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FROM MONOTONICITY OF A CLASS OF BESSEL DISTRIBUTION FUNCTIONS TO NEW BOUNDS FOR RELATED FUNCTIONALS

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Dedicated to Academician Gradimir V. Milovanović on the occasion of his 75th birthday

Abstract. In this note we prove a monotonicity result with respect to the parameter ν of the cumulative distribution function for the McKay I_ν Bessel distribution and uniform upper bounds for a bilinear expression containing modified Bessel function of the first kind I_ν. Certain implications, among others with the Horn function Φ_2 and for the Gaussian hypergeometric function close the exposition.

1. Introduction

The first results about probability distributions involving Bessel functions can be traced back to the early work of McKay [4] in 1932 who considered two classes of continuous distributions called Bessel distributions.

For reader's convenience, let us recall the definition of the modified Bessel function of the first kind I_ν of the order ν [6, p. 249, Eq. **10.25.2**]

$$I_\nu(z) = \sum_{k\geq 0} \frac{1}{\Gamma(\nu+k+1)\,k!}\left(\frac{z}{2}\right)^{2k+\nu}, \quad \mathrm{Re}\,(\nu) > -1,\; z \in \mathbb{C}\,.$$

On a standard probability space $(\Omega, \mathcal{F}, \mathsf{P})$ we consider a random variable (r.v.) ξ which follows a distribution which is a McNolty's variant of the McKay I_ν Bessel law.

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This means that ξ is a nonnegative r.v. with the following probability density function (density in short) [5, p. 496, Eq. (13)]

$$f_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac12\right)}\,\mathrm{e}^{-bx}x^\nu I_\nu(ax), \quad x \geq 0.$$

The density f_I depends on three real parameters a, b, ν, where $\nu > -1/2$ and $b > a > 0$.

The corresponding distribution function of ξ is as follows:

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac12\right)}\int_0^x \mathrm{e}^{-bt}t^\nu I_\nu(at)\,\mathrm{d}t, \quad x \geq 0. \tag{1.1}$$

In the sequel we use any of the notations $\xi \sim \mathrm{McKayI}(a,b,\nu)$, $\xi \sim f_I(x;a,b;\nu)$, $\xi \sim F_I(x;a,b;\nu)$.

Recently, Jankov Maširević and Pogány [2] reported on the expression of the distribution function F_I, see (1.1), in terms of the Horn confluent hypergeometric function [8, p. 25, Eq. (17)]

$$\Phi_2(b,b';c;x,y) = \sum_{m,n\geq 0}\frac{(b)_m(b')_n}{(c)_{m+n}}\cdot\frac{x^m\,y^n}{m!\,n!}, \quad \max\{|x|,|y|\} < +\infty\,.$$

So, for all $\nu > -1/2$, $b > a > 0$ and for all $x \geq 0$ this result is [2, p. 149, Theorem 3]

$$F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}x^{2\nu+1}}{\Gamma(2\nu+2)}\,\Phi_2\Big(\nu+\tfrac12, \nu+\tfrac12; 2\nu+2; (a-b)x, -(a+b)x\Big). \tag{1.2}$$

It is natural to ask about important characteristics of the Bessel distribution (1.1). While, as we know, the positive integer order moments play a great role in Probability and Statistics, here we can find an explicit expression for the moment m_s of order s, for $s \in \mathbb{C}$. Thus,

$$m_s = \mathsf{E}\,[\xi^s] = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac12\right)}\int_0^{+\infty}\mathrm{e}^{-bx}x^{\nu+s}\,I_\nu(ax)\mathrm{d}x\,.$$

We see that up to a constant factor, m_s is the Laplace transform of the input function $x^{\nu+s}\,I_\nu(ax)$. Applying a result [7, p. 313, Eq. 3.15.1.2.] for complex valued μ, ν, p, α, we obtain

$$\int_0^{+\infty}\mathrm{e}^{-px}x^\mu\,I_\nu(\alpha x)\,\mathrm{d}x = \frac{\alpha^\nu\,\Gamma(\mu+\nu+1)}{2^\nu p^{\mu+\nu+1}\Gamma(\nu+1)}\,{}_2F_1\left[\begin{matrix}\frac12(\mu+\nu+1), \frac12(\mu+\nu)+1\\ \nu+1\end{matrix}\,\Bigg|\,\frac{\alpha^2}{p^2}\right].$$

This formula is valid for all μ, ν, p, α, provided $\mathrm{Re}\,(\mu+\nu) > -1$, $\mathrm{Re}\,(p) > |\mathrm{Re}\,(\alpha)|$. Now together with the Legendre duplication formula for the gamma function, we conclude that for all $\mathrm{Re}\,(s) > -2\nu-1$ there holds

$$m_s = \frac{(b^2-a^2)^{\nu+1/2}\Gamma(2\nu+s+1)}{\Gamma(2\nu+1)\,b^{2\nu+s+1}}\,{}_2F_1\left[\begin{matrix}\nu+\frac12(s+1), \nu+\frac{s}{2}+1\\ \nu+1\end{matrix}\,\Bigg|\,\frac{a^2}{b^2}\right]. \tag{1.3}$$

One of our goals is to prove the monotonicity of the distribution function F_I with respect to ν. This result implies an attractive uniform bound upon a bilinear function built with modified Bessel functions of the first kind which orders are contiguous with the input parameter ν occuring in $\mathrm{McKayI}(a, b, \nu)$. We end the presentation with Turán type inequalities for Gauss hypergeometric function derived by certain moment inequalities.

2. Main Results

Sun et al. in [9] proved the next integral inequality. Let X and Y be positive independent random variables (r.v.), where X is absolutely continuous with density function f_X, while Y is arbitrary, either continuous or discrete; no density at the latter case. Let further, $g : (0, +\infty) \to (0, +\infty)$ be a nondecreasing positive function. Then, provided $F_Y(0) < 1$ and the integrals exist, compare [9, p. 1169, Lemma 1] (actually, this inequality is a consequence of the fact that if X and Y are positive r.v.s, $X + Y$ is *stochastically larger* than X), the following inequality holds true for each $x > 0$:

$$\int_x^{+\infty} g(t)\, f_{X+Y}(t)\,\mathrm{d}t > \int_x^{+\infty} g(t)\, f_X(t)\,\mathrm{d}t. \tag{2.1}$$

With the help of this inequality we prove a strict monotonicity of the generalized distribution function (1.2) and two consequences of this monotone behaviour of F_I.

Theorem 2.1. *For all $\nu_1 > -\frac{1}{2}$, $\nu_2 > -\frac{1}{2}$ and $b > a > 0$ there holds*

$$F_I\left(x; a, b; \nu_1 + \nu_2 + \frac{1}{2}\right) < F_I(x; a, b; \nu_1), \quad x \geq 0\,. \tag{2.2}$$

Moreover, for the same parameter range, the following inequality holds true

$$\frac{I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)}{I_{\nu_1}(ax) \mp I_{\nu_1+1}(ax)} < \frac{\Gamma(\nu_1+\nu_2+2)}{\Gamma(\nu_1+\frac{3}{2})} \left(\frac{2a}{(b^2-a^2)x}\right)^{\nu_2+1/2}.$$

Finally, for all $x > 0$ we have

$$x^{2\nu_2+1}\,\frac{\Phi_2^{[\nu_1+\nu_2+1]}(x)}{\Phi_2^{[\nu_1+\frac{1}{2}]}(x)} < \frac{\Gamma(2\nu_1+2\nu_2+3)}{(b^2-a^2)^{\nu_2+\frac{1}{2}}\,\Gamma(2\nu_1+2)}\,, \tag{2.3}$$

where we have used the quantity

$$\Phi_2^{[\eta]}(x) = \Phi_2\Big(\eta, \eta; 2\eta+1; (a-b)x, -(a+b)x\Big)\,.$$

Proof. The moment generating function of the r.v. $\xi \sim \mathrm{McKayI}(a, b; \nu)$ equals

$$\begin{aligned} M_\xi(s) = \mathsf{E}\,[\mathrm{e}^{s\xi}] &= \int_0^{+\infty} \mathrm{e}^{sx} f_I(x; a, b; \nu)\,\mathrm{d}x \\ &= \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{+\infty} \mathrm{e}^{-(b-s)x} x^\nu I_\nu(ax)\,\mathrm{d}x \\ &= \left(1 - \frac{s(2b-s)}{b^2-a^2}\right)^{-\nu-\frac{1}{2}}, \quad s \in \mathbb{R}, |b-s| > a, \end{aligned}$$

see again the Laplace transform [7, p. 313, Eq. **3.15.1.3**]. Clearly, the moment generating function M_ξ exists if we find a proper interval of zero, say $(-s_l, s_r)$, where $s_l > 0$, $s_r > 0$, such that for all $s \in (-s_l, s_r)$ it is $M_\xi(s) < +\infty$.

Now, letting $X \sim f_I(x; a, b; \nu_1)$ and $Y \sim f_I(x; a, b; \nu_2)$ be two independent r.v.s. Hence, the moment generating function of the r.v. $X + Y$ becomes

$$M_{X+Y}(s) = M_X(s)M_Y(s) = \Big(1 - \frac{s(2b-s)}{b^2-a^2}\Big)^{-\nu_1-\nu_2-1}, \quad |b-s| > a,$$

which implies that r.v. $X + Y \sim f_I(x; a, b; \nu_1 + \nu_2 + 1/2)$. Rewriting the inequality (2.1) in the form

$$\int_0^x g(t) f_{X+Y}(t)\, \mathrm{d}t < \int_0^x g(t) f_X(t)\, \mathrm{d}t, \tag{2.4}$$

and taking $g(x) = 1$ for all $x > 0$ we conclude

$$\int_0^x f_I(t; a, b; \nu_1 + \nu_2 + 1/2)\, \mathrm{d}t < \int_0^x f_I(t; a, b; \nu_1)\, \mathrm{d}t,$$

which is equivalent to the first stated result.

As to the second inequality, observe that from (2.4) there follows

$$\frac{(b^2-a^2)^{\nu_2+1/2}\Gamma(\nu_1+1/2)}{(2a)^{\nu_2+1/2}\Gamma(\nu_1+\nu_2+1)} \int_0^x g(t) \mathrm{e}^{-bt}\, t^{\nu_1+\nu_2+1/2} I_{\nu_1+\nu_2+1/2}(at)\, \mathrm{d}t$$
$$< \int_0^x g(t)\mathrm{e}^{-bt}\, t^{\nu_1} I_{\nu_1}(at)\, \mathrm{d}t,$$

and choosing the positive non-decreasing function $g(x) = \mathrm{e}^{(b\pm a)x}$ we conclude

$$\frac{(b^2-a^2)^{\nu_2+1/2}\Gamma(\nu_1+1/2)}{(2a)^{\nu_2+1/2}\Gamma(\nu_1+\nu_2+1)} \int_0^x \mathrm{e}^{\pm at} t^{\nu_1+\nu_2+1/2} I_{\nu_1+\nu_2+1/2}(at)\, \mathrm{d}t$$
$$< \int_0^x \mathrm{e}^{\pm at} t^{\nu_1} I_{\nu_1}(at)\, \mathrm{d}t.$$

By virtue of [6, p. 259, Eq. **10.43.7**]

$$\int_0^x \mathrm{e}^{\pm t} t^\nu I_\nu(t)\, \mathrm{d}t = \frac{\mathrm{e}^{\pm x} x^{\nu+1}}{2\nu+1}(I_\nu(x) \mp I_{\nu+1}(x)), \quad \mathrm{Re}\,(\nu) > -1/2\,,$$

and applying the substitution $at \mapsto u$ it follows that

$$\frac{\Gamma(\nu_1+3/2)}{\Gamma(\nu_1+\nu_2+2)} \left(\frac{(b^2-a^2)x}{2a}\right)^{\nu_2+1/2} \big(I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)\big)$$
$$< I_{\nu_1}(ax) \mp I_{\nu_1+1}(ax).$$

The rest is obvious.

Finally, inserting the Horn function representation (1.2) of the distribution function F_I into (2.2), we arrive at (2.3). □

To close the exposition we apply the well-known Turán inequality for the raw moments $m_s = \mathsf{E}[\xi^s]$, $s > 0$, of non-negative random variables [3, p. 28, Eqs. (1.4.6)] $m_{s+r}^2 \le m_s\, m_{s+2r}$, $s, r > 0$, which is an immediate consequence of the CBS inequality. Firstly, we define the Turánian ratio for the moment m_s with respect to the increment $r > 0$ as

$$\mathfrak{T}_r(m_s) := \frac{m_{s+r}^2}{m_s \cdot m_{s+2r}}\,,$$

which one transforms the previous inequality into

$$\mathfrak{T}_r(m_s) \le 1. \tag{2.5}$$

To establish the bounding inequality for the Gaussian hypergeometric function ${}_2F_1$, we insert into (2.5) the expression (1.3).

Proposition 2.1. *For all $b > a > 0$, $\nu > -1/2$ and $s, r > 0$ we have*

$$\frac{\Big\{{}_2F_1[s+r]\Big\}^2}{{}_2F_1[s] \cdot {}_2F_1[s+2r]} \le \frac{\Gamma(2\nu+s+1)\Gamma(2\nu+s+2r+1)}{\Gamma^2(2\nu+s+r+1)}\,,$$

where the abbreviation

$${}_2F_1[s] := {}_2F_1\left[\begin{matrix} \nu+\frac{1}{2}(s+1),\, \nu+\frac{s}{2}+1 \\ \nu+1 \end{matrix}\,\Bigg|\,\frac{a^2}{b^2}\right].$$

However, to derive another bound for ${}_2F_1[s]$ we take into account the integral moment inequality [1, p. 143, Theorem 192]

$$\mathfrak{M}_r(h,p) < \mathfrak{M}_s(h,p), \quad 0 < r < s, \tag{2.6}$$

where

$$\mathfrak{M}_r(h,p) = \int_\alpha^\beta h^r(t)\, p(t)\, \mathrm{d}t\,,$$

for a suitable, integrable non-negative input function h, the integration interval (α, β) is either finite or infinite, and the non-negative weight function p has integral $\int_\alpha^\beta p(t)\, \mathrm{d}t = 1$. In our case the shorthand $\mathfrak{M}_s(x^s, f_I) = (m_s)^{1/s}$ is adopted to the McKayI(a, b, ν) distribution, $(\alpha, \beta) = \mathbb{R}_+$. Inserting m_s from (1.3) into moment inequality (2.6) we obtain the following result.

Proposition 2.2. *For all $b > a > 0$, $\nu > -1/2$ and $s > r > 0$ there holds true*

$$\frac{\Big\{{}_2F_1[r]\Big\}^{1/r}}{\Big\{{}_2F_1[s]\Big\}^{1/s}} \le \left(1 - \frac{a^2}{b^2}\right)^{(\nu+1/2)(1/s-1/r)} \frac{(2\nu+1)_s^{1/s}}{(2\nu+1)_r^{1/r}}\,,$$

where the hypergeometric terms remain the same as in the previous proposition.

Remark 2.1. According to Lukacs [3, p. 393, a)] for all $0 < r \le s$ there holds the moment inequality $m_{s+r}^2 \le m_{2s} \cdot m_{2r}$. We notice that this inequality is implied by virtue of the CBS inequality, using re-scaling of the integrand in m_{s+r}. However, to imply another bound for ${}_2F_1[s]$ via this inequality and/or the Lyapunov inequality we leave to the interested reader.

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