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PROXIMAL POINT ALGORITHM FOR A COUNTABLE FAMILY OF WEIGHTED RESOLVENT AVERAGES

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Abstract. In this paper, we introduce a composite iterative algorithm for finding a common zero point of a countable family of weighted resolvent average of finite family of monotone operators in Hilbert spaces. We prove that the sequence generated by the iterative algorithm converges strongly to a common zero point. Finally, we apply our results to split common zero point problem.

1. Introduction

Monotone operator theory plays a central role in many areas of nonlinear functional analysis, nonlinear analysis and modern optimization. The literature on monotone operator theory is quiet rich. During the last five decades, monotone operators and their applications in so many branches of mathematics, have received a lot of attention (see [3] and [6] and the references cited therein). Monotone operator theory was first introduced by George Minty to aid in the abstract study of electrical networks [9], then in the setting of partial differential equations by Felix Browder and his school [4]. Maximal monotone operators rapidly found uses for subgradients, optimization, variational inequalities, algorithms, mathematical economics, and much more.

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. The notation $T : H \to H$ means that the operator (also called mapping) T maps every point in H to a point $T(x)$ in H. The notation $A : H \multimap H$ means that A is a set-valued operator (mapping) from H to H, i.e., A maps every point $x \in H$ to a set $A(x) \subseteq H$. Then A is characterized by its *graph*

$$\operatorname{gra} A = \{(x, u) \in H \times H : u \in A(x)\}.$$

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The *domain* and the *range* of A are

$$\operatorname{dom} A = \{x \in H : A(x) \neq \emptyset\} \text{ and } \operatorname{ran} A = A(H),$$

respectively. A set-valued operator $A : H \multimap H$ is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \text{ for all } (x, u), (y, v) \in \operatorname{gra} A.$$

A monotone operator A is called *maximal monotone* if there exists no monotone operator B such that $\operatorname{gra} A$ is a proper subset of $\operatorname{gra} B$. The *resolvent* of A is the mapping $J_{\lambda A} = (\lambda A + \mathrm{Id})^{-1}$ for all $\lambda > 0$.

Let us consider the zero point problem for monotone operator A on a real Hilbert space H, i.e., finding a point $x \in \operatorname{dom} A$ such that $0 \in A(x)$. It was first introduced by Martinet [8] in 1970. Rockafellar [12] extended the proximal point algorithm of Martinet by generating a sequence $\{x_n\}$ such that

$$x_{n+1} = J_{s_n A} x_n + e_n, \ n \in \mathbb{N},$$

for arbitrary point $x_0 \in H$, where $\{e_n\}$ is a sequence of errors and $\{s_n\} \subseteq (0, \infty)$. The sequence $\{x_n\}$ is known to converge weakly to a point such that $0 \in A(x)$, if $\liminf_{n\to\infty} s_n > 0$ and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, see [12], but fails in general to converge strongly [5]. Recently, Xu [13] investigated a modified version of the initial proximal point algorithm studied by Rockafellar with $x_0 \in H$ chosen arbitrary,

$$x_{n+1} = \beta_n x_0 + (1 - \beta_n) J_{s_n A} x_n + e_n, \ n \in \mathbb{N},$$

where $\{e_n\}$ is the error sequence. For $\{e_n\}$ summable, it was proved that [13] $\{x_n\}$ is strongly convergent if $s_n \to \infty$ and $\beta_n \subseteq (0, 1)$ with $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$.

Recently, Marino and Rugiano [7] introduced the following iteration process: For chosen $x_0 \in H$ construct a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) T(\alpha_n x_n + (1 - \alpha_n) x_{n+1}), \ n \in \mathbb{N},$$

where $\alpha_n, \beta_n \in (0, 1)$ and f is a k-contraction mapping on H. They showed that this process converges strongly to unique fixed point of the contraction $P_{\mathrm{Fix}(T)}$.

In 2014, Mongkolkeha, Cho and Kumam [10], defined the following iterative scheme, by $x_0 \in H$ and

$$\begin{cases} z_n = (1 - \gamma_n) x_n + \gamma_n U x_n, \\ y_n = (1 - \beta_n) T x_n + \beta_n S z_n, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n y_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0, 1)$. They show that if $\liminf (1 - \alpha_n)\alpha_n > 0$, $\liminf (1 - \beta_n)\beta_n > 0$ and $\sum_{n\in\mathbb{N}} \gamma_n < \infty$ then $\{x_n\}$ converges weakly to $\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$.

In this paper, we introduce a composite iteration for a countable family of weighted resolvent average of a finite family of monotone operators as follows:

$$\begin{cases} y_n = \beta_n x_n + \sum\limits_{i=1}^{n} (\beta_{i-1} - \beta_i) J_{R(\boldsymbol{A_i}, \boldsymbol{\lambda_i})} x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (\mathrm{Id} - \mu \alpha_n B) y_n, \end{cases} \tag{1.1}$$

where B is a l-Lipschitz and η-strongly monotone operator and f is a k-Lipschitz mapping on H. We prove, under certain appropriate assumption on sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subseteq (0,1]$ and $\beta_0 = 1$, that $\{x_n\}$ converges strongly to a zero point of the resolvent average of the family.

2. Preliminaries

The operator $T : H \to H$ is called *l-Lipschitz continuous* if there exists a constant $l > 0$ such that

$$\|Tx - Ty\| \leq l\|x - y\|, \text{ for all } x, y \in H.$$

The operator $T : H \to H$ is said to be *nonexpansive* if it is Lipschitz continuous with constant 1, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in H.$$

The operator $T : H \to H$ is called *firmly nonexpansive* if

$$\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2, \text{ for all } x, y \in H.$$

Clearly, every firmly nonexpansive mapping is nonexpansive and the converse generally is not true (see [2, Example 4.17]).

A point $x \in H$ is said to be a *fixed point* of an operator $T : H \to H$, if $Tx = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$, i.e.,

$$\mathrm{Fix}(T) = \{x \in H : Tx = x\}.$$

Let K be a closed convex subset of H. Then for every point $x \in H$, there exists a unique *nearest point* in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \leq \|x - y\|, \text{ for all } y \in K.$$

The operator P_K is called *metric projection* of H onto K. It is well known that $P_K(x)$ is nonexpansive. The metric projection $P_K(x)$ is characterized by $P_K(x) \in K$ and

$$\langle u - P_K(x), x - P_K(x)\rangle \leq 0, \text{ for all } u \in K.$$

A mapping $f : H \to H$ is said to be *k-contraction* on H if there exists a constant $k \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \text{ for all } x, y \in H.$$

A sequence of points $\{x_n\}$ in a Hilbert space H is said to *converge weakly* to a point x in H if

$$\langle x_n, y\rangle \to \langle x, y\rangle, \text{ for all } y \in H;$$

in symbols, $x_n \rightharpoonup x$.

An operator $B : H \to H$ is called *η-strongly monotone* on H if there exists a constant $\eta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \eta\|x - y\|^2, \text{ for all } x, y \in H.$$

These basic definitions are also have presented in various parts of the book [2]. Now, we recall some properties of monotone operators.

Proposition 2.1. [2, Proposition 23.7] *Suppose that $\lambda > 0$ and $A : H \multimap H$ is a set-valued mapping. Then*

(i) *if A is monotone, then $J_{\lambda A}$ is single-valued and firmly nonexpansive;*
(ii) *if A is maximal monotone, then $J_{\lambda A}$ is single-valued and firmly nonexpansive and its domain is all of H;*
(iii) $0 \in A(x)$ *if and only if* $x \in \operatorname{Fix}(J_{\lambda A})$*. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto $Z = A^{-1}(0)$ is well defined whenever $Z \neq \emptyset$.*

We recall (see [1]) the definition of the resolvent average. To this end, we assume that $m \in \mathbb{N}$ and $I = \{1, 2, \dots, m\}$. For every $i \in I$, let $A_i : H \multimap H$ be a set-valued mapping, let $\lambda_i > 0$ be such that $\sum_{i \in I} \lambda_i = 1$. We set $\boldsymbol{A} = (A_1, \dots, A_m)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$.

Definition 2.1. [1, Definition 1.4] The $\boldsymbol{\lambda}$-*weighted resolvent average* of $\boldsymbol{A}$ is defined by

$$R(\boldsymbol{A}, \boldsymbol{\lambda}) = \Big(\sum_{i \in I} \lambda_i (A_i + \mathrm{Id})^{-1} \Big)^{-1} - \mathrm{Id}\,. \tag{2.1}$$

The equation (2.1) is equivalent to the following equation (see [1]):

$$J_{R(\boldsymbol{A}, \boldsymbol{\lambda})} = \sum_{i \in I} \lambda_i J_{A_i}.$$

Proposition 2.2. [1, Theorem 2.5] *Suppose that for each $i \in I$, $A_i : H \multimap H$ is monotone and $x \in H$. If $\bigcap_{i \in I} A_i^{-1}(\{0\}) \neq \emptyset$, then*

$$(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\}) = \bigcap_{i \in I} A_i^{-1}(\{0\}).$$

Proposition 2.3. [1, Theorem 2.2] *Suppose that for each $i \in I$, $A_i : H \multimap H$ is a set-valued mapping. Then*

$$\Big(R(\boldsymbol{A}, \boldsymbol{\lambda})\Big)^{-1} = R(\boldsymbol{A}^{-1}, \boldsymbol{\lambda}).$$

Lemma 2.1. [1, Theorem 2.11] *Let $A_i : H \multimap H$ be monotone for each $i \in I$. Then $R(\boldsymbol{A}, \boldsymbol{\lambda})$ is monotone and*

$$\operatorname{dom} J_{R(\boldsymbol{A}, \boldsymbol{\lambda})} = \bigcap_{i \in I} \operatorname{dom} J_{A_i}.$$

3. Main Results

In this section, we introduce a new proximal point algorithm for a countable family of weighted resolvent averages of finite family of monotone operators and its convergence analysis is given. First we present some useful lemmas.

Lemma 3.1. *Let $B : H \to H$ be an l-Lipschitz and η-strongly monotone operator. Let $0 < \mu < 2\eta/l^2$ and $\tau = \mu(\eta - \mu l^2/2)$. Then $I - \mu B$ is $(1-\tau)$-contraction.*

Proof. By assumption, we have

$$\begin{aligned}\|(I-\mu B)x-(I-\mu B)y\|^2 &= \langle (I-\mu B)x-(I-\mu B)y, (I-\mu B)x-(I-\mu B)y\rangle \\ &= \|x-y\|^2 - 2\mu\,\langle Bx-By, x-y\rangle + \mu^2\|Bx-By\|^2 \\ &\le \|x-y\|^2 - 2\mu\eta\|x-y\|^2 + \mu^2 l^2\|x-y\|^2 \\ &= (1-2\mu\eta+\mu^2 l^2)\|x-y\|^2 \\ &\le (1-2\mu(\eta-\frac{1}{2}\mu l^2)+\mu^2(\eta-\frac{1}{2}\mu l^2)^2)\|x-y\|^2 \\ &= (1-\mu(\eta-\frac{1}{2}\mu l^2))^2\|x-y\|^2.\end{aligned}$$

Therefore, $\|(I-\mu B)(x-y)\| \le (1-\tau)\|x-y\|$. □

Remark 3.1. It should be noticed that the contraction constant $1-\tau$ in the above lemma is sharp. For example, let $B = \mathrm{Id}$. Then $l = \eta = 1$ and $0 < \mu < 2$. On the contrary suppose $1-\tau$ is not sharp. Then there exists $\varepsilon > 0$ (small enough) such that $A := I - \mu B$ is $(1-\tau-\varepsilon)$-contraction. Let $\mu = \sqrt{\varepsilon}$. Then A is $(1-\sqrt{\varepsilon})$-contraction. On the other hand,

$$1-\tau-\varepsilon = 1-\sqrt{\varepsilon}(1-\sqrt{\varepsilon}/2)-\varepsilon = 1-\sqrt{\varepsilon}-\varepsilon/2 < 1-\sqrt{\varepsilon}.$$

Lemma 3.2. *For each $n \in \mathbb{N}$, let $\{A_{n,j} : H \multimap H\}_{j\in I_n}$ be a finite family of monotone operators, $\boldsymbol{A_n} = (A_{n,1}, \dots, A_{n,m_n})$ with $\bigcap_{n\in\mathbb{N}}(R(\boldsymbol{A_n}, \boldsymbol{\lambda_n}))^{-1}(\{0\}) \neq \emptyset$, where $I_n = \{1, 2, \dots, m_n\}$, $m_n \in \mathbb{N}$, $\lambda_{n,j} > 0$ and $\boldsymbol{\lambda_n} = (\lambda_{n,1}, \dots, \lambda_{n,m_n})$ with $\sum_{j\in I_n}\lambda_{n,j} = 1$. Let $B : H \to H$ be an l-Lipschitz and η-strongly monotone operator. Assume that f is a k-Lipschitz mapping on H. Let $0 < \mu < 2\eta/l^2$, $0 < \gamma < \tau/k$ with $\tau = \mu(2\eta - \mu l^2)$ and $\{\beta_n\}$ be a strictly decreasing sequence on $(0, 1]$. Let $\{x_n\}$ be the sequence generated by (1.1). Then $\Big\{\|x_n - z\| : n \in \mathbb{N}\Big\}$ is bounded for each $z \in \bigcap_{n\in\mathbb{N}}(R(\boldsymbol{A_n}, \boldsymbol{\lambda_n}))^{-1}(\{0\})$. Consequently, $\{x_n\}$ and $\{\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - x_n\| : n \in \mathbb{N}\}_{i\in\mathbb{N}}$ are bounded.*

Proof. Let $z \in \bigcap_{n\in\mathbb{N}}(R(\boldsymbol{A_n}, \boldsymbol{\lambda_n}))^{-1}(\{0\})$ be arbitrary and fixed. By using the Proposition 2.1 and the triangle inequality, we have

$$\begin{aligned}\|y_n - z\| &= \|\beta_n x_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - z\| \\ &= \|\beta_n(x_n - z) + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\Big(J_{R(\boldsymbol{A_i}\boldsymbol{\lambda_i})}x_n - z\Big)\| \\ &\le \beta_n\|x_n - z\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - z\| \\ &\le \beta_n\|x_n - z\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|x_n - z\| \qquad (3.1)\end{aligned}$$

$$\begin{aligned}&\le \beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\|\\&= \|x_n - z\|.\end{aligned}$$

By using the triangle inequality, Lemma 3.1 and (3.1), we obtain

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n\gamma f(x_n) + (\mathrm{Id} - \mu\alpha_n B)y_n - z\|\\&= \|\alpha_n\gamma f(x_n) + (\mathrm{Id} - \mu\alpha_n B)y_n - (\mathrm{Id} - \mu\alpha_n B)z - \mu\alpha_n Bz\|\\&= \|\alpha_n(\gamma f(x_n) - \mu Bz) + (\mathrm{Id} - \mu\alpha_n B)y_n - (\mathrm{Id} - \mu\alpha_n B)z)\|\\&\le \alpha_n(\|(\gamma f(x_n) - \gamma f(z)\| + \|\gamma f(z) - \mu Bz\|)\\&\quad + \|(\mathrm{Id} - \mu\alpha_n B)y_n - (\mathrm{Id} - \mu\alpha_n B)z)\|\\&\le k\gamma\alpha_n\|x_n - z\| + \alpha_n\|\gamma f(z) - \mu Bz\| + (1-\alpha_n\tau)\|y_n - z\|\\&\le k\gamma\alpha_n\|x_n - z\| + \alpha_n\|\gamma f(z) - \mu Bz\| + (1-\alpha_n\tau)\|x_n - z\|\\&= (1-\alpha_n(\tau - k\gamma))\|x_n - z\| + \alpha_n(\tau - k\gamma)\frac{\|\gamma f(z) - \mu Bz\|}{\tau - k\gamma}\\&\le \max\left\{\|x_n - z\|, \frac{\|\gamma f(z) - \mu Bz\|}{\tau - k\gamma}\right\}.\end{aligned}$$

This shows by induction that

$$\|x_{n+1} - z\| \le \max\left\{\|x_1 - z\|, \frac{\|\gamma f(z) - \mu Bz\|}{\tau - k\gamma}\right\}.$$

Therefore, $\{\|x_n - z\| : n \in \mathbb{N}\}$ is bounded for each $z \in \bigcap_{n\in\mathbb{N}}(R(\boldsymbol{A_n}, \boldsymbol{\lambda_n}))^{-1}(\{0\})$. Hence $\{x_n\}$ is bounded. Finally, it follows from nonexpansivity of resolvent of $R(\boldsymbol{A_i}, \boldsymbol{\lambda_i})$ for each $i \in \mathbb{N}$, that

$$\begin{aligned}\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - x_n\| &\le \|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - z\| + \|x_n - z\|\\&\le 2\|x_n - z\|.\end{aligned}$$

Therefore, $\{\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n - x_n\| : n \in \mathbb{N}\}_{i\in\mathbb{N}}$ is bounded. We conclude that $\{y_n\}$ and $\{J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n\}_{i\in\mathbb{N}}$ are bounded. □

Lemma 3.3. [13, Lemma 2.5] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \rho_n,\ n \ge 0,$$

where $\{\gamma_n\}$,$\{\beta_n\}$ and $\{\delta_n\}$ satisfy the conditions:

(i) $\gamma_n \subseteq [0,1], \sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$;
(iii) $\rho_n \ge 0$ *for all* $n \ge 0$ *with* $\sum_{n=0}^{\infty}\rho_n < \infty$;

then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.4. [2, Theorem 4.17] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $\mathrm{Id} - T$ is demi-closed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 3.5. [11] *There holds the following inequality:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \text{ for all } x, y \in H.$$

Theorem 3.1. *For each $n \in \mathbb{N}$, let $\{A_{n,j} : H \multimap H\}_{j\in I_n}$ be a finite family of monotone operators, $\boldsymbol{A_n} = (A_{n,1}, \dots, A_{n,m_n})$ with $Z := \bigcap_{n\in\mathbb{N}}(R(\boldsymbol{A_n}, \boldsymbol{\lambda_n}))^{-1}(\{0\}) \neq \emptyset$, where $I_n = \{1, 2, \dots, m_n\}$, $m_n \in \mathbb{N}$, $\lambda_{n,j} > 0$ and $\boldsymbol{\lambda_n} = (\lambda_{n,1}, \dots, \lambda_{n,m_n})$ with $\sum_{j\in I_n} \lambda_{n,j} = 1$. Let B be an l-Lipschitz and η-strongly monotone operator. Assume that f is a k-Lipschitz mapping on H. Let $0 < \mu < 2\eta/l^2$, $0 < \gamma < \tau/k$ with $\tau = \mu(2\eta - \mu l^2)$ and $\{\beta_n\}$ be a strictly decreasing sequence on $(0, 1]$. Let $\{x_n\}$ be the sequence generated by (1.1). Assume that the following conditions hold:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \Big|\beta_{n+1} - \beta_n\Big| < \infty$;

(iii) $\sum_{n=1}^{\infty} \Big|\alpha_{n+1} - \alpha_n\Big| < \infty$.

Then

(a) $\lim\limits_{n\to\infty} \|x_n - J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})} x_n\| = 0$ *for every* $i \in \mathbb{N}$;

(b) $\{x_n\}$ *converges strongly to* $z = P_Z(\gamma f + (\mathrm{Id} - \mu B))(z)$.

Proof. (a) It follows from Lemma 3.2 that $\{x_n\}$ is bounded. First, we claim that

$$\|x_{n+1} - x_n\| \to 0.$$

We observe that

$$\begin{cases} y_n = \beta_n x_n + \sum\limits_{i=1}^{n} (\beta_{i-1} - \beta_i) J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})} x_n, \\ y_{n-1} = \beta_{n-1} x_{n-1} + \sum\limits_{i=1}^{n-1} (\beta_{i-1} - \beta_i) J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})} x_{n-1}. \end{cases}$$

Then

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \beta_n \|x_n - x_{n-1}\| + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})} x_n - J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})} x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\beta_n - \beta_{n-1}| \|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})} x_{n-1}\| \\ &\le \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})} x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\beta_n - \beta_{n-1}| \|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})} x_{n-1}\|. \end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n \gamma f(x_n) + (\mathrm{Id} - \mu \alpha_n B) y_n - \alpha_{n-1} \gamma f(x_{n-1}) - (\mathrm{Id} - \mu \alpha_{n-1} B) y_{n-1} \\ &= \alpha_n \gamma (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) + (\mathrm{Id} - \mu \alpha_n B) y_n \\ &\quad - (\mathrm{Id} - \mu \alpha_n B) y_{n-1} - (\alpha_n - \alpha_{n-1}) \mu B y_{n-1}. \end{aligned}$$

By Lemma 3.1 and (3.2), we obtain

$$\|x_{n+1} - x_n\| \le \alpha_n \gamma k \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + (1 - \alpha_n \tau) \|y_n - y_{n-1}\|$$

$$
\begin{aligned}
&+\mu|\alpha_n-\alpha_{n-1}|\|By_{n-1}\|\\
&\leq \alpha_n\gamma k\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\gamma\|f(x_{n-1})\|+By_{n-1}\|)\\
&\quad+(1-\alpha_n\tau)\|x_n-x_{n-1}\|\\
&\quad+(1-\alpha_n\tau)|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})}x_{n-1}\|)\\
&\leq (1-\alpha_n(\tau-\gamma k))\|x_n-x_{n-1}\|\\
&\quad+(1-\alpha_n\tau)|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})}x_{n-1}\|)\\
&\quad+|\alpha_n-\alpha_{n-1}|(\gamma\|f(x_{n-1})\|+\mu\|By_{n-1}\|)\\
&\leq (1-\alpha_n(\tau-\gamma k))\|x_n-x_{n-1}\|\\
&\quad+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})}x_{n-1}\|)\\
&\quad+|\alpha_n-\alpha_{n-1}|(\gamma\|f(x_{n-1})\|+\mu\|By_{n-1}\|)\\
&\leq (1-\gamma_n)\|x_n-x_{n-1}\|+\rho_n,
\end{aligned}
$$

where $N=\sup\{\|x_n-x_{n-1}\|:n\in\mathbb{N}\}$, $\gamma_n=\alpha_n(\tau-\gamma k)$, and

$$\rho_n=|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|J_{R(\boldsymbol{A_n},\boldsymbol{\lambda_n})}x_{n-1}\|)+|\alpha_n-\alpha_{n-1}|(\gamma\|f(x_{n-1})\|+\mu\|By_{n-1}\|).$$

By conditions (i)-(iii), we have $\gamma_n\to 0$, $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\sum_{n=1}^{\infty}\rho_n<\infty$. Hence, it follows from Lemma 3.3 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

For each $z\in\bigcap_{i\in\mathbb{N}}(R(\boldsymbol{A_i},\boldsymbol{\lambda_i}))^{-1}(\{0\})$, since for each $i\in\mathbb{N}$, $J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}$ is nonexpansive, we have

$$
\begin{aligned}
\|x_n-z\|^2\geq \|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-z\|^2&=\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n+x_n-z\|^2\\
&=\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n\|^2+\|x_n-z\|^2+2\Big\langle J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n,x_n-z\Big\rangle,
\end{aligned}
$$

which implies that

$$\frac{1}{2}\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n\|^2\leq\Big\langle x_n-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n,x_n-z\Big\rangle.$$

Then

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n\|^2\\
&\leq\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\Big\langle x_n-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n,x_n-z\Big\rangle\\
&=\Big\langle(1-\beta_n)x_n-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n,x_n-z\Big\rangle\\
&=\langle x_n-y_n,x_n-z\rangle\\
&=\langle x_n-x_{n+1},x_n-z\rangle+\langle x_{n+1}-y_n,x_n-z\rangle\\
&=\langle x_n-x_{n+1},x_n-z\rangle+\alpha_n\langle\gamma f(x_n)-\mu By_n,x_n-z\rangle\\
&\leq\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|\gamma f(x_n)-\mu By_n\|\|x_n-z\|,
\end{aligned}
$$

hence, by condition (i) and (3.3), we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n-x_n\|^2=0.$$

Since $\{\beta_n\}$ is strictly decreasing, for each $i\in\mathbb{N}$, we get

$$\lim_{n\to\infty}\|x_n-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_n\|=0.$$

(b) First, we show that there exists a unique $z\in Z$ such that $z=P_Z(\gamma f+(\mathrm{Id}-B))(z)$. Since Z is nonempty, closed and convex, the projection P_Z is well defined. Since P_Z is nonexpansive and f and B are respectively k-Lipschitz and l-Lipschitz, for each $x,y\in H$, we get

$$\begin{aligned}&\|P_Z(\gamma f+(\mathrm{Id}-\mu B))(x)-P_Z(\gamma f+(\mathrm{Id}-\mu B))(y)\|\\ \le&\|(\gamma f+(\mathrm{Id}-\mu B))(x)-(\gamma f+(\mathrm{Id}-\mu B))(y)\|\\ \le&\|\gamma f(x)-\gamma f(y)\|+\|(\mathrm{Id}-\mu B)(x-y)\|\\ \le&\gamma k\|x-y\|+(1-\tau)\|x-y\|\\ \le&(1-(\tau-\gamma k))\|x-y\|.\end{aligned}$$

Banach's Contraction Principle guaranties that $P_Z(\gamma f+(\mathrm{Id}-\mu B))$ has a unique fixed point. That is, there exists a unique element $z\in Z$ such that

$$z=P_Z(\gamma f+(\mathrm{Id}-\mu B))(z). \tag{3.4}$$

Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(z)-\mu Bz,x_n-z\rangle\le 0,$$

where $z=P_Z(\gamma f+(\mathrm{Id}-\mu B))(z)$.

It follows from Lemma 3.2 that there exists a point $\omega\in H$ and subsequence $\{x_{n_\alpha}\}$ of $\{x_n\}$ such that $x_{n_\alpha}\rightharpoonup\omega$ and

$$\lim_{\alpha\to\infty}\langle\gamma f(z)-\mu Bz,x_{n_\alpha}-z\rangle=\limsup_{n\to\infty}\langle\gamma f(z)-\mu Bz,x_n-z\rangle, \tag{3.5}$$

We show that $\omega\in Z$. To see this, for every $i\in\mathbb{N}$, we have

$$\begin{aligned}\|x_{n_\alpha}-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}\omega\|&\le\|x_{n_\alpha}-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_{n_\alpha}\|+\|J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_{n_\alpha}-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}\omega\|\\&\le\|x_{n_\alpha}-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}x_{n_\alpha}\|+\|x_{n_\alpha}-\omega\|,\end{aligned}$$

which implies that

$$\limsup_{\alpha\to\infty}\|x_{n_\alpha}-J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}\omega\|\le\limsup_{\alpha\to\infty}\|x_{n_\alpha}-\omega\|.$$

By Lemma 3.4, we obtain $\omega=J_{R(\boldsymbol{A_i},\boldsymbol{\lambda_i})}\omega$ for every $i\in\mathbb{N}$. Hence $\omega\in Z$.

Since Z is closed and convex, by (3.4) and (3.5), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f-\mu B)z,x_n-z\rangle&=\lim_{\alpha\to\infty}\langle(\gamma f-\mu B)z,x_{n_\alpha}-z\rangle\\&=\langle(\gamma f+(\mathrm{Id}-\mu B))z-z,\omega-z\rangle\le 0.\end{aligned}$$

Finally, we show that $x_n \to P_Z(\gamma f + (\mathrm{Id} - \mu B))(z)$. By using Lemma 3.1 and Lemma 3.5, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n \gamma f(x_n) + (\mathrm{Id} - \mu\alpha_n B)y_n - z\|^2 \\
&= \|(\mathrm{Id} - \mu\alpha_n B)y_n - (\mathrm{Id} - \mu\alpha_n B)z + \alpha_n(\gamma f(x_n) - Bz)\|^2 \\
&\le \|(\mathrm{Id} - \mu\alpha_n B)y_n - (\mathrm{Id} - \mu\alpha_n B)z\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu Bz, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n\tau)^2\|(y_n - z\|^2 + 2\alpha_n\gamma \langle f(x_n) - f(z), x_{n+1} - z\rangle \\
&\quad + 2\alpha_n \langle \gamma f(z) - \mu Bz, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n\tau)^2\|(x_n - z\|^2 + \alpha_n k\gamma(\|(x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n \langle \gamma f(z) - \mu Bz, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{(1 - \alpha_n\tau)^2 + \alpha_n k\gamma}{1 - \alpha_n\gamma k}\|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma k} \langle \gamma f(z) - \mu Bz, x_{n+1} - z\rangle \\
&\le \left(1 - \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k}\right)\|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k}\left(\frac{1}{\tau - \gamma k} \langle \gamma f(z) - \mu Bz, x_{n+1} - z\rangle\right) \\
&\le (1 - \gamma_n)\|x_n - z\|^2 + \gamma_n\delta_n,
\end{aligned}$$

where $N = \sup\{\|x_n - z\|^2 : n \in \mathbb{N}\}$, $\gamma_n = \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k}$, and

$$\delta_n = \frac{1}{\tau - \gamma k} \langle \gamma f(z) - \mu Bz, x_{n+1} - z\rangle.$$

By assumption $\gamma_n \to 0$, $\sum_{n\in\mathbb{N}} \gamma_n = \infty$ and we have $\limsup_{n\to\infty} \delta_n \le 0$. Hence, it follows from Lemma 3.3 that $\{x_n\}$ converges strongly to $z = P_Z(\gamma f + (\mathrm{Id} - \mu B))(z)$. ☐

Corollary 3.1. *Let $\boldsymbol{A} = \{A_i : H \multimap H\}_{i\in I}$ be a finite family of monotone operators with $R(\boldsymbol{A}, \boldsymbol{\lambda})^{-1}(\{0\}) \neq \emptyset$, where $\lambda_i > 0$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ with $\sum_{i\in I} \lambda_i = 1$. Let B be an l-Lipschitz and η-strongly monotone operator. Assume that f is a k-Lipschitz mapping on H. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (\mathrm{Id} - \mu\alpha_n B)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n,$$

where $\beta_0 = 1$, $\{\beta_n\}, \{\alpha_n\} \subseteq (0, 1]$, $0 < \mu < \frac{2\eta}{l^2}$ and $0 < \gamma < \frac{\tau}{k}$ with $\tau = \mu(2\eta - \mu l^2)$. If

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty$;

then $\{x_n\}$ converges strongly to $z = P_Z(\gamma f + (\mathrm{Id} - \mu B))(z)$.

References

[1] S. Bartz, H. H. Bauschke, S. M. Moffat and X. Wang, *The resolvent average of monotone operators: dominant and recessive properties*, SIAM J. Optim. **26** (2016), 602–634.

[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Berlin, 2011.

[3] J. M. Borwein, *Fifty years of maximal monotonicity*, Optim. Lett. **4** (2010), 473–490.

[4] F. E. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.

[5] O. Guler, *On the convergence of proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[6] N. Hadjisavvas, S. Komlosi and S. Schaible, *Handbook of Generalized Convexity and Generalized Monotonicity*, Springer, New York, 2005.

[7] G. Marino and A. Rugiano, *Strong convergence of a generalized viscosity implicit midpoint rule for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal. **17** (2016), 2255–2275.

[8] B. Martinet, *Regularisation dinequations variationnelles par approximations successives*, Recherche Operationnelle **4** (1970), 154–158.

[9] G. J. Minty, *Monotone networks*, Proceedings of the Royal Society of London **257** (1960), 194–212.

[10] C. Mongkolkeha, Y. Cho and P. Kumam, *Weak convergence theorems of iterative sequences in Hilbert spaces*, J. Nonlinear Convex Anal. **15** (2014), 1303–1318.

[11] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[12] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[13] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), 240–256.

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