Kragujevac Journal of Mathematics
Volume 50(6) (2026), Pages 941–955.

THE CURVELET TRANSFORM ON FUNCTION SPACES

DREMA LHAMU[1], SUNIL KUMAR SINGH[2,*], AWNIYA KUMAR[3], AND C. P. PANDEY[4]

Abstract. In this paper, we delve into the comprehensive exploration of the continuous curvelet transform (CCT), an advanced iteration of the continuous wavelet transform. Renowned for its applications in diverse mathematical realms such as signal analysis, image processing, and seismic exploration, the CCT holds significant promise. Our focus is on an in-depth examination of the CCT's properties within function spaces, i.e., in Sobolev spaces $H^s(\mathbb{R}^2)$, $W^{m,p}(\mathbb{R}^2)$, the weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$, the generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$, Besov space $B^{\alpha,q}_p(\mathbb{R}^2)$, weighted Besov space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, Hardy space $H^p(\mathbb{R}^2)$ and $BMO(\mathbb{R}^2)$ space. Through investigation, we uncover valuable insights into the continuity and boundedness of the CCT within these function spaces.

1. Introduction

In higher dimensions, wavelets struggle to handle discontinuities along curves due to poor orientation management. To address this limitation, Candés and Donoho [1,2] introduced the curvelet transform.

Curvelets are efficient tools for managing discontinuities along curves. The curvelet transforms has been used in a variety of applications during the last two decades. Starck et al. have shown applications of the CCT in image de-noising [3], astronomical image representation [4], and color image enhancement [5], while Choi et al. [6] and Nencini et al. have examined image fusion using the CCT [7]. Jero et al. accomplished ECG steganography with the CCT [8]. Dong et al. studied image fusion methods based on the CCT [9], whereas Singh et al. recently studied watermarking techniques utilizing the CCT [10]. However, literature on the theoretical aspects of curvelet

Key words and phrases. Curvelet transform, Sobolev space, Besov space, Hardy space, BMO space.
2020 *Mathematics Subject Classification.* Primary: 46F12. Secondary: 46E35, 30H25, 30H10.
https://doi.org/10.46793/KgJMat2606.941L
Received: March 16, 2024.
Accepted: July 01, 2024.

transforms in spaces like function spaces, Bochner spaces, and quaternion spaces has been lacking.

Roopkumar and collaborators identified this research gap, extending the concept of curvelet transforms to tempered Boehmians [11], Boehmians [12], tempered distributions [13], and studying curvelet transforms on periodic distributions [14]. Additionally, Akila and Roopkumar explored the quaternionic version of curvelet transforms [15]. While the curvelet transform traditionally uses the Fourier transform for intermediate calculations, associating the linear canonical transform with curvelet transforms has been suggested to yield better results [16–18].

Recently, Khan [19] examined the properties of the linear canonical curvelet transform in the quaternionic domain. In 2-D, the localization operator and wavelet multipliers have been studied in the context of linear canonical curvelet transforms by Catana et al. [20], while Starck et al. [21] explored curvelets on the sphere with applications in astronomy. Similar studies for the second-generation curvelet transform have been conducted by Chan et al. [22]. Sharma et al. [23] have shown that a partial differential equation can be solved numerically by curvelet transforms.

Building upon this existing literature, we aim to address the following unanswered problems.

• Can the concept of curvelet transform be extended to Sobolev spaces and weighted Sobolev space?

• Can we extend the concept of curvelet transform to generalized Sobolev spaces?

• Furthermore, can we extend the concept of curvelet transform to Besov space, Hardy space and BMO spaces?

These problems are essential for further study of the topological properties of the functions or signals and the curvelet transforms. The discontinuous signals can be approximated using mollifiers in these function spaces, and then their respective applications can be examined. In this paper, we have addressed these problems and obtained important inequalities.

This paper addresses questions on curvelet transforms through a structured exploration organized into four sections. Section 1 introduces curvelet transforms and reviews relevant literature. Section 2 delves into generalized Sobolev spaces, enriching the theoretical foundation. Furthermore, the curvelet transforms extended to Sobolev and weighted Sobolev spaces, exploring continuity. In Section 3, the continuous extension of the curvelet transform to Besov space and weighted Besov space is discussed. In Section 4 and 5, the continuity of curvelet transform in Hardy and BMO space is discussed. Following that, Section 6 wraps up the work by summarizing major findings and contributions. This systematic approach offers a clear understanding of curvelet transforms in diverse mathematical contexts.

Definition 1.1 (The Fourier Transform)**.** The Fourier transform of a function $f \in L^1(\mathbb{R}^2)$ is defined by

$$\mathscr{F}(f)(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^2} f(\mathbf{t}) e^{-i\langle \mathbf{t}, \boldsymbol{\xi}\rangle} d\mathbf{t}.$$

Definition 1.2 (The Continuous curvelet transform)**.** Consider two functions $W : (0,+\infty) \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ supported in $(0.5, 2)$ and $(-1, 1)$, respectively, and satisfying the following conditions:

$$\int_{1/2}^{2} W^2(r)\,\frac{dr}{r} = 1, \tag{1.1}$$

$$\int_{-1}^{1} V^2(t)\,dt = 1. \tag{1.2}$$

The functions W and V are called radial window and angular window respectively. The conditions given in equations (1.1) and (1.2) are admissibility conditions for radial and angular windows. A basic curvelet $\gamma_{a,\mathbf{0},0}$ $\gamma_{a,\mathbf{0},0}$ is defined by:

$$\hat{\gamma}_{a,\mathbf{0},0}(r,\omega) = W(ar)\,V(\omega/\sqrt{a})\,a^{3/4}, \quad 0 < a < a_0.$$

The family of curvelets is defined by

$$\gamma_{a,\mathbf{b},\theta}(\mathbf{x}) = \gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{x} - \mathbf{b})),$$

where a is positive scaling parameter, $\theta \in [0, 2\pi)$ is rotation parameter and $\mathbf{b} \in \mathbb{R}^2$ is translation parameter. For $\mathbf{u} \equiv (u_1, u_2) \in \mathbb{R}^2$, the rotation operator $R_\theta(\mathbf{u}) = (u_1 \cos\theta - u_2 \sin\theta, u_1 \sin\theta + u_2 \cos\theta)$. The continuous curvelet transform (CCT) of a function $f \in L^2(\mathbb{R}^2)$ is defined as follows [2]

$$\begin{aligned}(\Gamma_\gamma f)(a, \mathbf{b}, \theta) &= \langle f, \gamma_{a,\mathbf{b},\theta}\rangle \\ &= \int_{\mathbb{R}^2} f(\mathbf{t})\overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{t})}d\mathbf{t}, \quad 0 < a < a_0 < \pi^2,\ \mathbf{b} \in \mathbb{R}^2, \theta \in [0, 2\pi).\end{aligned}$$

Here, the choice of coarsest fixed scale $a_0 \le \pi^2$ is essential for derivation of reconstruction formula (see [12]).

Theorem 1.1. ([1, Theorem 1, p. 167]). *For a function $f \in L^2(\mathbb{R}^2)$, with $\hat{f}(\xi) = 0$ for all $\xi < \frac{2}{a_0}$, the reconstruction formula is given by*

$$f(\boldsymbol{x}) = \int (\Gamma_\gamma f)(a, \boldsymbol{b}, \theta)\,\gamma_{a,\boldsymbol{b},\theta}\,\frac{da}{a^3}\,d\boldsymbol{b}\,d\theta,$$

which is valid for high frequency. Parseval formula for functions having high-frequency is given by

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int |(\Gamma_\gamma f)(a, \boldsymbol{b}, \theta)|^2\,\frac{da}{a^3}\,d\boldsymbol{b}\,d\theta.$$

Theorem 1.2. *If $f \in L^2(\mathbb{R}^2)$, then the following results hold.*

(a) (*Linearity*) $(\Gamma_\gamma(Af + Bg))(a, \boldsymbol{b}, \theta) = A(\Gamma_\gamma f)(a, \boldsymbol{b}, \theta) + B(\Gamma_\gamma g)(a, \boldsymbol{b}, \theta)$, *where A and B are scalars.*

(b) (*Shifting*) $(\Gamma_\gamma T_{\boldsymbol{c}} f)(a, \boldsymbol{b}, \theta) = (\Gamma_\gamma f)(a, \boldsymbol{b}\text{-}\boldsymbol{c}, \theta)$, *where $T_{\boldsymbol{c}} f(\boldsymbol{t}) = f(\boldsymbol{t}\text{-}\boldsymbol{c})$, for $\boldsymbol{t},\boldsymbol{c} \in \mathbb{R}^2$.*

Example 1.1. For the Dirac delta function, we can find the following:

(a) $(\Gamma_\gamma \delta)(a, \mathbf{b}, \theta) = \overline{\gamma}_{a,\mathbf{b},\theta}(\mathbf{0})$;

(b) $(\Gamma_\gamma T_{\mathbf{c}}\delta)(a,\mathbf{b},\theta) = \overline{\gamma}_{a,\mathbf{b},\theta}(\mathbf{c})$.

Example 1.2. The CCT of $f(\mathbf{t}) = 1$ is $\overline{\hat{\gamma}}_{a,\mathbf{b},\theta}(\mathbf{0})$.

2. The Continuous Curvelet Transform on Sobolev Space

Let us recall the basic definitions which are required for Sobolev space on $\mathbb{R}^2$.

Definition 2.1 ([24])**.** A distribution is a continuous linear functional defined on test function space $\mathscr{D}(\mathbb{R}^2) := \{\phi \in C_K^\infty(\mathbb{R}^2) : \phi(x) \in \mathbb{C}\}$, where $C_K^\infty(\mathbb{R}^2)$ denotes the space of infinitely differentiable functions having compact support K. The collection of such distributions form linear space and is denoted by $\mathscr{D}'(\mathbb{R}^2)$. If for each multi-index α and $\phi_j \in C^\infty(\mathbb{R}^2)$, the $D^\alpha \phi_j \to 0$ uniformly on every compact subset of $\mathbb{R}^2$, then the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ is said to be a convergent sequence on $C^\infty(\mathbb{R}^2)$ with limit 0. The space of such convergent sequences is denoted by $\mathscr{E}(\mathbb{R}^2)$. The collection of compactly supported distributions is denote by $\mathscr{E}'(\mathbb{R}^2)$.

Definition 2.2. Let $\phi \in C^\infty(\mathbb{R}^2)$ be a rapidly decreasing function with

$$\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\phi) = \sup_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}^{\boldsymbol{\alpha}} D^{\boldsymbol{\beta}} \phi(\mathbf{x})| < +\infty \text{ for all multi-indices } \boldsymbol{\alpha} = (\alpha_1,\alpha_2) \text{ and} \boldsymbol{\beta} = (\beta_1,\beta_2).$$

The collection of such functions ϕ is called Schwartz space and is denoted by $S(\mathbb{R}^2)$. The continuous linear functionals on $S(\mathbb{R}^2)$ are tempered distributions and the space of tempered distributions is denoted by $S'(\mathbb{R}^2)$.

Definition 2.3 (The Sobolev space $H^s(\mathbb{R}^2)$)**.** The space containing all such tempered distributions f, i.e., $f \in S'(\mathbb{R}^2)$ having property:

$$\left(1 + |\boldsymbol{\eta}|^2\right)^{s/2} \mathscr{F}\{f\}(\boldsymbol{\eta}) \in L^2(\mathbb{R}^2), \quad \text{for all } s \in \mathbb{R},$$

is called Sobolev space and it is denoted by $H^s(\mathbb{R}^2)$. The inner-product on $H^s(\mathbb{R}^2)$ is defined by

$$\langle f, g\rangle_{H^s} = \int_{\mathbb{R}^2} (1 + |\boldsymbol{\eta}|^2)^s \mathscr{F}\{f\}(\boldsymbol{\eta}) \overline{\mathscr{F}\{g\}(\boldsymbol{\eta})}\, d\boldsymbol{\xi}.$$

The norm induced by above inner-product is given by

$$||f||_{H^s} = \left(\int_{\mathbb{R}^2} (1 + |\boldsymbol{\eta}|^2)^s |\mathscr{F}\{f\}(\boldsymbol{\eta})|^2\, d\boldsymbol{\eta}\right)^{\frac{1}{2}} < +\infty.$$

Let us recall the definition of generalized Sobolev space defined in [25, 26].

Definition 2.4 (Generalized Sobolev space)**.** The collection of continuous real-valued functions ω on $\mathbb{R}^2$ satisfying the following conditions:

(i) $0 = \omega(\mathbf{0}) \le \omega(\boldsymbol{\xi} + \boldsymbol{\eta}) \le \omega(\boldsymbol{\xi}) + \omega(\boldsymbol{\eta})$;

(ii) $\displaystyle\int_{\mathbb{R}^2} \frac{\omega(\boldsymbol{\xi})\, d\boldsymbol{\xi}}{(1 + |\boldsymbol{\xi}|)^3} < +\infty$;

(iii) $a + b\log(1 + |\boldsymbol{\xi}|) \le \omega(\boldsymbol{\xi})$, $a \in \mathbb{R}$, $b \in (0, +\infty)$, is denoted by $\mathscr{M}$.

The set $\mathscr{M}_c$ consists of all $\omega \in \mathscr{M}$ such that $\omega(\boldsymbol{\xi}) = \sigma(|\boldsymbol{\xi}|)$, where σ concave on $[0, +\infty)$.

Definition 2.5 ([27])**.** For $\omega \in \mathscr{M}_c$, the Bjorck-space $S_\omega(\mathbb{R}^2)$ is the set of all functions $\phi \in L^1(\mathbb{R}^2)$ such that $\phi, \hat{\phi} \in C^\infty$ and for each multi-indices $\boldsymbol{\alpha}$ and each non-negative number λ

$$p_{\boldsymbol{\alpha},\lambda}(\phi) = \sup_{\mathbf{x}} e^{\lambda\omega(\mathbf{x})} \left| D^{\boldsymbol{\alpha}} \phi(\boldsymbol{x}) \right| < +\infty$$

and

$$\pi_{\boldsymbol{\alpha},\lambda}(\phi) = \sup_{\boldsymbol{\xi}} e^{\lambda\omega(\boldsymbol{\xi})} \left| D^{\boldsymbol{\alpha}} \hat{\phi}(\boldsymbol{\xi}) \right| < +\infty.$$

The dual of S_ω is denoted by S'_ω, the elements of which are called ultradistributions. We may found its various properties in [25].

Now, we consider a continuous weight function w on $\mathbb{R}^2$ with the following properties. There exist $\lambda > 0$ and $C, D, E > 0$, such that, for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^2$, $t \in \mathbb{R}$, $|t| < 1$ and $\omega \in \mathscr{M}_c$

$$\begin{aligned} w(\boldsymbol{\xi}) &\leq C e^{\lambda\omega(\boldsymbol{\xi})}, \\ w(\boldsymbol{\xi} + \boldsymbol{\eta}) &< D\left(w(\boldsymbol{\xi}) + w(\boldsymbol{\eta})\right), \\ w(t\,\boldsymbol{\xi}) &< E\, w(\boldsymbol{\xi}). \end{aligned} \tag{2.1}$$

Definition 2.6 (Generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$ [28])**.** The generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$ is defined as the set of all ultradistributions $f \in S'_\omega$ such that

$$||f||^2_{H^\omega_w(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\hat{f}(\boldsymbol{\xi})|^2 w(\boldsymbol{\xi}) d\boldsymbol{\xi} < +\infty.$$

Theorem 2.1. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. Then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : H^\omega_w \to H^\omega_w$$

is continuous and

$$||(\Gamma_\gamma f)(a, \cdot, \theta)||^2_{H^\omega_w(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^\omega_w(\mathbb{R}^2)}.$$

Proof. Since, $\left|\mathscr{F}\left(\left(\Gamma_\gamma f\right)(a, \cdot, \theta)\right)(\boldsymbol{\xi})\right|^2 = \left|\hat{f}(\boldsymbol{\xi})\right|^2 \left|\hat{\gamma}_{a,\mathbf{0},\theta}(\boldsymbol{\xi})\right|^2$. Therefore,

$$\begin{aligned} ||(\Gamma_\gamma f)(a, \cdot, \theta)||^2_{H^\omega_w(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \left| \mathscr{F}\left((\Gamma_\gamma f)(a, \cdot, \theta)\right)(\boldsymbol{\xi}) \right|^2 w(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^2} \left|\hat{f}(\boldsymbol{\xi})\right|^2 \left|\overline{\hat{\gamma}}_{a,\mathbf{0},\theta}(\boldsymbol{\xi}))\right|^2 w(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ &\leq ||\gamma_{a,\mathbf{0},0}||^2_{L_1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \left|\hat{f}(\boldsymbol{\xi})\right|^2 w(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ &= ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^\omega_w(\mathbb{R}^2)}. \end{aligned}$$

□

Corollary 2.1. *If* $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$ *and* $f, g \in H^{\omega}_{w}(\mathbb{R}^2)$, *then for fixed* $a > 0$, *the following estimate holds*

$$\begin{aligned}||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||^2_{H^{\omega}_{w}(\mathbb{R}^2)} &\leq ||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^{\omega}_{w}(\mathbb{R}^2)} \\ &\quad + ||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f-g||^2_{H^{\omega}_{w}(\mathbb{R}^2)}.\end{aligned}$$

Since, the spaces $H^{\omega}_{w}(\mathbb{R}^2)$ reduce to Sobolev space $H^s(\mathbb{R}^2)$ for weight function $w(\boldsymbol{\xi}) = (1+|\boldsymbol{\xi}|^2)^s$, $s \in \mathbb{R}$. Therefore, we have the following result for space $H^s(\mathbb{R}^2)$.

Theorem 2.2. *Let* $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. *If* $f \in S'(\mathbb{R}^2)$, *then, for fixed* $a > 0$, *the curvelet transform*

$$\Gamma_\gamma : H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{H^s(\mathbb{R}^2)} \leq \; ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)} ||f||^2_{H^s(\mathbb{R}^2)}.$$

Corollary 2.2. *If* $\phi_{a,\mathbf{0},0}, \gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$, *then for the curvelet transforms* $(\Gamma_\gamma f)$ *and* $(\Gamma_\phi g)$ *with admissible and curvelets* $\phi_{a,\mathbf{0},0}, \gamma_{a,\mathbf{0},0}$ *and* $f, g \in H^s(\mathbb{R}^2), s \in \mathbb{R}$, *the following estimate holds*

$$\begin{aligned}||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||_{H^s(\mathbb{R}^2)} &\leq ||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)}\, ||f||_{H^s(\mathbb{R}^2)} \\ &\quad + ||\phi_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)}\, ||f-g||_{H^s(\mathbb{R}^2)}.\end{aligned}$$

Definition 2.7 (The Sobolev space $W^{m,p}(\mathbb{R}^2)$ [27]). Let $1 \leq p \leq \infty$ and $m \in \mathbb{N} \cup \{0\}$. The Sobolev space $W^{m,p}(\mathbb{R}^2)$ is defined by

$$W^{m,p}(\mathbb{R}^2) = \Big\{ f \in \mathscr{D}'(\mathbb{R}^2) : D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^2) \text{ for all } |\boldsymbol{\alpha}| \leq m \Big\}$$

and equipped with the norm

$$||f||_{W^{m,p}(\mathbb{R}^2)} = \left(\sum_{|\boldsymbol{\alpha}| \leq m} ||D^{\boldsymbol{\alpha}} f||^p_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < +\infty,$$

and $||f||_{W^{m,\infty}} = \sup_{|\boldsymbol{\alpha}| \leq m} ||D^{\boldsymbol{\alpha}} f||_{L^\infty(\mathbb{R}^2)}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, and α_1, α_2 are non-negative integers, and partial derivatives $D^{\boldsymbol{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}$ in distributional sense.

Definition 2.8 (The weighted L^p space [29, 30]). Let κ be a weight function, i.e., a non-negative locally integrable function. For $1 \leq p < +\infty$, the weighted $L^p_\kappa(\mathbb{R}^2)$ space is defined as the set of all measurable functions f on $\mathbb{R}^2$ such that

$$\|f\|_{L^p_\kappa(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^p \, \kappa(\mathbf{x}) d^2\mathbf{x} \right)^{1/p} < +\infty.$$

Theorem 2.3 (Weighted Young's Inequality [31]). *Suppose* κ *be a weight function for which there exists another weight function* w *such that*

$$\kappa(\boldsymbol{x} + \boldsymbol{y}) \leq C\, w(\boldsymbol{x})\, \kappa(\boldsymbol{y}), \quad \textit{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2, \tag{2.2}$$

here C is a constant. Let $f \in L^p_\kappa(\mathbb{R}^2)$, $g \in L^1_{w^{1/p}}(\mathbb{R}^2)$, $1 < p < +\infty$. Then, we have the following inequality

$$\|f * g\|_{L^p_\kappa(\mathbb{R}^2)} \leq C\,||f||_{L^p_\kappa(\mathbb{R}^2)}\,||g||_{L^1_{w^{1/p}}(\mathbb{R}^2)}. \tag{2.3}$$

Definition 2.9 (The weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$ [30])**.** Let m be a non-negative integer and $1 \leq p < +\infty$. The weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$ is defined as the set of all $f \in \mathscr{D}'(\mathbb{R}^2)$ with distributional derivatives $D^{\boldsymbol{\alpha}} f \in L^p_\kappa(\mathbb{R}^2)$ for $|\boldsymbol{\alpha}| \leq m$. The norm of f in $W^{m,p}_\kappa(\mathbb{R}^2)$ is defined as

$$\|f\|_{W^{m,p}_\kappa(\mathbb{R}^2)} = \left(\sum_{|\boldsymbol{\alpha}|\leq m} \|D^{\boldsymbol{\alpha}} f\|^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{1/p}.$$

Theorem 2.4. *Suppose that κ, w are weight functions that satisfy* (2.2)*. If $\gamma_{a,\mathbf{0},0} \in L^1_{w^{1/p}}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : W^{m,p}_\kappa(\mathbb{R}^2) \to W^{m,p}_\kappa(\mathbb{R}^2)$$

is continuous and

$$\begin{aligned} ||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\leq m} ||D^{\boldsymbol{\alpha}}_{\boldsymbol{b}}(\Gamma_\gamma f)(a,\boldsymbol{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\leq C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f||_{W^{m,p}_\kappa(\mathbb{R}^2)}. \end{aligned}$$

Proof. Since $f \in W^{m,p}_\kappa(\mathbb{R}^2)$, therefore, for all $|\boldsymbol{\alpha}| \leq m$, $D^{\boldsymbol{\alpha}} f \in L^p_\kappa(\mathbb{R}^2)$. Using weighted Young's inequality, we have

$$\begin{aligned} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(\Gamma_\gamma f)(a,\mathbf{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)} &= ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(f(\cdot) * \overline{\gamma_{a,\mathbf{0},\theta}(\cdot)})(\mathbf{b})||^p_{L^p_\kappa(\mathbb{R}^2)} \\ &= ||\left(\overline{\gamma}_{a,\mathbf{0},\theta}(\cdot) * D^{\boldsymbol{\alpha}}_{\mathbf{b}}\, f(\cdot)\right)(\mathbf{b})\,||^p_{L^p_\kappa(\mathbb{R}^2)} \\ &\leq C^p\,||\gamma_{a,\mathbf{0},0}||^p_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f\,||^p_{L^p_\kappa(\mathbb{R}^2)} \end{aligned}$$

and hence,

$$\begin{aligned} ||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\leq m} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(\Gamma_\gamma f)(a,\mathbf{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{|\boldsymbol{\alpha}|\leq m} C^p\,||\gamma_{a,\mathbf{0},0}||^p_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\leq C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\left(\sum_{|\boldsymbol{\alpha}|\leq m} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &= C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f||_{W^{m,p}_\kappa(\mathbb{R}^2)}. \end{aligned}$$

□

Corollary 2.3. *Suppose that* κ, w *are weight functions that satisfy* (2.2). *If* $\gamma_{a,\mathbf{0},0}, \phi_{a,\mathbf{0},0} \in L^1_{w^{1/p}}(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$ *and* $f, g \in W^{m,p}_{\kappa}(\mathbb{R}^2)$, *then, for fixed* $a > 0$, *the following estimate holds*

$$\begin{aligned}&||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)}\\ \leq& C\left(||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f||_{W^{m,p}_\kappa(\mathbb{R}^2)} + ||\phi_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f-g||_{W^{m,p}_\kappa(\mathbb{R}^2)}\right).\end{aligned}$$

The space $W^{m,p}_\kappa(\mathbb{R}^2)$ reduces to Sobolev space $W^{m,p}(\mathbb{R}^2)$ for $\kappa = 1$. Hence, we have the following result for $W^{m,p}(\mathbb{R}^2)$ space.

Theorem 2.5. *Let* $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. *Then, for fixed* $a > 0$, *the curvelet transform*

$$\Gamma_\gamma : W^{m,p}(\mathbb{R}^2) \to W^{m,p}(\mathbb{R}^2)$$

is a continuous map and

$$\begin{aligned}||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\leq m} ||D^{\boldsymbol{\alpha}}_{\boldsymbol{b}}(\Gamma_\gamma f)(a,\boldsymbol{b},\theta)||^p_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{p}}\\ &\leq ||\gamma_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)}\,||f||_{W^{m,p}(\mathbb{R}^2)}.\end{aligned}$$

3. Curvelet Transform on Besov space

Russian Mathematician, Oleg Vladimirovich Besov has defined a new Banach space $B^{\alpha,q}_p$ with quasi-norm to study the regularity and smoothness of functions in 1961. To commemorate 'O. V. Besov', the function space is known as Besov space. This function space has many applications in study of PDEs, fluid dynamics and quantum mechanics. Some existing works related to wavelet analysis in Besov space can be found in [24, 29, 31–33].

Let us recall the definition of notions related to Besov space. The modulus of smoothness for the function $f \in L^p(\mathbb{R}^2)$ is defined by $w_p(f,\mathbf{h}) = \|f(\cdot+\mathbf{h}) - f(\cdot)\|_{L^p(\mathbb{R}^2)}$, where $0 \neq \mathbf{h} \in \mathbb{R}^2$.

Definition 3.1 (Besov space)**.** For $1 \leq p, q \leq +\infty$ and $\alpha \in (0,1)$, the Besov space $B^{\alpha,q}_p(\mathbb{R}^2)$ is defined as

$$B^{\alpha,q}_p(\mathbb{R}^2) = \left\{f \in L^p(\mathbb{R}^2) : \int_{\mathbb{R}^n} [\omega_p(f,\mathbf{h})]^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} < +\infty\right\},$$

for $1 \leq p < +\infty$ and for $q = +\infty$

$$B^{\alpha,\infty}_p(\mathbb{R}^2) = \left\{f \in L^p(\mathbb{R}^2) : |\mathbf{h}|^{-\alpha}\omega_p(f,\mathbf{h}) \in L^\infty(\mathbb{R}^2)\right\},$$

where $|\mathbf{h}|$ is an Euclidean norm of $\mathbf{h} \in \mathbb{R}^2$. The space $B_p^{\alpha,q}(\mathbb{R}^2)$ is Banach space with norms

$$\|f\|_p^{\alpha,q} = \|f\|_{L^p(\mathbb{R}^2)} + \left(\int_{\mathbb{R}^n} [\omega_p(f,\mathbf{h})]^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}}, \quad \text{for } q < +\infty,$$
$$\|f\|_p^{\alpha,\infty} = \|f\|_{L^p(\mathbb{R}^2)} + \||\mathbf{h}|^{-\alpha}\omega_p(f,\mathbf{h})\|_\infty, \quad \text{for } q = +\infty.$$

For $f \in L^p_\kappa(\mathbb{R}^2)$, the modulus of smoothness is defined as:

$$w_{p,\kappa}(f,\mathbf{h}) = \|f(\cdot + \mathbf{h}) - f(\cdot)\|_{L^p_\kappa(\mathbb{R}^2)},$$

where κ is a weight function and $\mathbf{h}$ is non-zero element of $\mathbb{R}^2$.

Definition 3.2 (Weighted Besov space)**.** For $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, the weighted Besov space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, $0 < \alpha < 1$, is defined as

$$B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2) = \left\{ f \in L^p_\kappa(\mathbb{R}^2) : \int_{\mathbb{R}^2} (w_{p,\kappa}(f,\mathbf{h}))^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} < \infty \right\}, \quad \text{for all } 1 \leq q < +\infty,$$

and

$$B^{\alpha,\infty}_{p,\kappa}(\mathbb{R}^2) = \left\{ f \in L^p_\kappa(\mathbb{R}^2) : |\mathbf{h}|^{-\alpha} w_{p,\kappa} \in L^\infty\left(\mathbb{R}^2\right)\right\}, \quad \text{for } q = +\infty.$$

It is easy to see that the space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, $1 \leq q < +\infty$, is a Banach space associated with the norm defined by

$$||f||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)} = ||f||_{L^p_\kappa(\mathbb{R}^2)} + \left(\int_{\mathbb{R}^2} (w_{p,\kappa}(f,\mathbf{h}))^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}},$$

and if $q = +\infty$,

$$||f||_{B^{\alpha,\infty}_{p,\kappa}(\mathbb{R}^2)} = ||f||_{L^p_\kappa(\mathbb{R}^2)} + \left\||\mathbf{h}|^{-\alpha} w_{p,\kappa}(f,\mathbf{h})\right\|_\infty.$$

Theorem 3.1. *Let* $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. *If* $f \in B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, *then, for fixed* $a > 0$, *the curvelet transform*

$$\Gamma_\gamma : B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2) \to B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)} ||f||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}.$$

Proof. By change of variable, we have

$$(\Gamma_\gamma f)(a,\mathbf{b},\theta) = \int_{\mathbb{R}^2} f(\mathbf{t})\overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{t}-\mathbf{b}))} d\mathbf{t}$$
$$= \int_{\mathbb{R}^2} f(\mathbf{u}+\mathbf{b})\overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})} d\mathbf{u}.$$

Now, smoothness function for curvelet transform is given by:

$$
\begin{aligned}
&\omega_{p,\kappa}(\Gamma_\gamma f)(a,\cdot,\theta),\mathbf{h}) \\
=&\|(\Gamma_\gamma f)(a,\cdot+\mathbf{h},\theta)-(\Gamma_\gamma f)(a,\cdot,\theta)\|_{L^p_\kappa(\mathbb{R}^2)} \\
=&\left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2}(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))\,\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{u}\right|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}} \\
\le&\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))\,\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\right|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}}d\mathbf{u} \\
=&\int_{\mathbb{R}^2}\left|\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\right|\left(\int_{\mathbb{R}^2}|(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}}d\mathbf{u} \\
\le&\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\,\omega_{p,\kappa}(f,\mathbf{h}).
\end{aligned}
$$

Hence, for $q<+\infty$, we have

$$\left(\int_{\mathbb{R}^2}[\omega_p(\Gamma_\gamma f)(a,\mathbf{0},\theta),\mathbf{h})]^q\frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}}\le\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\|f\|_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}. \qquad \square$$

Corollary 3.1. *If $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$ and $f,g\in B^{\alpha,q}_p(\mathbb{R}^2)$, then for fixed $a>0$, the following estimate holds*

$$
\begin{aligned}
||(\Gamma_\gamma f)(a,\cdot,\theta)-(\Gamma_\phi g)(a,\cdot,\theta)||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}\le{}&||\gamma_{a,\mathbf{0},0}-\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)} \\
&+||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f-g||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}.
\end{aligned}
$$

For $\kappa(x)=1$, the weighted Besov space reduces to the Besov space, yielding the following theorem.

Theorem 3.2. *Let $\gamma_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$. If $f\in B^{\alpha,q}_p(\mathbb{R}^2)$, then, for fixed $a>0$, the curvelet transform*

$$\Gamma_\gamma : B^{\alpha,q}_p(\mathbb{R}^2)\to B^{\alpha,q}_p(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}\le\ ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}||f||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}.$$

Corollary 3.2. *If $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$ and $f,g\in B^{\alpha,q}_p(\mathbb{R}^2)$, then for fixed $a>0$, the following estimate holds*

$$
\begin{aligned}
||(\Gamma_\gamma f)(a,\cdot,\theta)-(\Gamma_\phi g)(a,\cdot,\theta)||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}\le{}&||\gamma_{a,\mathbf{0},0}-\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f||^2_{B^{\alpha,q}_p(\mathbb{R}^2)} \\
&+||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f-g||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}.
\end{aligned}
$$

4. Curvelet Transform on Hardy space

In the early 20th century, Hardy and Littlewood's work on Hardy spaces was primarily focused on understanding the properties of analytic functions and their behavior on the boundary of the domain of analyticity. Their collaboration resulted in the development of the classical Hardy spaces, H^p, which are defined as spaces of analytic functions for which the p-norm of the function is finite on certain domains, such as the unit disk in the complex plane. This space has application in PDEs, Harmonic analysis, function spaces and operator theory (see [30, 34–36]).

Definition 4.1 (Hardy space)**.** Hardy Space $H^p(\mathbb{R}^2)$ is defined as the space of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H^p(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \sup_{t>0} |(f * \varphi_t)(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}, \tag{4.1}$$

where $\varphi_t = t^{-n}\varphi\left(\frac{\mathbf{x}}{t}\right)$, $t > 0$, $\mathbf{x} \in \mathbb{R}^2$, and φ be a function in the Schwartz space such that $\int_{\mathbb{R}^2} \varphi(\mathbf{x})d\mathbf{x} \neq 0$.

Theorem 4.1. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. If $f \in H^p(\mathbb{R}^2)$, then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : H^p(\mathbb{R}^2) \to H^p(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a, \cdot, \theta)||^2_{H^p(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)} ||f||^2_{H^p(\mathbb{R}^2)}.$$

Proof. Invoking change of variable in the definition of curvelet transform, we have

$$\begin{aligned}((\Gamma_\gamma f)(a, \cdot, \theta) * \varphi_t)(\mathbf{b}) &= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})} \left(\int_{\mathbb{R}^2} f(\mathbf{u}+\mathbf{b}\text{-}\mathbf{x})\varphi_t(\mathbf{x})d\mathbf{x}\right) d\mathbf{u} \\ &= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})}(f * \varphi_t)(\mathbf{u} + \mathbf{b})d\mathbf{u}.\end{aligned}$$

Therefore, the application of Minkowski inequality yields

$$\begin{aligned}\|(\Gamma_\gamma f)(a, \cdot, \theta)\|_{H^p} &= \left(\int_{\mathbb{R}^2} \sup_{t>0} |((\Gamma_\gamma f)(a, \cdot, \theta) * \varphi_t)(\mathbf{b})|^p d\mathbf{b}\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{u}))}(f * \varphi_t)(\mathbf{u} + \mathbf{b})d\mathbf{u}\right|^p d\mathbf{b}\right)^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^2} \left(\sup_{t>0} \int_{\mathbb{R}^2} \left|\overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{u}))}(f * \varphi_t)(\mathbf{u} + \mathbf{b})\right|^p d\mathbf{b}\right)^{\frac{1}{p}} d\mathbf{u} \\ &= \int_{\mathbb{R}^2} |\overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})}| \left(\int_{\mathbb{R}^2} \sup_{t>0} |(f * \varphi_t)(\mathbf{u} + \mathbf{b})|^p d\mathbf{b}\right)^{\frac{1}{p}} d\mathbf{u} \\ &\leq \|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)} \|f\|_{H^p(\mathbb{R}^2)}.\end{aligned}$$

□

5. Curvelet Transfrom on BMO Space

The "Bounded Mean Oscillation space" (BMO space) was defined by F. John and L. Nirenberg in 1961 and it is dual space of Hardy space H^1.

Definition 5.1. The space BMO($\mathbb{R}^2$) is defined as the space of all functions $f \in L^1_{loc}(\mathbb{R}^2)$ such that

$$\|f\|_{BMO(\mathbb{R}^2)} = \sup_{B\subset\mathbb{R}^2} \frac{1}{|B|}\int_B |f - f_B|\, d\mathbf{x} < +\infty,$$

where the supremum is taken over all disk B in $\mathbb{R}^2$, and f_B is the mean value of the function f on B defined by $f_B = \frac{1}{|B|}\int_B f(\mathbf{y})d\mathbf{y}$ for each disk $B \subset \mathbb{R}^2$.

Theorem 5.1. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$. If $f \in BMO(\mathbb{R}^2)$. Then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : BMO(\mathbb{R}^2) \to BMO(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{BMO(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}||f||^2_{BMO(\mathbb{R}^2)}.$$

Proof. For an arbitrary disk B contained in $\mathbb{R}^2$, we have

$$\begin{aligned}\int_B(\Gamma_\gamma f)(a,\mathbf{b},\theta)d\mathbf{b} &\leq \int_{\mathbb{R}^2}\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\left(\int_B f(\mathbf{u}+\mathbf{b})d\mathbf{b}\right)d\mathbf{u}\\ &= \int_{\mathbb{R}^2}\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\left(\int_Q f(\mathbf{y})d\mathbf{y}\right)d\mathbf{u},\end{aligned}$$

where $Q = \mathbf{u} + B$. Since, $Q \subset \operatorname{supp}\gamma_{a,\mathbf{0},0} + B \subseteq \mathbb{R}^2$ is compact set and $f \in L^1_{loc}(\mathbb{R}^2)$. It follows that

$$\int_B((\Gamma_\gamma f)(a,\cdot,\theta) * \varphi_t)(\mathbf{b})d\mathbf{b} \leq K\int_{\mathbb{R}^2}\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})d\mathbf{u} = K\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)} < \infty,$$

and hence, $(\Gamma_\gamma f)(a,\cdot,\theta) \in L^1_{loc}(\mathbb{R}^2)$. Using Fubini's theorem, we have

$$\Gamma_{f_B}(a,\cdot,\theta) = \int_{\mathbb{R}^2}\left(\frac{1}{B}\int_B f(\mathbf{u}+\mathbf{b})\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{b}\right)d\mathbf{u} = \int_{\mathbb{R}^2} f_Q\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{u}.$$

Applying Minkowski's inequality, we obtain

$$\begin{aligned}\|(\Gamma_\gamma f)(a,\cdot,\theta)\|_{BMO(\mathbb{R}^2)} &= \sup_{B\subset\mathbb{R}^2}\frac{1}{B}\int_B|(\Gamma_\gamma f)(a,\mathbf{b},\theta) - \Gamma_{f_B}(a,\mathbf{b},\theta)|d\mathbf{b}\\ &\leq \sup_{B\subset\mathbb{R}^2}\frac{1}{B}\int_B\left(\int_{\mathbb{R}^2}|(f(\mathbf{u}+\mathbf{b}) - f_Q)\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}|d\mathbf{u}\right)d\mathbf{b}\\ &= \int_{\mathbb{R}^2}|\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}|\left(\sup_{Q\subset\mathbb{R}^2}\frac{1}{Q}\int_Q|f(\mathbf{y}) - f_Q|d\mathbf{y}\right)d\mathbf{u}\\ &\leq \|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\|f\|_{BMO(\mathbb{R}^2)}.\end{aligned}$$

□

6. Conclusion

The questions posed in the introduction have been addressed and answered affirmatively in this paper, contributing to the existing literature on curvelet transforms. The research gap identified in the introduction has been successfully bridged through the continuous extension of the curvelet transform to functional spaces, i.e., Sobolev space, weighted Sobolev space, generalized Sobolev space, Besov space, weighted

Besov space, Hardy space and BMO space. The continuity of curvelet transform in these spaces provides the basis for applications like solution of partial differential equations in these spaces.

The theorems presented for the aforementioned spaces provide valuable insights into the behaviour of the curvelet transform across different functional domains. These results offer bounds that enhance our understanding of the curvelet transform's applicability and effectiveness in diverse mathematical spaces.

The successful extension of the curvelet transform to these spaces opens avenues for exploring its applications in a broader range of mathematical and scientific disciplines. Future research may delve deeper into the implications and potential advancements stemming from this extended framework, paving the way for innovative applications and theoretical developments in the field.

Acknowledgements. The author extends sincere thanks to the referees for their valuable comments, which have significantly improved this paper.

References

[1] E. J. Candès and D. L. Donoho, *Continuous curvelet transform I. Resolution of the wavefront set*, Appl. Comput. Harmon. Anal. **19**(2) (2005), 162–197. `https://doi.org/10.1016/j.acha.2005.02.003`

[2] E. J. Candès and D. L. Donoho, *Continuous curvelet transform II. Discretization and frames*, Appl. Comput. Harmon. Anal. **19**(2) (2005), 198–222. `https://doi.org/10.1016/j.acha.2005.02.004`

[3] J-L. Starck, E. J. Candés and D. L. Donoho. *The curvelet transform for image denoising*, IEEE Trans. Image Process. **11**(6) (2002), 670–684. `https://doi.org/10.1109/TIP.2002.1014998`

[4] J-L. Starck, D. L. Donoho and E. J. Candés. *Astronomical image representation by the curvelet transform.* Astronomy & Astrophysics **398**(2) (2003), 785–800. `https://doi.org/10.1051/0004-6361:20021571`.

[5] J. L. Starck, F. Murtagh, E. J. Candés and D. L. Donoho. *Gray and color image contrast enhancement by the curvelet transform*, IEEE Trans. Image Process. **12**(6) (2003), 706–717. `https://doi.org/10.1109/TIP.2003.813140`

[6] M. Choi, R. Y. Kim, M. R. Nam and H. O. Kim. *Fusion of multispectral and panchromatic satellite images using the curvelet transform*, IEEE Geoscience and Remote Sensing Letters **2**(2) (2005), 136–140. `https://doi.org/10.1109/LGRS.2005.845313`

[7] F. Nencini, A. Garzelli, S. Baronti and L. Alparone *Remote sensing image fusion using the curvelet transform*, Inf. Fusion **8**(2) (2007), 143–156. `https://doi.org/10.1016/j.inffus.2006.02.001`

[8] S. E. Jero, P. Ramu and S. Ramakrishnan. *ECG steganography using curvelet transform*, Biomedical Signal Processing and Control **22** (2015), 161–169. `https://doi.org/10.1016/j.bspc.2015.07.004`

[9] L. Dong, Q. Yang, H. Wu and H. Xiao, *High quality multi-spectral and panchromatic image fusion technologies based on curvelet transform*, Neurocomputing **159** (2015), 268–274. `https://doi.org/10.1016/j.neucom.2015.01.050`

[10] B, Singh and M. K. Sharma, *Watermarking technique for document images using discrete curvelet transform and discrete cosine transform*, Multimedia Tools and Applications (2024), 1–25. `https://doi.org/10.1007/s11042-024-18770-3`

[11] S. M. Rajendran and R. Roopkumar, *Curvelet transform on tempered Boehmians*, Integration **3**(4) (2012), 365–380.

[12] S. M. Rajendran and R. Rajakumar, *Curvelet transform for Boehmians*, Arab. J. Math. Sci. **20**(2) (2014), 264–279. https://doi.org/10.1016/j.ajmsc.2013.10.001

[13] S. M. Rajendran and R. Roopkumar, *Curvelet transform on tempered distributions*, Asian-Eur. J. Math. **8**(2) (2015), Article ID 1550031. https://doi.org/10.1142/S179355711550031X

[14] S. M. Rajendran and R. Roopkumar, *Curvelet transform on periodic distributions*, Integral Transforms Spec. Funct. **25**(1) (2014), 874–887. https://doi.org/10.1080/10652469.2014.938237

[15] L. Akila and R. Roopkumar, *Quaternionic curvelet transform*, Optik **131** (2017), 255–266. https://doi.org/10.1016/j.ijleo.2016.11.011

[16] A. Y. Tantary and F. A. Shah, *An intertwining of curvelet and linear canonical transforms*, J. Math. (2020), 1–14, Article ID 8814998. https://doi.org/10.1155/2020/8814998

[17] A. A. Khan and K. Ravikumar, *Linear canonical curvelet transform and the associated Heisenberg-type inequalities*, Int. J. Geom. Methods Mod. Phys. **18**(7) (2021), Article ID 2150100. https://doi.org/10.1142/S0219887821501000

[18] A. A. Khan, *Some inequalities for linear canonical curvelet transform*, International Journal of Nonlinear Analysis and Applications **14**(1) (2023), 2361–2372.

[19] A. A. Khan, *Quaternion linear canonical curvelet transform*, Palest. J. Math. **12**(1) (2023), 645–660.

[20] V. Catană and M-G. Scumpu, *Localization operators and wavelet multipliers involving two-dimensional linear canonical curvelet transform*, Pseudo-Differ. Oper. Appl. **14**(4) (2023), Article ID 53. https://doi.org/10.1007/s11868-023-00547-1

[21] J-L. Starck, Y. Moudden, P. Abrial and M. Nguyen, *Wavelets, ridgelets and curvelets on the sphere*, Astronomy & Astrophysics **446**(3) (2006), 1191–1204. https://doi.org/10.1051/0004-6361:20053246

[22] J. Y. Chan, B.Leistedt, T. D. Kitching and J. D. McEwen, *Second-generation curvelets on the sphere*, IEEE Trans. Signal Process **65**(1) (2016), 5–14. https://doi.org/10.1109/TSP.2016.2600506

[23] D. Sharma, K. Goyal and R. K. Singla, *A curvelet method for numerical solution of partial differential equations*, Appl. Numer. Math. **148** (2020), 28–44. https://doi.org/10.1016/j.apnum.2019.08.029

[24] D. Lhamu and S. K. Singh, *The quaternion Fourier and wavelet transforms on spaces of functions and distributions*, Res. Math. Sci. **7**(2020), 1–15. https://doi.org/10.1007/s40687-020-00209-4

[25] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1966), 351–407. https://doi.org/10.1007/BF02590963

[26] R. S. Pathak, *The Wavelet Transform*, Atlantis Press/World Scientific, France, 2009.

[27] S. K. Singh, and B. Kalita, *The S-transform on Sobolev spaces*, Journal of Analysis & Number Theory **4**(2) (2016), 125–131. https://doi:10.18576/jant/040207

[28] R. S. Pathak, *Wavelets in a generalized Sobolev space*, Comput. Math. Appl. **49** (2005), 823–839. https://doi.org/10.1016/j.camwa.2004.07.021

[29] N. M. Chuong and D. V. Duong, *Boundedness of the wavelet Integral operator on weighted function spaces*, Russ. J. Math. Phys. **20**(2) (2013), 268–275. https://doi.org/10.1134/S1061920813030023

[30] S. Kumar, Singh and B. Kalita, *The S-transform on Hardy spaces and its duals*, International Journal of Analysis and Applications **7**(2) (2015), 171–178.

[31] D. Lhamu and S. K. Singh, *Besov norms of the continuous wavelet transform in variable Lebesgue space*, J. Pseudo-Differ. Oper. Appl. **11**(4) (2020), 1537–1548. https://doi.org/10.1007/s11868-020-00361-z

[32] A. Pathak and S. Pandey, *Besov type spaces associated with Lebedev-Skalskaya wavelet transform*, Math. Methods Appl. Sci. **46**(14) (2023), 15626–15640. https://doi.org/10.1002/mma.9416
[33] H. Q. Bui, *Weighted Young's inequality and convolution theorems on weighted Besov spaces*, Math. Nachr. **170**(1) (1994), 25–37. https://doi.org/10.1002/mana.19941700104
[34] B. Kalita and S. K. Singh, *The fractional S-transform on BMO and Hardy spaces,* Investigations in Mathematics Learning **5**(1) (2015), 113–122.
[35] R. S. Pathak, *A Course in Distribution Theory and Applications*, Narosa Publishing House, New Delhi, India, 2001.
[36] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer-Verlag, Berlin Heidelberg, New York, 1983. https://doi.org/10.1007/978-3-642-96750-4
[37] A. Kumar, S. K. Singh and S. K. Singh, *A note on Moritoh transforms*, Creat. Math. Inform. **33**(2) (2024), 185–201. https://doi.org/10.37193/CMI.2024.02.05

[1]Department of Mathematics,
Jawaharlal Nehru College,
Pasighat-791102, Arunachal Pradesh, India
Email address: dremalhamu114@gmail.com
ORCID iD: https://orcid.org/0000-0001-9915-084X

[2]Department of Mathematics,
Babasaheb Bhimrao Ambedkar University
Lucknow-226025, India
Email address: sks_math@yahoo.com
ORCID iD: https://orcid.org/0000-0001-9120-5279

[3]Unitedworld Institute of Technology,
Karnavati University, Gandhinagar, Gujarat, India
Email address: awniyakumar554@gmail.com
ORCID iD: https://orcid.org/0000-0002-5117-4575

[4]Department of Mathematics,
North Eastern Regional Institute of Science and Technology
Nirjuli, Arunachal Pradesh-791109, India
Email address: drcppandey@gmail.com
ORCID iD: https://orcid.org/0000-0002-4835-3441

[*]Corresponding Author