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# A STUDY ON LACUNARY STATISTICAL CONVERGENCE OF MULTISET SEQUENCES

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Abstract. Statistical convergence developed rapidly after being defined independently by Fast and Steinhaus in 1951 and was studied in many fields. One of them is lacunary statistical convergence and it was defined by Fridy and Orhan in 1993. On the other hand, although there are various studies on multisets, which are sets that can repeat elements, the convergence of multiset sequences was defined by Pachilangode in 2021. In this study, lacunary statistical convergence of multiset sequences is examined and releated examples and theorems are given.

## 1. Introduction

We know that in classical set theory an element is written only once in the set. Besides, in our daily life, we see a lot of situations where it is necessary to write more than one element. Some of these situations are computer coding, element formulas, and phone numbers. In each example, there are same numbers and same molecules that play different roles. If these numbers are used once rather than multiple times, it is clear that there will be problems. Weyl explained this situation as there can be more than one white ball, more than one red ball, and more than one green ball in the same sack and he tried to apply his notion of multiset (a set with an equivalence relation) to a variety of problems in physics, chemistry, and genetics [29]. For this reason, multisets have been found interesting and studied in many disciplines such as mathematics, physics, philosophy, logic, linguistics, computer science, etc. for many years. Looking at the literature, it is seen that studies on multisets date back to the 1970s. Many researchers have studied these sets under various names such as bags,

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occurrence set, weighted set, etc. Manna and Waldinger developed an elementary theory of bags using a primitive binary insertion symbol $\odot$ [19]. If an atom $u$ has multiplicity $n \geq 0$ in bag $x$, then $u$ has multiplicity $n+1$ in bag $u \odot x$. Their theory BAG admits only finite collections of atoms (no hierarchy of bags) and is developed to the point of a simple algebra of bags. On the other hand, Bender [1], Lake [17], Hickman [13], Meyer [20], and Monro [22] investigated some important properties of multisets. In 1981, Knuth [16] studied computer programing and multisets. Blizard studied on multisets in his doctoral thesis [4], [3], [2]. In the 2000s, important studies were carried out on multisets. Syropoulos published "Mathematics of multisets" in 2001 [28], Singh published "An overview of the application of multiset" in 2007 [26], Majumdar published "Soft multisets" in 2012 [18], Nazmul published "On multisets and multigroups" in 2013 [23] and İbrahim published "Multigroup actions on multisets" in 2017 [14].

Now we can give some basic information about multisets. As we said, the same element in a multiset can be written multiple times and plays a different role in each write. The order in which the element is written is not important, but it is very important how many times the elements are repeated in the set. For example, $\{2,3,5,3,4,2,2,4\}$ and $\{5,3,4,4,3,2,2,2\}$ sets are the same. We write $\{2,3,5,3,4,2,2,4\}$ multiset by $\{2,3,4,5\}_{3,2,2,1}$ or $\{2|3, 3|2, 4|2, 5|1\}$ and it means 2 appearing 3 times, 3 appearing 2 times, 4 appearing 2 times and 5 appearing 1 times. The cardinality of a multiset is the sum of the multiplicities of its elements. Despite the long-term studies on multisets, studies on multiset sequences are quite new. In 2021, Pachilangode and John defined usual convergence of multiset sequences [24] and Debnath and Debnath defined statistical convergence of multiset sequences [5].

**Definition 1.1** ([24])**.** Let $X$ be a set. A sequence in which all the terms are multisets is known as a multiset sequence. For any sequence $x=(x_i) \in X$, a multiset sequence is defined by

$$M = \{x_i | c_i : x_i \in X, c_i \in \mathbb{N}_0\}.$$

We can give the following two examples from Pachilangode's study to better understand multiset sequences.

*Example* 1.1 ([24]). Let $N_n = \{1|1, 2|2, \ldots, n|n\}$. Then $\{N_n\}$ is an multiset sequence and $n^{th}$ terms has $\frac{n(n+1)}{2}$ elements.

*Example* 1.2 ([24]). The prime factorises $n$ completely, and let $F_n$ be the multiset of these factors, including 1. Then, $F_1 = \{1\}$, $F_2 = \{1,2\}$, $F_3 = \{1,3\}$, $F_4 = \{1,2,2\}$ and $F_{36} = \{1,2,2,3,3\}$. In this case, $\{F_n\}$ is an multiset sequence.

In 1935, Zygmund first mentioned the idea of statistical convergence in her monograph in Warsaw but it was formally introduced by Fast [7] and Steinhaus [27], independently. Later on, Schoenberg studied statistical convergence as a summability method [25]. After this date, it is seen that statistical convergence has been studied in many mathematical fields [6, 8, 9, 21, 30]. This concept was also studied with ideals, weak convergence, modulus functions, complex uncertain sequences [11, 12, 15].

**Definition 1.2.** Let $A \subseteq \mathbb{N}$, $A_n = \{k \in A : k \leq n\}$ and $|A_n|$ gives the cardinality of $A_n$. Then, $d(A) = \lim_{n\to+\infty} \frac{|A_n|}{n}$ is natural density of the set $A$.

**Definition 1.3** ([7])**.** A number sequence $(x_i)$ is statistically convergent to $L$ provided that for every $\varepsilon > 0$, $d(\{i \leq n : |x_i - L| \geq \varepsilon\}) = 0$. In this case we write $st-\lim x_i = L$ and usually the set of statistically convergent sequences is denoted by $S$.

Considering the definition of natural density, this definition can also be expressed as for every $\varepsilon > 0$,

$$\lim_{n\to+\infty} \frac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| = 0.$$

Lacunary statistical convergence was defined by Fridy and Orhan in 1993 [10]. Before giving this definition, let's remind the definition of a lacunary sequence.

**Definition 1.4.** A lacunary sequence is an increasing integer sequence $\theta = (i_r)$ such that $i_0 = 0$ and $h_r = i_r - i_{r-1} \to +\infty$ as $r \to +\infty$. The intervals $J_r = (i_{r-1}, i_r]$ are determined by $\theta$ and the ratio is determined $q_r = \frac{i_r}{i_{r-1}}$.

*Example* 1.3. $\theta = (r^2)$ is a lacunary sequence because $i_0 = 0$ and $h_r = i_r - i_{r-1} \to +\infty$ as $r \to +\infty$.

*Example* 1.4. $\theta = (r)$ is not a lacunary sequence because $i_0 = 0$ but $h_r = i_r - i_{r-1} = 1$ for all $r = 0, 1, \ldots$

**Definition 1.5** ([10])**.** Let $\theta = (i_r)$ be a lacunary sequence. The number sequence $x = (x_i)$ is lacunary statistically convergent (or $S_\theta-$convergent) to $L$ if for every $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{1}{h_r} |\{i \in J_r : |x_i - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim x_i = L$ and usually the set of lacunary statistically convergent sequences is denoted by $S_\theta$.

Another concept closely related to statistical convergence is strong Cesáro summability:

$$|C_1| := \left\{ x : \text{for some } L,\ \lim_{n\to+\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_i - L| \right) = 0 \right\}.$$

Similarly, there is a close relationship between strong Cesáro summability and $N_\theta$ space:

$$N_\theta := \left\{ x : \text{for some } L,\ \lim_{r\to+\infty} \left( \frac{1}{h_r} \sum_{i\in J_r} |x_i - L| \right) = 0 \right\}.$$

After all this information, we can give definitions about the usual convergence and statistical convergence of multiset sequences. Throughout the paper, we study with multiset sequences of real numbers.

**Definition 1.6** ([5])**.** Let $\mathbb{N}_0$ is the set of non-negative integers. The set

$$m\mathbb{R} = \{M = mx = x_i|c_i : x_i \in \mathbb{R} \text{ and } c_i \in \mathbb{N}_0\},$$

is called multiset of real numbers.

Due to repetitive elements of the multisets, it is necessery to define a new metric in order to work on multisets. Let $(X, d)$ be a metric space and $M$ be a multiset in this space. The $d$ metric is not very functional on $M$ because of the repetitive elements of $M$. Hence, if a new $d_M$ metric is defined on $M$, then $(M, d_M)$ is a metric space. In this study, it is defined as

$$d_M(mx, my) = d_M(x_i|c_i, y_i|t_i) = \sqrt{(x_i - y_i)^2 + (c_i - t_i)^2},$$

where $d_M : M \times M \to \mathbb{R}$ for each $i \in \mathbb{N}$. It is easily seen that $d_M$ satisfies the metric conditions with Minkowsky inequality.

**Definition 1.7.** A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is convergent to $l|c$ if given the set

$$\lim_{i\to+\infty} d_M(x_i|c_i, l|c) = \lim_{i\to+\infty} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, $x_i \to l$ and $c_i \to c$, i.e., for each $\varepsilon > 0$, $|x_i - l| < \varepsilon$ and $|c_i - c| < \varepsilon$.

**Definition 1.8.** ([5]) Let $x = (x_i)$ be a real sequence and $c = (c_i)$ be a sequence of $\mathbb{N}_0$. A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is statistically convergent to $l|c$ of $m\mathbb{R}$ if given for any $\varepsilon > 0$,

$$\lim_{n\to+\infty} \frac{1}{n} \{i \le n : d_M(x_i|c_i, l|c) \ge \varepsilon\} = \lim_{n\to+\infty} \frac{1}{n} \left|\left\{i \le n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \ge \varepsilon\right\}\right| \\ = 0.$$

The set of all statistically convergent multiset sequences is denoted by $S^{l/c}$ and is written by $mx \to l|c(S)$.

*Example* 1.5 ([5]). Consider a multisequence $mx = (x_i|c_i)$, given by

$$x_i = \begin{cases} i, & i = n^2;\ n = 1, 2, 3, \ldots, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \quad c_i = \begin{cases} i, & i = n^3;\ n = 1, 2, 3, \ldots, \\ 5, & \text{otherwise.} \end{cases}.$$

Then for any $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n\to+\infty} \tfrac{1}{n} \left|\left\{i \le n : \sqrt{(x_i - 1)^2 + (c_i - 5)^2} \ge \varepsilon\right\}\right| \\ \le & \lim_{n\to+\infty} \tfrac{1}{n} \left(n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}}\right) = 0. \end{aligned}$$

Therefore, the multisequence $mx$ statistically converges to $1|5$.

## 2. Main Results

We aim to define lacunary statistical convergence based on Debnath's study on statistical convergence for multiset sequences. For this purpose, we need following definitions.

**Definition 2.1.** Let $\theta = (i_r)$ be a lacunary sequence and $mx = (x_i|c_i)$ be a multiset sequence of $m\mathbb{R}$. $mx$ is said to be lacunary statistically convergent to $l/c$, if for each $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| = 0.$$

In this case, we write $mx \to l/c(S_\theta)$. The set of all lacunary statistically convergent multiset sequences is symbolized as $S_\theta^{l/c}$.

**Definition 2.2.** Let $mx = (x_i|c_i)$ be a multiset sequence of $m\mathbb{R}$. $mx$ is said to be statistically Cesáro summable to $l/c$ if for each $\varepsilon > 0$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, we write $mx \to l/c(\sigma)$. The set of all statistically Cesáro summable multiset sequences is symbolized as $\sigma^{l/c}$.

**Definition 2.3.** Let $\theta = (i_r)$ be a lacunary sequence and $mx = (x_i|c_i)$ be a multiset sequence of $m\mathbb{R}$. $mx$ is said to be lacunary strongly summable to $l/c$ if for each $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{1}{h_r} \sum_{i\in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, we write $mx \to l/c(N_\theta)$. The set of all lacunary strongly summable multiset sequences is symbolized as $N_\theta^{l/c}$.

Boundedness plays an important role in our results and proofs of theorems. So let's give the definition of bounded multiset sequence.

**Definition 2.4** ([5])**.** A multiset sequence $mx = (x_i|c_i)$ is said to be bounded provided that there exists a non-negative real number $B$ such that $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$.

**Theorem 2.1.** *Let $\theta = (i_r)$ be a lacunary sequence.*

*i) For any multiset sequence $mx = (x_i|c_i)$, $mx \in N_\theta^{l/c}$ implies $mx \in S_\theta^{l/c}$.*

*ii) If $mx = (x_i|c_i)$ is a bounded multiset sequence then, $mx \in S_\theta^{l/c}$ implies $mx \in N_\theta^{l/c}$.*

*Proof.* $i)$ Let $mx \in N_\theta^{l/c}$ and $\varepsilon > 0$ be given. Then,

$$\sum_{i\in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \sum_{\substack{i\in J_r \\ \sqrt{(x_i-l)^2+(c_i-c)^2}\geq\varepsilon}} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$

$$\geq \varepsilon \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|,$$

and so,

$$\frac{1}{\varepsilon h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

If we take the limit of both sides

$$\lim_{r \to +\infty} \frac{1}{\varepsilon h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \lim_{r \to +\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

Then, we have the proof.

*ii*) This part shows in which case the inverse of *i*) is valid. Assume that $mx \in S_\theta^{l/c}$ and $mx$ be bounded. Then, there exists a non-negative real number $B$ such that $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$ for all $i \in \mathbb{N}$. At the same time, from the fact that $c, c_i \in \mathbb{N}_0$ and $x_i \to l$, we have,

$$\sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \sqrt{x_i^2 + (c_i - 1)^2} \leq B.$$

Hence,

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} =& \frac{1}{h_r} \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2}}} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &+ \frac{1}{h_r} \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} < \frac{\varepsilon}{2}}} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ \leq& \frac{B}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

In that case

$$\lim_{r \to +\infty} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \lim_{r \to +\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right|.$$

Since the limit of the right side is zero, the left side is also zero. So, the proof is completed. □

Now, let us investigate the relation between the spaces $S^{l/c}$ and $S_\theta^{l/c}$ with the following theorem.

**Theorem 2.2.** *Let $\theta = (i_r)$ be a lacunary sequence.*

*i) If $\liminf q_r > 1$, then $mx \in S^{l/c}$ implies $x \in S_\theta^{l/c}$.*

*ii) If $\limsup_r q_r < +\infty$, then $mx \in S_\theta^{l/c}$ implies $mx \in S^{l/c}$.*

*Proof.* *i)* Assume that $\liminf q_r > 1$. In this case we know that for sufficiently large $r$ there exists $\lambda > 0$ such that $q_r \geq 1+\lambda$. This implies that,

$$\frac{h_r}{i_r} \geq \frac{\lambda}{1+\lambda}.$$

Since $mx \in S^{l/c}$, for each $\varepsilon > 0$ and sufficiently large $r$ from the definition of $J_r$ we have,

$$\begin{aligned}
&\frac{1}{i_r}\left|\left\{i \leq i_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right| \\
\geq&\frac{1}{i_r}\left|\left\{i \in J_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right| \\
\geq&\frac{\lambda}{1+\lambda}\cdot\frac{1}{h_r}\left|\left\{i \in J_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence, from the limit of both sides,

$$\begin{aligned}
&\lim_r \frac{1}{h_r}\left|\left\{i \in J_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right| \\
\leq&\left(\frac{1+\lambda}{\lambda}\right)\lim_{r\to+\infty}\frac{1}{i_r}\left|\left\{i \leq i_r : \left|\sqrt{(x_i-l)^2+(c_i-c)^2}\right| \geq \varepsilon\right\}\right| = 0.
\end{aligned}$$

Then, we have the proof.

*ii)* Now, suppose that $\limsup_r q_r < +\infty$ then, there is a $K > 0$ such that $q_r < K$ for all $r$. Let $mx \in S_\theta^{l/c}$. In order to facilitate transactions define the set for $\varepsilon > 0$,

$$N_r := \left|\left\{i \in J_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right|.$$

From the definition of limit in $S_\theta^{l/c}$, for $\varepsilon > 0$ there is an $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \varepsilon, \quad \text{for each } r > r_0.$$

Now, let $C := \max\{N_r : 1 \leq r \leq r_0\}$ and let $i_{r-1} < n \leq i_r$. Then,

$$\begin{aligned}
&\frac{1}{n}\left|\left\{i \leq n : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right| \\
\leq&\frac{1}{i_{r-1}}\left|\left\{i \leq i_r : \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon\right\}\right| \\
=&\frac{1}{i_{r-1}}\{N_1+N_2+\cdots+N_{r_0}\}+\frac{i_{r_0+1}-i_{r_0}}{i_{r-1}}\cdot\frac{1}{h_{r_0+1}}N_{r_0+1}+\cdots+\frac{i_r-i_{r-1}}{i_{r-1}}\cdot\frac{1}{h_r}N_r \\
<&\frac{C}{i_{r-1}}r_0+\varepsilon\left\{\frac{i_{r_0+1}-i_{r_0}}{i_{r-1}}+\cdots+\frac{i_r-i_{r-1}}{i_{r-1}}\right\} \\
=&\frac{C}{i_{r-1}}r_0+\varepsilon\frac{i_r-i_{r_0}}{i_{r-1}}=\frac{C}{i_{r-1}}r_0+\varepsilon\left\{\frac{i_r}{i_{r-1}}-\frac{i_{r_0}}{i_{r-1}}\right\}
\end{aligned}$$

$$<\frac{C}{i_{r-1}}r_0+\varepsilon q_r<\frac{C}{i_{r-1}}r_0+\varepsilon K.$$

From this result we have the proof. □

The following theorem gives the relations between $\sigma^{l/c}$ and $N_\theta^{l/c}$.

**Theorem 2.3.** *Let $\theta=(i_r)$ be a lacunary sequence.*

*i) If $\theta$ satisfies $\liminf q_r>1$, then $mx\in\sigma^{l/c}$ implies $mx\in N_\theta^{l/c}$.*

*ii) If $\theta$ satisfies $\limsup_r q_r<+\infty$, then $mx\in N_\theta^{l/c}$ implies $mx\in\sigma^{l/c}$.*

*Proof.* $i)$ Assume that $\liminf_r q_r>1$. Then, there exists $\lambda>0$ such that $q_r\geq 1+\lambda$ for sufficiently large $r$. Since $h_r=i_r-i_{r-1}$, we have $\frac{i_r}{i_{r-1}}\geq 1+\lambda$ for sufficiently large $r$ which implies that $\frac{h_r}{i_r}\geq\frac{\lambda}{1+\lambda}$

$$\begin{aligned}\frac{1}{i_r}\sum_{i=1}^{i_r}\sqrt{(x_i-l)^2+(c_i-c)^2}\geq&\frac{1}{i_r}\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}\\ =&\left(\frac{h_r}{i_r}\right)\frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}\\ \geq&\left(\frac{\lambda}{1+\lambda}\right)\frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}.\end{aligned}$$

We know that from assumption $\lim_{i_r\to+\infty}\frac{1}{i_r}\sum_{i=1}^{i_r}\sqrt{(x_i-l)^2+(c_i-c)^2}=0$. Therefore, $\lim_{r\to+\infty}\frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}=0$ which implies $mx\in N_\theta^{l/c}$.

$ii)$ If $\limsup q_r<+\infty$, then there exists $K>0$ such that $q_r<K$ for all $r\geq 1$. Let $mx\in N_\theta^{l/c}$ and denote $L_r=\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}$. By the definition of convergence when $r>r_0$ for every $\varepsilon>0$,

$$\frac{L_r}{h_r}=\frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}<\varepsilon.$$

Choose $D:=\max\{L_r:1\leq r\leq r_0\}$ and $i_{r-1}<n<i_r$.

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\ \leq&\frac{1}{i_{r-1}}\sum_{i=1}^{i_r}\sqrt{(x_i-l)^2+(c_i-c)^2}\\ =&\frac{1}{i_{r-1}}\Bigg(\sum_{i\in J_1}\sqrt{(x_i-l)^2+(c_i-c)^2}+\sum_{i\in J_2}\sqrt{(x_i-l)^2+(c_i-c)^2}\\ &+\cdots+\sum_{i\in J_r}\sqrt{(x_i-l)^2+(c_i-c)^2}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{i_{r-1}}\left\{L_1+L_2+\cdots+L_{r_0}\right\}+\frac{i_{r_0+1}-i_{r_0}}{i_{r-1}}\cdot\frac{1}{h_{r_0+1}}L_{r_0+1}+\cdots+\frac{i_r-i_{r-1}}{i_{r-1}}\cdot\frac{1}{h_r}L_r\\
&<\frac{D}{i_{r-1}}r_0+\varepsilon\left\{\frac{i_{r_0+1}-i_{r_0}}{i_{r-1}}+\cdots+\frac{i_r-i_{r-1}}{i_{r-1}}\right\}\\
&=\frac{D}{i_{r-1}}r_0+\varepsilon\frac{i_r-i_{r_0}}{i_{r-1}}=\frac{D}{i_{r-1}}r_0+\left\{\frac{i_r}{i_{r-1}}-\frac{i_{r_0}}{i_{r-1}}\right\}\\
&<\frac{D}{i_{r-1}}r_0+\varepsilon q_r<\frac{D}{i_{r-1}}r_0+\varepsilon K.
\end{aligned}
$$

This completes the proof of the theorem. □

**Theorem 2.4.** *i) For any multiset sequence $mx=(x_i/c_i)$, $mx\in\sigma^{l/c}$ implies $mx\in S^{l/c}$.*

*ii) If $mx$ is a bounded multiset sequence, then $mx\in S^{l/c}$ implies $mx\in\sigma^{l/c}$.*

*Proof. i)* We can prove this theorem in a similar way to the proof of Theorem 2.1. Let $mx\in\sigma^{l/c}$ and $\varepsilon>0$ be given. Then,

$$
\begin{aligned}
\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\geq &\sum_{\substack{i=1\\ \sqrt{(x_i-l)^2+(c_i-c)^2}\geq\varepsilon}}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\
&\geq\varepsilon\left|\left\{i\leq n:\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\varepsilon\right\}\right|,
\end{aligned}
$$

and so,

$$
\frac{1}{\varepsilon n}\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\frac{1}{n}\left|\left\{i\leq n:\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\varepsilon\right\}\right|.
$$

If we take the limit of both sides,

$$
\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{\varepsilon n}\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\
\geq&\lim_{n\to+\infty}\frac{1}{n}\left|\left\{i\leq n:\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\varepsilon\right\}\right|.
\end{aligned}
$$

Then, we have the proof.

*ii)* Now suppose that $mx\in S^{l/c}$ and $mx$ are bounded. Then, there exists a non-negative real number $B$ such that $\sqrt{x_i^2+(c_i-1)^2}\leq B$ for all $i\in\mathbb{N}$. We also know that,

$$
\sqrt{(x_i-l)^2+(c_i-c)^2}\ \leq\ \sqrt{x_i^2+(c_i-1)^2}\leq B.
$$

Hence,

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2} =& \frac{1}{n}\sum_{\substack{i=1\\ \left|\sqrt{(x_i-l)^2+(c_i-c)^2}\right|\geq\frac{\varepsilon}{2}}}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\
&+\frac{1}{n}\sum_{\substack{i=1\\ \left|\sqrt{(x_i-l)^2+(c_i-c)^2}\right|<\frac{\varepsilon}{2}}}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\
\leq& \frac{B}{n}\left|\left\{i\leq n:\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2},
\end{aligned}
$$

is obtained. In that case

$$
\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{(x_i-l)^2+(c_i-c)^2}\\
\leq&\lim_{n\to+\infty}\frac{1}{n}\left|\left\{i\leq n:\sqrt{(x_i-l)^2+(c_i-c)^2}\geq\frac{\varepsilon}{2}\right\}\right|=0.
\end{aligned}
$$

So, the proof is completed. □

## 3. Conclusions

In our daily life, we often encounter situations where an element of a set must be written more than once in the set. Some of these situations are computer coding, element formulas, and phone numbers. These sets are multisets and for this reason, multisets have been found interesting and studied in many disciplines such as mathematics, physics, philosophy, logic, linguistics, computer science, etc. for many years. In this situation, multiset sequences are also interesting and the studies on this subject are quite new. For this purpose, in this paper we introduce the lacunary statistical convergence of multiset sequences and we investigate some important relations.

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