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GENERALIZED DERIVATIONS ASSOCIATE WITH HOCHSCHILD 2-COCYCLES ON A CLASS OF BANACH ALGEBRAS

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Abstract. We study a new type of generalized derivations associated with Hochschild 2-cocycles which was introduced by Nakajima (Turk. J. Math. **30** (2006), 403–411). We investigate generalized derivable maps at (commutative) zero product associated with Hochschild 2-cocycles on a class of Banach algebras. We also prove that every generalized Jordan derivation of this type from C^*-algebra A into a Banach A-bimodule M is a generalized derivation.

1. Introduction and Preliminaries

Let A be an algebra and M be an A-bimodule. A linear map $\delta : A \to M$ is said to be a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$, and is called a *Jordan derivation* if $\delta(x^2) = \delta(x)x + x\delta(x)$, for all $x \in A$. By the usual polarization the Jordan derivation identity is equivalent to assuming that

$$\delta(x \circ y) = \delta(x)y + y\delta(x) + x\delta(y) + \delta(y)x, \quad x, y \in A,$$

where '$\circ$' denotes the Jordan product $x \circ y = xy + yx$ on A.

Obviously, each derivation is a Jordan derivation, but the converse is fails in general, see [6, 12]. Herstein [11] showed that each Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Johnson in [12] proved that every continuous Jordan derivation from a C^*-algebra A into a Banach A-bimodule M is a derivation. It is shown that every Jordan derivation on nest algebras is an inner derivation [17]. Recall that a ring A is called *prime* if $aAb = 0$ implies that $a = 0$ or $b = 0$.

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Recently, Nakajima [16] introduced a new type of generalized derivations as follows. A bilinear map $\phi : A \times A \to M$ is said to be a *Hochschild* 2*-cocycle* if

$$x\phi(y,z) - \phi(xy,z) + \phi(x,yz) - \phi(x,y)z = 0, \quad x,y,z \in A.$$

A linear map $\delta : A \to M$ is said to be a *generalized derivation* if there is a 2-cocycle ϕ such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \phi(x,y), \quad x,y \in A, \tag{1.1}$$

and it is called a *generalized Jordan derivation* if

$$\delta(x^2) = \delta(x)x + x\delta(x) + \phi(x,x), \quad x \in A.$$

We denote it by (δ, ϕ). If $\phi = 0$, then they are the usual derivations and Jordan derivations, respectively. If we set $\phi(x,y) = -x\xi y$ for some $\xi \in M$, then we obtain

$$\delta(xy) = \delta(x)y + x\delta(y) - x\xi y, \quad x,y \in A, \tag{1.2}$$

which was introduced in [15]. Note that if A is unital and M is a unital A-bimodule, then (1.2) can be written as

$$\delta(xy) = \delta(x)y + x\delta(y) - x\delta(1)y, \quad x,y \in A. \tag{1.3}$$

It is shown in [16] that the usual generalized derivations defined in [3, 15], left centralizers and (σ,τ)-derivations are also generalized derivations in above sense.

Clearly, generalized derivations are generalized Jordan derivations, however, there exist generalized Jordan derivations that are not generalized derivations. Nakajima in [16] showed, under suitable conditions, that every generalized Jordan derivation (δ, ϕ) from a 2-torsion free ring into itself is a generalized derivation. In [14], the authors proved that every generalized Jordan derivation (δ, ϕ) on triangular algebra is a generalized derivation. The analogous result was obtained for von Neumann algebras as follows.

Theorem 1.1 ([13, Theorem 2.4])**.** *Let A be a von Neumann algebra and let M be a Banach A-bimodule. If (δ, ϕ) is a generalized Jordan derivation such that δ is continuous and ϕ is continuous in the first component, then (δ, ϕ) is a generalized derivation.*

Recall that a *von Neumann algebra* is a weakly closed, self-adjoint algebra of operators on a complex Hilbert space $\mathcal{H}$ containing the identity operator.

The linear map δ is called *derivable map* at $w \in A$ if

$$\delta(xy) = \delta(x)y + x\delta(y),$$

for all $x, y \in A$ with $xy = w$, and it is called *derivable map at commutative w-product* if the derivation identity holds true for all $x, y \in A$ with $xy = yx = w$.

Characterizing derivable maps on rings and algebras at a point $w \in A$ is maybe one of the most studied linear preserver problems. For example, [1,2,4,5,9] considered the case when $w = 0$. The derivable maps at commutative zero products have also been studied in several papers, for instance, see [2, 10, 19, 20], and the references therein.

A linear map δ is called *generalized derivable map* at $w \in A$ if the equality (1.1) holds true for all $x, y \in A$ with $xy = w$.

Zhou in [18], showed that if δ is a generalized derivable map at $w \in A$, where w is a left or right separating point of M, then δ is a generalized Jordan derivation associated with a Hochschild 2-cocycle ϕ. Generalized derivable maps at zero point associated with Hochschild 2-cocycles on CSL algebras are discussed in [13].

A complex Banach algebra A is said to have *property* ($\mathbb{B}$) if for every continuous bilinear map $\psi : A \times A \to X$, where X is an arbitrary Banach space, the condition

$$x, y \in A, \quad xy = 0 \text{ implies } \psi(x, y) = 0,$$

implies that

$$\psi(xy, z) = \psi(x, yz), \quad x, y, z \in A.$$

This concept was introduced in [1] and has since turned out to be applicable and powerful for characterizing linear maps through the action on zero products.

It is worth noting that C^*-algebras, group algebras $L^1(G)$ of locally compact group G and Banach algebras that are generated by idempotents have this property, [1].

Recall that the Banach algebra A is generated by idempotents if $\mathfrak{J}(A)$, the subalgebra of A generated algebraically by all idempotents in A, is dense in A.

Motivated by the above studies, in this paper, we consider the subsequent conditions on a linear map $\delta : A \to M$ associated with Hochschild 2-cocycle map ϕ;

($\mathbb{D}1$) $x, y \in A,\ xy = 0$ implies $\delta(x)y + x\delta(y) + \phi(x, y) = 0$,
($\mathbb{D}2$) $x, y \in A,\ xy = yx = 0$ implies $\delta(x)y + x\delta(y) + \phi(x, y) = 0$.

Our purpose is to characterize the maps (δ, ϕ) satisfying ($\mathbb{D}1$) or ($\mathbb{D}2$) in terms of generalized derivations. We also prove, under mild conditions, that every generalized Jordan derivation from a C^*-algebra A into a Banach A-bimodule M is a generalized derivation. This result generalizes Theorem 1.1, and implies Johnson's result.

Throughout this paper, A is a unital Banach algebra with unit 1, and M is a unital Banach A-bimodule.

2. Generalized Derivable Maps at Zero Products

Let A be a Banach algebra with property ($\mathbb{B}$). In this section, we characterize maps (δ, ϕ) from A into M that satisfy condition ($\mathbb{D}1$).

It is clear that if (δ, ϕ) is a generalized derivation, then conditions ($\mathbb{D}1$) and ($\mathbb{D}2$) hold true, but in general, the converse fails. The following example illustrates this fact.

Example 2.1. Let

$$A = \left\{ \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{C} \right\}.$$

We make $M = \mathbb{C}$ an A-bimodule by defining

$$x\lambda = x_1\lambda, \quad \lambda x = \lambda x_1, \quad \lambda \in \mathbb{C},\ x = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix} \in A.$$

Let $\phi : A \times A \to M$ be a continuous bilinear map defined by

$$\phi\left(\begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}, \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix}\right) = x_1 y_1.$$

Then, ϕ is Hochschild 2-cocycle. Define $\delta : A \to M$ via $\delta(x) = x_1$ for all $x = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$ in A. Clearly,

$$\delta(xy) = \delta(x)y + x\delta(y) + \phi(x, y),$$

for all $x, y \in A$ with $xy = 0$, or $xy = yx = 0$. Therefore, (δ, ϕ) satisfies $(\mathbb{D}1)$ and $(\mathbb{D}2)$, however, (δ, ϕ) is not a generalized derivation.

Recall that the centre of A-bimodule M is defined as

$$Z_A(M) = \{\xi \in M : \xi x = x\xi \text{ for all } x \in A\}.$$

If $A = M$, then $Z_A(M)$ will be denoted by $Z(A)$ as a centre of A.

We commence with the following lemma which plays a key role in this sequel.

Lemma 2.1. *Let ϕ be a Hochschild 2-cocycle map. If $\phi(1, 1) \in Z_A(M)$, then*

$$\phi(x, 1) = \phi(1, x) \quad \text{and} \quad \phi(xy, 1)z = x\phi(yz, 1), \quad x, y, z \in A.$$

Proof. By assumption

$$x\phi(y, z) - \phi(xy, z) + \phi(x, yz) - \phi(x, y)z = 0, \quad x, y, z \in A. \tag{2.1}$$

Setting $x = y = 1$ in (2.1), we obtain

$$\phi(1, z) = \phi(1, 1)z, \quad z \in A.$$

Taking $y = z = 1$ in (2.1), we arrive at

$$x\phi(1, 1) = \phi(x, 1), \quad x \in A.$$

Since $\phi(1, 1) \in Z_A$(M), it follows from the above equalities that

$$\phi(x, 1) = \phi(1, x), \quad x \in A. \tag{2.2}$$

By taking $x = 1$ and $z = 1$ in (2.1), respectively, we obtain

$$\phi(1, yz) = \phi(1, y)z \quad \text{and} \quad \phi(xy, 1) = x\phi(y, 1). \tag{2.3}$$

From (2.2) and (2.3) we conclude that

$$\phi(xy, 1)z = x\phi(y, 1)z = x\phi(1, y)z = x\phi(1, yz) = x\phi(yz, 1),$$

for all $x, y, z \in A$. □

Our first main theorem is the following.

Theorem 2.1. *Let (δ, ϕ) be a generalized derivable map at zero product. If*

(i) $\phi(1, 1) \in Z_A(M)$,

(ii) *δ and ϕ are continuous,* *then* δ *is a generalized derivation associated with Hochschild* 2*-cocycle* θ*, where*

$$\theta(x,y) = \phi(x,y) - \phi(xy,1) - x\delta(1)y, \quad x,y \in A.$$

Proof. Define a continuous bilinear mapping $\psi : A \times A \to M$ by

$$\psi(x,y) = \delta(x)y + x\delta(y) + \phi(x,y), \quad x,y \in A.$$

Then, $\psi(x,y) = 0$ whenever $xy = 0$, and so property $(\mathbb{B})$ gives

$$\psi(xy,z) = \psi(x,yz), \quad x,y,z \in A,$$

that is,

$$\delta(xy)z + xy\delta(z) + \phi(xy,z) = \delta(x)yz + x\delta(yz) + \phi(x,yz). \tag{2.4}$$

Taking $z = 1$ in (2.4), we arrive at

$$\delta(xy) = \delta(x)y + x\delta(y) + \phi(x,y) - \phi(xy,1) - xy\delta(1), \quad x,y \in A. \tag{2.5}$$

By setting $x = 1$ in (2.5), since it follows from Lemma 2.1 that $\phi(1,y) = \phi(y,1)$, we get $\delta(1)y = y\delta(1)$ for all $y \in A$. Thus, (2.5) can be written as $\delta(xy) = \delta(x)y + x\delta(y) + \theta(x,y)$ for all $x,y \in A$, where

$$\theta(x,y) = \phi(x,y) - \phi(xy,1) - x\delta(1)y.$$

Clearly, θ is a continuous bilinear map. Using Lemma 2.1, we see that

$$x\theta(y,z) - \theta(xy,z) + \theta(x,yz) - \theta(x,y)z = -x\phi(yz,1) + \phi(xy,1)z = 0.$$

Therefore, θ is a Hochschild 2-cocycle. This finishes the proof. □

By taking $\phi = 0$ in Theorem 2.1, we obtain the next result.

Corollary 2.1. *If* $\delta : A \to M$ *is a continuous linear derivable map at zero product, then* δ *is a generalized derivation of type* (1.3)*.*

Next we show that with extra condition that $\phi(1,1) = -\delta(1)$, the map θ in Theorem 2.1 is nothing other than ϕ.

Corollary 2.2. *Let* (δ,ϕ) *be a generalized derivable map at zero product. Suppose that*

(i) $\phi(1,1) \in Z_A(M)$,
(ii) $\phi(1,1) = -\delta(1)$,
(iii) δ *and* ϕ *are continuous.*

Then, (δ,ϕ) *is a generalized derivation.*

Proof. By Lemma 2.1, we have

$$\phi(xy,1)z = x\phi(yz,1), \quad x,y,z \in A. \tag{2.6}$$

Setting $z = 1$, and $x = y = 1$ in (2.6), respectively, we arrive at

$$\phi(xy,1) = x\phi(y,1) \quad \text{and} \quad \phi(1,1)z = \phi(z,1),$$

for all $x, y, z \in A$. Therefore,

$$\phi(xy, 1) = x\phi(y, 1) = x\phi(1, 1)y = -x\delta(1)y, \quad x, y \in A.$$

Thus,

$$\theta(x, y) = \phi(x, y) - \phi(xy, 1) - x\delta(1)y = \phi(x, y),$$

and hence (δ, ϕ) is a generalized derivation by Theorem 2.1. □

We pointed out that Corollary 2.2 applied for unital C^*-algebras. Furthermore, it covers all unital Banach algebra that are generated by idempotent such as topologically simple Banach algebras containing a non-trivial idempotent and matrix algebra $M_n(B)$ of $n \times n$ matrices over a unital Banach algebra B.

The next example shows that the condition $\phi(1, 1) = -\delta(1)$ in the preceding corollary cannot be removed.

Example 2.2. Let $C(X)$ denote the space of all continuous functions defined on compact Hausdorff space X, and take $A = C(X)$.

Define $\delta : A \to A$ and $\phi : A \times A \to A$ by

$$\Big(\delta(f)\Big)x = f(x) \quad \Big(\phi(f, g)\Big)x = f(x)g(x), \quad f, g \in A, x \in X.$$

Then, for every $f, g \in A$ with $fg = 0$ we have $\delta(fg) = \delta(f)g + f\delta(g) + \phi(f, g)$, however, (δ, ϕ) is not a generalized derivation. Note that $\phi(1, 1) \neq -\delta(1)$.

An A-bimodule M is called *symmetric* if $Z_A(M) = M$. For example, if A is commutative Banach algebra, then A^* is a symmetric Banach A-bimodule with the following module structures:

$$(f \cdot x)y = f(xy), \quad (x \cdot f)y = f(yx), \quad x, y \in A, f \in A^*.$$

Similarly, A^n, the n-th dual module of A is symmetric. Thus, we get the following result.

Corollary 2.3. *Let $\delta : A \to A^n$ be a generalized derivable map at zero product associated with a Hochschild 2-cocycle map ϕ. If A is commutative and*

(i) $\phi(1, 1) = -\delta(1)$,

(ii) *δ and ϕ are continuous,*

then (δ, ϕ) is a generalized derivation.

Let $B(X)$ be the operator algebra of all bounded linear operators on Banach space X. A *standard operator algebra* is any subalgebra of $B(X)$ which contains the identity, and the ideal $F(X)$ of all finite rank operators. It is well known that $F(X)$ is dense in $B(X)$ with respect to the strong operator topology (denoted by "SOT", for short).

Note that standard operator algebras does not have property ($\mathbb{B}$), in general. However, the next lemma can be useful to characterize derivable maps at zero products on such algebras.

Lemma 2.2 ([7, Theorem 4.1])**.** *Let V be a unital algebra. If ψ is a bilinear map from $V \times V$ into a vector space X such that*

$$x, y \in V, \quad xy = 0 \text{ implies } \psi(x, y) = 0,$$

then

$$\psi(x, u) = \psi(xu, 1) \quad \text{and} \quad \psi(u, x) = \psi(1, ux),$$

for all $x \in V$ and $u \in \mathfrak{J}(V)$.

Theorem 2.2. *Let V be a standard operator algebra and $\delta : V \to B(X)$ be a generalized derivable map at zero product associated with Hochschild 2-cocycle ϕ. If $\phi(1, 1) = -\delta(1)$, then (δ, ϕ) is a generalized derivation.*

Proof. Define a bilinear mapping $\psi : V \times V \to B(X)$ by

$$\psi(x, y) = \delta(x)y + x\delta(y) + \phi(x, y), \quad x, y \in V.$$

Then $xy = 0$ implies that $\psi(x, y) = 0$. Applying Lemma 2.2, we obtain

$$\begin{aligned} \delta(x)p + x\delta(p) + \phi(x, p) &= \psi(x, p) = \psi(xp, 1) \\ &= \delta(xp) + xp\delta(1) + \phi(xp, 1), \end{aligned}$$

for all $x \in V$ and every rank-one idempotent $p \in V$.

Since each element $u \in F(X)$ is a linear combination of rank-one idempotents, we get

$$\delta(xu) = \delta(x)u + x\delta(u) + \phi(x, u) - \phi(xu, 1) - xu\delta(1), \tag{2.7}$$

for all $x \in V$ and $u \in F(X)$. As $\phi(xu, 1) = xu\phi(1, 1) = -xu\delta(1)$, so (2.7) gives

$$\delta(xu) = \delta(x)u + x\delta(u) + \phi(x, u), \quad x \in V, u \in F(X). \tag{2.8}$$

Let $x, y \in V$. By applying (2.8), we obtain

$$\delta(xyu) = \delta(xy)u + xy\delta(u) + \phi(xy, u), \quad u \in F(X).$$

Replacing u by yu in (2.8), we have

$$\begin{aligned} \delta(xyu) &= \delta(x)yu + x\delta(yu) + \phi(x, yu) \\ &= \delta(x)yu + x\Big(\delta(y)u + y\delta(u) + \phi(y, u)\Big) + \phi(x, yu). \end{aligned}$$

By comparing the two expressions for $\delta(xyu)$, we arrive at

$$\delta(xy)u = \delta(x)yu + x\delta(y)u + x\phi(y, u) - \phi(xy, u) + \phi(x, yu),$$

for all $x, y \in V$ and $u \in F(X)$. Noticing that

$$x\phi(y, u) - \phi(xy, u) + \phi(x, yu) = \phi(x, y)u,$$

therefore,

$$\delta(xy)u = \delta(x)yu + x\delta(y)u + \phi(x, y)u, \quad x, y \in V.$$

Since $F(X)$ is dense in $B(X)$ with respect to SOT, there is a net u_i in $F(X)$ such that $u_i \to 1$. Thus,

$$\delta(xy)u_i = \delta(x)yu_i + x\delta(y)u_i + \phi(x,y)u_i,$$

for every $x, y \in V$. From the separate continuity of product in $B(X)$, we get

$$\delta(xy) = \delta(x)y + x\delta(y) + \phi(x,y), \quad x, y \in V.$$

Consequently, (δ, ϕ) is a generalized derivation. □

As a consequence of Theorem 2.2 the next result follows.

Corollary 2.4. *Let V be a standard operator algebra and $\delta : V \to B(X)$ be a linear derivable map at zero product. If $\delta(1) = 0$, then δ is a derivation.*

3. Generalized Derivable Maps at Commutative Zero Products

This section is devoted to characterizing generalized derivable maps at commutative zero products, i.e., maps (δ, ϕ) that satisfy condition $(\mathbb{D}2)$.

We start with the following lemma that vanish at commutative zero products.

Lemma 3.1 ([4, Lemma 2.2])**.** *Let V be a unital algebra. If ψ is a bilinear mapping from $V \times V$ into a vector space X such that*

$$x, y \in V, \quad xy = yx = 0 \textit{ implies } \psi(x,y) = 0,$$

then

$$\psi(x,u) + \psi(u,x) = \psi(xu,1) + \psi(1,ux),$$

for all $x \in V$ and $u \in \mathfrak{J}(V)$.

Our main theorem in this section is indicated as follows.

Theorem 3.1. *Let A be a unital Banach algebra with property $A = \overline{\mathfrak{J}(A)}$ and let (δ, ϕ) be a generalized derivable map at commutative zero product. Suppose that*

(i) *$\phi(1,1) \in Z_A(M)$,*
(ii) *δ and ϕ are continuous.*

Then, (δ, θ) is a generalized Jordan derivation, where

$$\theta(x,y) = \phi(x,y) - \phi(xy,1) - x\delta(1)y.$$

Additionally, if $\phi(1,1) = -\delta(1)$, then $\theta = \phi$.

Proof. Define a continuous bilinear mapping $\psi : A \times A \to M$ by

$$\psi(x,y) = \delta(x)y + x\delta(y) + \phi(x,y), \quad x, y \in A.$$

Then, $\psi(x,y) = 0$ whenever $xy = yx = 0$. Applying Lemma 3.1, we get

$$\psi(x,y) + \psi(y,x) = \psi(xy,1) + \psi(1,yx),$$

for all $x, y \in A$. This means that

$$\delta(x \circ y) = \delta(x)y + x\delta(y) + \delta(y)x + y\delta(x) + \phi(x, y) + \phi(y, x) - xy\delta(1) - \delta(1)xy - \phi(xy, 1) - \phi(1, yx).$$

Interchanging y by x, yields that

$$2\Big(\delta(x)x + x\delta(x)\Big) + 2\phi(x, x) = \delta(1)x^2 + x^2\delta(1) + 2\delta(x^2) + 2\phi(x^2, 1). \tag{3.1}$$

Let p be an idempotent in A. Replacing x by p in (3.1), we get

$$2\Big(\delta(p)p + p\delta(p)\Big) + 2\phi(p, p) = \delta(1)p + p\delta(1) + 2\delta(p) + 2\phi(p, 1). \tag{3.2}$$

We multiply (3.2) on the left by p to obtain

$$2p\delta(p)p + 2p\phi(p, p) = p\delta(1)p + p\delta(1) + 2p\phi(p, 1). \tag{3.3}$$

Similarly, by multiplying (3.2) on the right by p, we arrive at

$$2p\delta(p)p + 2\phi(p, p)p = p\delta(1)p + \delta(1)p + 2\phi(p, 1)p. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$2p\phi(p, p) - 2\phi(p, p)p = p\delta(1) - \delta(1)p + 2p\phi(p, 1) - 2\phi(p, 1)p, \tag{3.5}$$

for every idempotent $p \in A$.

Since ϕ is 2-cocycle, we obtain $p\phi(p, p) = \phi(p, p)p$. On the other hand, by Lemma 2.1,

$$\phi(p, 1)p = p\phi(p, 1).$$

Therefore, by (3.5) we get $p\delta(1) = \delta(1)p$ for every idempotent $p \in A$. As A is generated by idempotent, we have $\delta(1) \in Z_A(M)$. Thus, (3.1) implies that

$$\delta(x^2) = \delta(x)x + x\delta(x) + \theta(x, x), \quad x \in A, \tag{3.6}$$

where $\theta(x, x) = \phi(x, x) - \phi(x^2, 1) - x\delta(1)x$. Consequently, (δ, θ) is a generalized Jordan derivation associated with 2-cocycle map θ, defined by

$$\theta(x, y) = \phi(x, y) - \phi(xy, 1) - x\delta(1)y, \quad x, y \in A.$$

If $\phi(1, 1) = -\delta(1)$, then

$$\phi(xy, 1) = -x\delta(1)y,$$

as is done in Corollary 2.2, therefore we get $\theta(x, y) = \phi(x, y)$. This completes the proof. □

Corollary 3.1. *Let A be a von Neumann algebra and let (δ, ϕ) be a generalized derivable map at commutative zero product. If*

(i) $\phi(1, 1) \in Z_A(M)$,
(ii) $\phi(1, 1) = -\delta(1)$,
(iii) *δ and ϕ are continuous,*

then (δ, ϕ) is a generalized derivation.

Proof. Since the linear span of projections is norm dense in a von Neumann algebra A, by Theorem 3.1, (δ, ϕ) is a generalized Jordan derivation. On account of Theorem 1.1, (δ, ϕ) is a generalized derivation. □

Combining Theorem 3.1 and [16, Theorem 6], we obtain the next result.

Corollary 3.2. *Let A be a unital prime Banach algebra with property $A = \overline{\mathfrak{J}(A)}$. Suppose that $\delta : A \to A$ is a generalized derivable map at commutative zero product associated with Hochschild 2-cocycle ϕ. If*

(i) $\phi(1,1) \in Z(A)$,
(ii) $\phi(1,1) = -\delta(1)$,
(iii) *δ and ϕ are continuous,*

then (δ, ϕ) is a generalized derivation.

It is well-known that on the second dual space A^{**} of a Banach algebra A there are two multiplications, called the first and second Arens products which make A^{**} into a Banach algebra [8]. These products, which we denote by $\Box$ and $\Diamond$, are defined by

$$\Phi \Box \Psi = \lim_i \lim_j a_i \cdot b_j, \quad \Psi \Diamond \Phi = \lim_j \lim_i a_i \cdot b_j, \quad \Phi, \Psi \in A^{**},$$

where $\{a_i\}_{i\in I}$ and $\{b_i\}_{j\in I}$ are nets in A that converge in the w^*-topology, to Φ and Ψ, respectively. If these products coincide on A^{**}, then A is said to be Arens regular. We regard A as a closed subalgebra of both $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$, and A is w^*-dense in A^{**}.

More precisely, according to [8], for each Banach A-bimodule M, M^{**} turns into a Banach A^{**}-bimodule where A^{**} equipped with the first Arens product. The module actions are defined by

$$\Phi \cdot u = w^* - \lim_i \lim_j a_i \cdot x_j, \quad u \cdot \Phi = w^* - \lim_j \lim_i x_j \cdot a_i, \quad \Phi \in A^{**}, u \in M^{**},$$

where $a_i \to \Phi$ and $x_j \to u$, in w^*-topologies.

We shall use the following basic facts about the w^*-continuity of the above defined products which the reader can find in [8, Proposition A.3.52].

(i) For all $\Phi \in A^{**}$ and $x \in A$, the maps $u \mapsto u \cdot \Phi$ and $u \longmapsto x \cdot u$ from M^{**} into itself are w^*-continuous.
(i) For all $u \in M^{**}$ and $\xi \in M$, the maps $\Phi \mapsto \Phi \cdot u$ and $\Phi \longmapsto \xi \cdot \Phi$ from A^{**} into M^{**} are w^*-w^*-continuous.

Note that by [8, Corollary 3.2.43], every continuous linear map from a C^*-algebra A into its dual A^* is weakly compact. This property entails that every continuous bilinear map ϕ from $A \times A$ into some Banach space M is Arens regular, which means that the two ways of extending to the second dual give the same result, that is

$$w^* - \lim_i \lim_j \phi(a_i, b_j) = w^* - \lim_j \lim_i \phi(a_i, b_j),$$

for all w^*-convergent nets $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in I}$ in A. One may refer to the monograph of Dales [8] for a full account of Arens product.

By [8, Corollary 3.2.37], every C^*-algebra A is Arens regular and A^{**} is a von Neumann algebra, thus by extending the continuous linear map $\delta : A \to M$ and continuous bilinear map $\phi : A \times A \to M$ to the second adjoints and applying Theorem 1.1, we get the next result.

Theorem 3.2. *Let A be a C^*-algebra and let M be a Banach A-bimodule. If (δ, ϕ) is a generalized Jordan derivation such that δ is continuous and ϕ is continuous in the first component, then (δ, ϕ) is a generalized derivation.*

It should be pointed out that by setting $\phi = 0$ in Theorem 3.2, we obtain a well-known result due to Johnson [12, Theorem 6.3].

The following interesting result now follows from Theorem 3.1. Of course, it can be obtained as a consequence of Theorem 3.2.

Theorem 3.3. *Let A be a unital C^*-algebra and (δ, ϕ) be a generalized derivable map at commutative zero product. Suppose that*

(i) $\phi(1, 1) \in Z_A(M)$,
(ii) $\phi(1, 1) = -\delta(1)$,
(iii) δ *and* ϕ *are continuous.*

Then, (δ, ϕ) is a generalized derivation.

From Theorem 3.3 we get the following result.

Corollary 3.3. *Let A be a unital C^*-algebra. If $\delta : A \to M$ is a continuous linear derivable map at commutative zero product such that $\delta(1) = 0$, then δ is a derivation.*

In what follows, we prove a similar result of Theorem 2.2 for generalized derivable maps at commutative zero products.

Theorem 3.4. *Let V be a standard operator algebra and $\delta : V \to B(X)$ be a generalized derivable map at commutative zero product associated with Hochschild 2-cocycle ϕ. If*

(i) $\phi(1, 1) = -\delta(1)$,
(ii) δ *and* ϕ *are SOT continuous,*

then (δ, ϕ) is a generalized Jordan derivation.

Proof. Define a bilinear mapping $\psi : V \times V \to B(X)$ by

$$\psi(x, y) = \delta(x)y + x\delta(y) + \phi(x, y), \quad x, y \in V.$$

Then $xy = yx = 0$ implies that $\psi(x, y) = 0$. Applying Lemma 2.2, we get

$$\psi(x, u) + \psi(u, x) = \psi(xu, 1) + \psi(1, ux),$$

for all $x \in V$ and $u \in \mathfrak{J}(V)$. That is,

$$\begin{aligned} \delta(x \circ u) = \delta(x)u + x\delta(u) + \delta(u)x + u\delta(x) + \phi(x, u) + \phi(u, x) \\ - xu\delta(1) - \delta(1)ux - \phi(xu, 1) - \phi(1, ux). \end{aligned} \tag{3.7}$$

On the other hand,

$$\phi(xu,1) = xu\phi(1,1) = -xu\delta(1) \quad \text{and} \quad \phi(1,ux) = \phi(1,1)ux = -\delta(1)ux.$$

Therefore, by (3.7), we arrive at

$$\delta(x \circ u) = \delta(x)u + x\delta(u) + \delta(u)x + u\delta(x) + \phi(x,u) + \phi(u,x), \tag{3.8}$$

for all $x \in V$ and $u \in \mathfrak{J}(V)$.

For every $y \in V$, there is a net u_i in $F(X)$ such that $u_i \to y$ with respect to SOT. Thus, by using (3.8) together condition (iii) we obtain

$$\delta(x \circ y) = \delta(x)y + x\delta(y) + \delta(y)x + y\delta(x) + \phi(x,y) + \phi(y,x), \quad x, y \in V.$$

Taking $y = x$, we have

$$\delta(x^2) = \delta(x)x + x\delta(x) + \phi(x,x), \quad x \in V.$$

Consequently, (δ, ϕ) is a generalized Jordan derivation. □

It should be pointed out that Theorem 3.4 remain valid if the algebra $B(X)$ replaced by V.

Corollary 3.4. *Let V be a standard operator algebra and $\delta : V \to V$ be a generalized derivable map at commutative zero product associated with Hochschild 2-cocycle ϕ. If*

(i) $\phi(1,1) = -\delta(1)$,
(ii) *δ and ϕ are SOT continuous,*

then (δ, ϕ) is a generalized derivation.

Proof. If V is commutative, then the result actually is Theorem 2.2. Note that in this case condition (ii) is not necessary. If V is not commutative, then, by Theorem 3.4,

$$\delta(x^2) = \delta(x)x + x\delta(x) + \phi(x,x),$$

for every $x \in V$. Since V is prime, it follows from [16, Theorem 6] that (δ, ϕ) is a generalized derivation. □

Obviously, if (δ, ϕ) is a generalized derivable map at Jordan zero product, that is, $x, y \in A$, $x \circ y = 0$ implies $\delta(x)y + x\delta(y) + \phi(x,y) = 0$, then δ satisfies condition ($\mathbb{D}2$). In this regard, in all results of this section the term commutative zero products could be replaced by Jordan zero products.

In view of Theorem 3.4, the question arises whether it is possible to remove the SOT continuity of δ and ϕ from the mention result.

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