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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY LINEAR MULTIPLIER FRACTIONAL q-DIFFERENTIAL OPERATOR

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ABSTRACT. This paper introduces a novel subclass of analytic and bi-univalent functions that are linked to a linear multiplier fractional q-differential operator, defined in the open unit disk $\mathbb{D}$. The authors establish the upper bounds for the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to this new subclass and its subclasses.

1. INTRODUCTION AND PRELIMINARIES

Let the class of functions $\mathcal{A}$ be of the form:

$$\eta(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1.1}$$

which are analytic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let S indicates the functions of all subclasses in $\mathcal{A}$, which are univalent in $\mathbb{D}$. Since univalent functions are one-to-one, they are invertible. Although the inverse functions of single-valued functions are inverse functions, they do not need to be defined for the entire unit disk $\mathbb{D}$. Certainly, according to Koebe's quarter theorem [1], the disk with radius $\frac{1}{4}$ is in the image $\mathbb{D}$. Thus, every univalent function η has an inverse η^{-1} that satisfies $\eta^{-1}(\eta(z)) = z$, $z \in \mathbb{D}$, and $\zeta(w) = \eta^{-1}(\eta(w)) = w$, $|w| < r_0(\eta)$, $r_0(\eta) \geq \frac{1}{4}$, where

$$\eta^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

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A function $\eta \in \mathcal{A}$ is bi-univalent in $\mathbb{D}$ if both $\eta(z)$ and $\eta^{-1}(z)$ are univalent $\mathbb{D}$. Let Σ be the class of bi-univalent functions on $\mathbb{D}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z}, \quad \log\frac{1}{1-z}, \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathbb{D}$ such as

$$\frac{2z-z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

The widely-cited by Srivastava et al. [2] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (see, for example, [3–23]).

If $|q| < 1$, the q-shifted factorial, also known as the q-Pochhammer symbol, is defined for all $n \in \mathbb{N}$ by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$

where a and q are complex numbers. When $n = +\infty$, the product becomes

$$(a;q)_{+\infty} = \prod_{k=0}^{+\infty}(1-aq^k).$$

If $|q| < 1$, then the product converges absolutely, and we can define the q-shifted factorial for $n = +\infty$ as the limit of the sequence of partial products

$$(a;q)_{+\infty} = \lim_{n\to+\infty}(a;q)_n = \lim_{n\to+\infty}\prod_{k=0}^{n-1}(1-aq^k).$$

Therefore, when $|q| < 1$, the q-shifted factorial remains meaningful for $n = +\infty$ as a convergent infinite product.

The q-gamma function is a q-analogue of the gamma function, defined by the recurrence relation $\Gamma_q(y+1) = [y]_q\Gamma_q(y)$, where $[y]_q = \frac{(1-q^y)}{(1-q)}$ is the q-analogue of y.

Jackson's [24] q-derivative and q-integral of a function η defined on a subset of $\mathbb{C}$ are given by

$$D_q^a\eta(x) = \frac{\eta(q^a x) - \eta(x)}{(1-q^a)x - x}, \quad I_q^a\eta(x) = (1-q^a)x\sum_{n=0}^{+\infty} q^{an}\eta(q^n x),$$

where $a \in \mathbb{C}$ is a fixed parameter. These operators are also known as the q-difference and q-integral operators, respectively. The theory of q-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, as well as geometric function theory of complex analysis. The application of q-calculus was initiated by Jackson [24]. Recently, many researchers studied q-calculus such as Srivastava et al.

[25], Muhammad and Darus [26], Kanas and Răducanu [27], (see also, [28–33]) and also the reference cited therein.

Definition 1.1 ([34])**.** The fractional integral operator $I^{\delta}_{q,z}$ of order $\delta > 0$, for the function $\eta(z)$ is defined by

$$I^{\delta}_{q,z} = D^{-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z - rq)_{1-\delta}\eta(r)d_qr,$$

where $\eta(z)$ is the analytic of the simply connected regions of the z plane containing the origin. Here, the term $(z - rq)_{\delta-1}$ is a q-binomial function defined by

$$(z - rq)_{\delta-1} = z^{\delta-1}\prod_{k=0}^{+\infty}\left[\frac{1-(\frac{rq}{z})q^k}{1-(\frac{rq}{z})q^{\delta}+k-1}\right] = z^{\delta}\ {}_1\phi_0\left[q^{-\delta+1}; -; q, \frac{rq^{\delta}}{z}\right].$$

Definition 1.2. The fractional q-derivative operator $D^{\delta}_{q,z}$ of a $\eta(z)$ of order $0 \le \delta < 1$, is defined by

$$D^{\delta}_{q,z}\eta(z) = D_{q,z}I^{1-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z - rq)_{-\delta}\eta(r)d_qr,$$

where $\eta(z)$ is suitably constrained and the multiplicity of $(z - rq)_{-\delta}$ is removed as in Definition 1.1 above.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional q-derivative for the function $\eta(z)$ of order δ is defined by

$$D^{\delta}_{q,z}\eta(z) = D^{n}_{q,z}I^{n-\delta}_{q,z}\eta(z),$$

where $n - 1 \le \delta < n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.4 ([35])**.** The definition of the fractional q-differintegral operator $\Omega^{\delta}_{q,z}$ is as follows. For a function $\eta(z)$ of the form (1.1), we define

$$\Omega^{\delta}_{q}\eta(z) = \Gamma_q(2-\delta)z^{\delta}D^{\delta}_{q,z}\eta(z),$$

where $D^{\delta}_{q,z}$ denotes the fractional δ order of the q-integral $\eta(z)$ when $-\infty < \delta < 0$ and the fractional δ order q-derivative of $\eta(z)$ if $0 < \delta < 2$.

The expression for $\Omega^{\delta}_{q}\eta(z)$ in terms of the coefficients a_k of the power series expansion of $\eta(z)$ is given by

$$\Omega^{\delta}_{q}\eta(z) = z + \sum_{k=2}^{+\infty}\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}a_kz^k.$$

Definition 1.5 ([34])**.** A linear multiplier fractional q-differintegral operator is defined as

$$\begin{aligned} \mathcal{L}_{q,\lambda}^{\delta,0}\eta(z) =& \eta(z), \\ \mathcal{L}_{q,\lambda}^{\delta,1}\eta(z) =& (1-\lambda)\Omega_q^{\delta}\eta(z) + \lambda z \mathcal{L}_q\left(\Omega_q^{\delta}\eta(z)\right), \\ \mathcal{L}_{q,\lambda}^{\delta,2}\eta(z) =& \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,1}\eta(z)\right), \\ & \vdots \\ \mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) =& \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,n-1}\eta(z)\right). \end{aligned} \tag{1.3}$$

We note that if $f \in \mathcal{A}$ is given by (1.1), then by (1.3), we have

$$\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) = z + \sum_{k=2}^{+\infty} C\left(k,\delta,\lambda,n,q\right) a_k z^k,$$

where

$$C\left(k,\delta,\lambda,n,q\right) = \left(\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}\Big[\left([k]_q - 1\right)\lambda + 1\Big]\right)^n.$$

We define two new subclasses of the function class Σ by utilizing the linear multiplier fractional q-differential operator of a function $\eta \in \mathcal{A}$. Then, we provide coefficient estimates for $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses of the function class Σ.

First, we have to follow the lemma to get the main results.

Lemma 1.1 ([36])**.** *Let $\mathcal{H}$ be the family of all functions $\mathfrak{h}$ that are analytic in the open unit disk $\mathbb{D}$ and satisfy $\mathfrak{h}(0) = 1$ and $\Re(\mathfrak{h}(z)) > 0$ for all $z \in \mathbb{D}$. If a function $\mathfrak{h} \in \mathcal{H}$ is given by $\mathfrak{h}(z) = 1 + d_1 z + d_2 z^2 + \cdots$ for $z \in \mathbb{D}$, then $|d_k| \leq 2$ for all $k \in \mathbb{N}$.*

2. Coefficient Bounds for the Function Class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$

Definition 2.1. A function $\eta(z)$ given by (1.1) is said to be in the class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$ if the following conditions are satisfied: $\eta \in \Sigma$ and

$$\left|\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau z D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right| < \frac{\alpha\pi}{2},$$

where $0 < \alpha \leq 1$, $0 \leq \tau < 1$, $\delta \leq 2$, $\lambda > 0$, $n \in \mathbb{N}_0$, $z \in \mathbb{D}$, and

$$\left|\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau w D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right| < \frac{\alpha\pi}{2},$$

where $0 < \alpha \leq 1$, $0 \leq \tau < 1$, $\delta \leq 2$, $\lambda > 0$, $n \in \mathbb{N}_0$, $w \in \mathbb{D}$ and function ζ is given by

$$\zeta(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2.1}$$

We note that the following hold.

(a) When we set $\delta = 0$, $\lambda = 1$, and $q \to 1^-$, the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$ reduces to the class $S_\Sigma^{n,\tau}(\alpha)$, where $0 < \alpha \leq 1$, $0 \leq \tau < 1$, and $n \in \mathbb{N}_0$. This class was previously introduced and studied by Jothibasu [37].

(b) If we set $\delta = 0$, $\lambda = 1$, $q \to 1^-$, $n = 0$, and $\tau = 0$ in the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$, it reduces to the class of strongly bi-starlike functions $S_\Sigma^\star(\alpha)$ of order α introduced and studied by Brannan and Taha [38], where $0 < \alpha \leq 1$.

Theorem 2.1. *Let $\eta(z)$ given by (1.1) be in the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$, $0 < \alpha \leq 1$, $0 \leq \tau < 1$, $\delta \leq 2$, $\lambda > 0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Y q(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)\left[\tau q + 1\right] + X^2(1-\alpha)^2(1-\tau)^2}} \tag{2.2}$$

and

$$|a_3| \leq \frac{4\alpha^2}{X^2 q^2 (1-\tau)^2} + \frac{2\alpha}{Y q(q+1)(1-\tau)}, \tag{2.3}$$

where $X = C(2, \delta, \lambda, n, q)$ and $Y = C(3, \delta, \lambda, n, q)$.

Proof. It follows from the Definition 2.1

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau z D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} = \left[s(z)\right]^\alpha \tag{2.4}$$

and

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau w D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} = \left[t(w)\right]^\alpha, \tag{2.5}$$

respectively, where $s(z)$ and $t(w)$ satisfy the following inequalities: $\mathfrak{R}(s(z)) > 0$, $z \in \mathbb{D}$, and $\mathfrak{R}(t(w)) > 0$, $w \in \mathbb{D}$.

Furthermore, the functions $s(z)$ and $t(w)$ have the forms

$$s(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \tag{2.6}$$

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots. \tag{2.7}$$

Now, equating the coefficients in (2.4) and (2.5), we get

$$a_2 X q(1-\tau) = \alpha s_1, \tag{2.8}$$

$$a_3 Y q(q+1)(1-\tau) - a_2^2 X^2 q(1-\tau)[\tau q + 1] = \alpha s_2 + \frac{\alpha(\alpha-1)}{2}s_1^2, \tag{2.9}$$

$$-a_2 X q(1-\tau) = \alpha t_1 \tag{2.10}$$

and

$$\begin{aligned} &- a_3 Y q(q+1)(1-\tau) + 2a_2^2 Y q(q+1)(1-\tau) - a_2^2 X^2 q(1-\tau)[\tau q + 1] \\ &= \alpha t_2 + \frac{\alpha(\alpha-1)}{2}t_1^2. \end{aligned} \tag{2.11}$$

From (2.8) and (2.10), we get

$$s_1 = -t_1 \tag{2.12}$$

and

$$2a_2^2 X^2 q^2 (1-\tau)^2 = \alpha^2 \left(s_1^2 + t_1^2\right). \tag{2.13}$$

From (2.9), (2.11) and (2.13), we obtain

$$a_2^2 = \frac{\alpha^2 (s_2 + t_2)}{2\alpha Y q(q+1)(1-\tau) - 2X^2 \alpha q (1-\tau)\left[\tau q + 1\right] + X^2 (1-\alpha) q^2 (1-\tau)^2}.$$

Applying Lemma 1.1 to the coefficients s_2 and t_2, we immediately get

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Y q(q+1)(1-\tau) - 2X^2 \alpha q (1-\tau)\left[\tau q + 1\right] + X^2 (1-\alpha) q^2 (1-\tau)^2}}.$$

This gives the value of $|a_2|$ as shown in (2.2)

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.9), we get

$$\begin{aligned} &2a_3 Y q(q+1)(1-\tau) - 2a_2^2 Y q(q+1)(1-\tau) \\ =&\alpha (s_2 - t_2) + \frac{\alpha(\alpha-1)}{2}\left(s_1^2 - t_1^2\right). \end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$|a_3| = \frac{\alpha^2 (s_1^2 + t_1^2)}{2X^2 q^2 (1-\tau)^2} + \frac{\alpha (s_2 - t_2)}{2Y q(q+1)(1-\tau)}.$$

Applying Lemma 1.1 again to the coefficients s_1, s_2, t_1 and t_2, we easily get

$$|a_3| \leq \frac{4\alpha^2}{X^2 q^2 (1-\tau)^2} + \frac{2\alpha}{Y q(q+1)(1-\tau)}.$$

This end the proof of Theorem 2.1. □

Utilizing the parameters setting of Definition 2.1 in the Theorem 2.1, we get the following corollaries.

Corollary 2.1. *If $\eta(z)$ given by (1.1) be in the class $S_\Sigma^{n,\tau}(\alpha)$, $0 < \alpha \leq 1$, $0 \leq \tau < 1$ and $n \in \mathbb{N}_0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1-\tau)3^n + \left[2\alpha(\tau^2 - 1) - (\alpha-1)(1-\tau)^2\right]2^{2n}}}$$

and

$$|a_3| \leq \frac{\alpha}{3^n (1-\tau)} + \frac{4\alpha^2}{2^{2n}(1-\tau)^2}.$$

Corollary 2.2. *If $\eta(z)$ given by (1.1) and in the class $S_\Sigma^\star(\alpha)$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}} \quad \text{and} \quad |a_3| \leq 4\alpha^2 + \alpha.$$

3. Coefficient Bounds for the Function Class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$

Definition 3.1. A function $\eta(z)$ given by (1.1) is said to be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ if the following conditions are satisfied: $\eta\in\Sigma$ and

$$\Re\left(\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right)>\gamma,$$

where $0\leq\gamma<1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $z\in\mathbb{D}$, and

$$\Re\left(\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right)>\gamma,$$

where $0\leq\gamma<1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $w\in\mathbb{D}$.

The function ζ is defined as given in equation (2.1).

(a) If we set $\delta=0$, $\lambda=1$, and $q\to 1^-$ in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, it reduces to the class $S_\Sigma^{n,\tau}(\gamma)$ introduced and studied by Jothibasu [37], where $0\leq\gamma<1$, $0\leq\tau<1$ and $n\in\mathbb{N}_0$.

(b) When $\delta=0$, $\lambda=1$, $q\to 1^-$, $n=0$ and $\tau=0$, the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ simplifies to the class of strongly bi-starlike functions $S_\Sigma^\star(\gamma)$ of order γ introduced and studied by Brannan and Taha [38].

Theorem 3.1. *Let $\eta(z)$ given by (1.1) be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, $0\leq\gamma<1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$. Then*

$$|a_2|\leq\sqrt{\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)-X^2q(1-\tau)[\tau q+1]}} \tag{3.1}$$

and

$$|a_3|\leq\frac{4(1-\gamma)^2}{X^2q^2(1-\tau)^2}+\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)}, \tag{3.2}$$

where $X=C(2,\delta,\lambda,n,q)$ and $Y=C(3,\delta,\lambda,n,q)$.

Proof. It follows from the Definition 3.1 that there exist $s(z)$ and $t(w)\in\mathcal{H}$ such that

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}=\gamma+(1-\gamma)\,s(z), \tag{3.3}$$

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}=\gamma+(1-\gamma)\,t(w), \tag{3.4}$$

where $s(z)$ and $t(w)$ in $\mathcal{H}$ and have the forms (2.6) and (2.7), respectively.

Equating the coefficients in (3.3) and (3.4) yields

$$a_2 X q\,(1-\tau) = (1-\gamma)\, s_1, \tag{3.5}$$

$$a_3 Y q(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\, s_2, \tag{3.6}$$

$$-a_2 X q\,(1-\tau) = (1-\gamma)\, t_1 \tag{3.7}$$

and

$$\begin{aligned} &- a_3 Y q(q+1)\,(1-\tau) + 2a_2^2 Y q(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] \\ &= (1-\gamma)\, t_2. \end{aligned} \tag{3.8}$$

From (3.5) and (3.7), we get $s_1 = -t_1$ and

$$2a_2^2 X^2 q^2\,(1-\tau)^2 = (1-\gamma)^2 \left(s_1^2 + t_1^2\right). \tag{3.9}$$

Also, from (3.6) and (3.8), we find that

$$2a_2^2 Y q(q+1)\,(1-\tau) - 2a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\,(s_2+t_2)\,.$$

Applying Lemma 1.1 to the coefficients s_2 and t_2, we immediately get

$$|a_2| \le \sqrt{\frac{2\,(1-\gamma)}{Y q(q+1)\,(1-\tau) - X^2 q\,(1-\tau)\,[\tau q+1]}},$$

which is the bound on $|a_2|$ as given in (3.1). Then, to get the limit of $|a_3|$ by subtracting (3.8) from (3.6),

$$2a_3 Y q(q+1)\,(1-\tau) - 2a_2^2 Y q(q+1)\,(1-\tau) = (1-\gamma)\,(s_2-t_2)\,,$$

or, equivalently

$$a_3 = a_2^2 + \frac{(1-\gamma)\,(s_2-t_2)}{2Y q(q+1)\,(1-\tau)}.$$

Substituting the values of a_2^2 into (3.9), we get

$$a_3 = \frac{(1-\gamma)^2\,(s_1^2+t_1^2)}{2X^2 q^2\,(1-\tau)^2} + \frac{(1-\gamma)\,(s_2-t_2)}{2Y q(q+1)\,(1-\tau)}.$$

After applying Lemma 1.1 to the coefficients s_1, s_2, t_1 and t_2, we get

$$|a_3| \le \frac{4\,(1-\gamma)^2}{X^2 q^2\,(1-\tau)^2} + \frac{2\,(1-\gamma)}{Y q(q+1)\,(1-\tau)}.$$

This completes the proof of Theorem 3.1. □

Utilizing the parameters setting of Definition 3.1 in the Theorem 3.1, we get the following corollaries.

Corollary 3.1. *If $\eta(z)$ given by (1.1) is in the class $S_{\Sigma}^{n,\tau}(\gamma)$, $0 \le \gamma < 1$, $0 \le \tau < 1$ and $n \in \mathbb{N}_0$, then*

$$|a_2| \le \sqrt{\frac{2\,(1-\gamma)}{2^{2n}\,(\tau^2-1) + 2\,(1-\tau)\,3^n}}$$

and

$$|a_3| \leq \frac{4\left(1-\gamma\right)^2}{2^{2n}\left(1-\tau\right)^2} + \frac{\left(1-\gamma\right)}{3^n\left(1-\tau\right)}.$$

Corollary 3.2. *If $\eta(z)$ given by (1.1) and in the class $S_{\Sigma}^{\star}(\gamma)$, $0 \leq \gamma < 1$, then*

$$|a_2| \leq \sqrt{2\left(1-\gamma\right)} \quad and \quad |a_3| \leq 4\left(1-\gamma\right)^2 + \left(1-\gamma\right).$$

4. Conclusions

The main contribution of this paper is the introduction of new subclasses of bi-univalent functions defined by the linear multiplier fractional q-differential operator. Additionally, we provide upper bounds for the coefficients $|a_2|$ and $|a_3|$ for functions belonging to this new subclass and its subclasses.

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