|u\|^{p^+}\right\} \leq \psi(u) \leq r.$$

Therefore, from (1.7), we deduce that

$$\|u\| \leq \nu. \tag{3.4}$$

Additionally, from Remark 1.2, we have the bound:

$$\theta(u) \leq C_0 \int_0^T |u(x)|^{s(x)} dx.$$

Hence, it follows from Remark 1.1 that there exists $C > 0$ such that

$$\theta(u) \leq C_0 \max\left\{C^{s^+}, C^{s^-}\right\} \max\left\{\|u\|^{s^+}, \|u\|^{s^-}\right\}.$$

This, together with (3.4) and (1.7), implies that

$$\frac{\theta(u)}{r} < \frac{s_0}{\sigma}.$$

So, from (3.3), we conclude that

$$\frac{\sup_{\psi(u)\leq r} \theta(u)}{r} < \frac{\theta(u_0)}{\psi(u_0)}.$$

Now, we need to prove that for each $\lambda \in \left(\frac{\psi(u_0)}{\theta(u_0)}, \frac{r}{\sup_{\psi(u)\leq r} \theta(u)}\right)$, the functional $I_\lambda = \psi - \lambda\theta$ is coercive.

Let $u \in E$ with $\|u\| > 1$. From Remarks 1.1 and 1.2, we have the estimate

$$\theta(u) \leq C_0 \max\left\{C^{s^+}, C^{s^-}\right\} \|u\|^{s^+}$$

and

$$\psi(u) \geq \frac{1}{p^+}\|u\|^{p^-}.$$

Thus,

$$\psi(u) - \lambda\theta(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda C_0 \max\left\{C^{s^+}, C^{s^-}\right\} \|u\|^{s^+}.$$

Since, from (H_2), we have $1 \leq s^+ < p^-$, and deduce that $I_\lambda = \psi - \lambda\theta$ is coercive.

Therefore, conditions (A_1) and (A_2) in Theorem 3.1 are satisfied. Hence, by Proposition 3.2 and Theorem 3.1, the functional I_λ admits at least three distinct critical points in E, which correspond to weak solutions of Problem (1.2).

This ends the proof. □

4. An Example

To illustrate the results obtained in this paper, we provide a relevant example.

Let $s_0 > 0$ and let $W \in C([0,T],\mathbb{R})$ be a function satisfying $W(t) \geq s_0$, for all $t \in [0,T]$.

Consider $\gamma > 1$ and an interval $(a,b) \subset (0,T)$. Let $u_0 \in E$ be a nonnegative function satisfying

$$u_0(t) \geq \left(\frac{\gamma}{T}\right)^{\frac{1}{\gamma}}, \quad \text{if } t \in (a,b) \text{ and } u_0(0) = u_0(T) = 0.$$

Let $\lambda > 0$, and suppose f and g are two continuous functions on $[0,T]$ satisfying

$$f(t,u) + \frac{\mu}{\lambda} g(t,u) = W(t)u^{\gamma-1}, \quad \text{for all } (t,u) \in [0,T] \times \mathbb{R}.$$

Clearly, $f + \frac{\mu}{\lambda} g$ is positively homogeneous of a degree $\gamma - 1$, and hypotheses (H_3) and (H_4) hold. Therefore, by applying Theorem 1.1 and using (1.3), we establish that the following problem

$$\begin{cases} {}_tD_T^{\alpha,\beta,\phi}\left(K_{p(t)}(u(t)) + L(t)K_{q(t)}(u(t))\right) = \lambda W(t)u^{\gamma-1}(t), & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases}$$

admits at least three weak solutions.

5. Conclusions

In this paper, we consider a double-phase problem (1.2) involving the ϕ-Hilfer fractional derivative and the $p(t)$-Laplacian operator. By combining a variational framework with the critical point theorem of Bonanno and Marano, we establish the existence of at least three distinct nontrivial solutions to Problem (1.2). To the extent known to us, this represents the first application of the Bonanno-Marano theorem to problems governed by the ϕ-Hilfer fractional derivative. Furthermore, the proposed methodology is sufficiently flexible to be extended to analogous problems in higher-dimensional settings.

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