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INTEGRAL BOUNDARY VALUE PROBLEMS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HADAMARD AND CAPUTO-HADAMARD FRACTIONAL DERIVATIVES

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Abstract. In this paper, we examine the existence and uniqueness of integral boundary value problem for implicit fractional differential equations (IFDE's) involving Hadamard and Caputo-Hadamard fractional derivative. We prove the existence and uniqueness results by utilizing Banach and Schauder's fixed point theorem. Finally, examples are introduced of our results.

1. Introduction

FDE's are considered to be a different model to integer differential equations. It has been proved by applying importance in the modeling of various fields of physical sciences, medicine, electronics and wave transformation [8, 16, 21, 23, 26]. The dominant techniques are the method of introducing a parameter for solving an implicit differential equations. In past three years, the most of research paper to developed existence and uniqueness of implicit FDE's involving various derivatives like the Caputo,Riemann-Liouville, Caputo-Hadamard, Hadamard, Hilfer-Hadamard fractional derivatives etc., (see [4–7, 9, 14, 15, 19, 20, 24]).

Caputo Hadamard fractional derivatives were studied in [12] by the authors F. Jarad, T. Abdeljawad and D. Baleanu, where a Caputo-type modification for Hadamard derivatives was introduced and studied. Later, more properties of Hadamard fractional derivatives were investigated in [1, 2, 10, 13].

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The applications of Hadamard fractional differential equations in mathematical physics cuold be found in [11, 17, 18, 22, 25]. In [3] the authors have studied Hilfer-Hadamard FDE's with variable-order fractional integral and fractional derivative. Motivated by the above cited work, we studies the solutions of existence and uniqueness results to the following implicit fractional differential equations with integral boundary conditions of the form

$$ {}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t) = g(t, x(t), {}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t)), \quad t \in \mathcal{J} := (b, \mathcal{T}), \tag{1.1} $$

$$ x(b) = 0,\ x(\mathcal{T}) = \lambda \int_0^{\sigma} x(s)ds, \quad b < \sigma < \mathcal{T}, \lambda \in \mathbb{R}, \tag{1.2} $$

where ${}^{\mathcal{H}}\mathcal{D}^{\vartheta}$ is the Hadamard fractional derivative of order $1 < \vartheta \leq 2$,

$$ {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t) = g(t, x(t), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)), \quad t \in \mathcal{J} := [b, \mathcal{T}], \tag{1.3} $$

$$ x(b) = 0,\ x(\mathcal{T}) = \lambda \int_0^{\sigma} x(s)ds, \quad b < \sigma < \mathcal{T}, \lambda \in \mathbb{R}, \tag{1.4} $$

where ${}^{\mathcal{CH}}\mathcal{D}^{\vartheta}$ is the Caputo-Hadamard fractional derivative of order $1 < \vartheta \leq 2$ and $g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this paper, Section 2, has definitions and some of the most important basic concepts of the fractional calculus. In Section 3, existence and uniqueness of solutions for integral boundary conditions of implicit fractional differential equations involving Hadamard fractional derivative and Caputo-Hadamard fractional derivatives are proved by utilizing Banach and Schauder's fixed point theorems. In Section 4, an illustrative examples are provided to explain of the results of the problem (1.1)–(1.4).

2. Basic Results

In this section, the some most important basic concepts, definitions and some supporting results are used in this paper. By $\mathcal{C}(\mathcal{J}, \mathbb{R})$ we denote the Banach space of all continuous functions form $\mathcal{J}$ into $\mathbb{R}$ with the norm $||x||_{\infty} = \sup\{|x(t)| : t \in \mathcal{J}\}$.

Definition 2.1 ([15])**.** The derivative of fractional order $\vartheta > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is given by

$$ \mathcal{D}^{\vartheta}_{0+}x(t) = \frac{1}{\Gamma(n - \vartheta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t - s)^{\vartheta - n + 1}} ds, $$

where $n = [\vartheta] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([15])**.** The **Hadamard fractional integral** of g is defined by

$$ {}_{\mathcal{H}}\mathcal{J}^{\vartheta}x(t) = \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\log \frac{t}{s}\right)^{\vartheta - 1} \frac{g(s)}{s} ds, \quad \vartheta > 0. $$

Definition 2.3 ([15])**.** The **Hadamard fractional derivative** of g is continuous function and further, $\log(\cdot) = \log_e(\cdot)$ is defined as

$$ {}_{\mathcal{H}}\mathcal{D}^{\vartheta}x(t) = \frac{1}{\Gamma(n - \vartheta)} \left(t\frac{d}{dt}\right)^n \int_b^t \left(\log \frac{t}{s}\right)^{n - \vartheta - 1} \frac{g(s)}{s} ds, $$

where $n-1<\vartheta<n$, $n=[\vartheta]+1$ and $[\vartheta]$ denotes the integer part of the real number ϑ.

Definition 2.4 ([12])**.** For at least n-times differentiable function g, the **Caputo-Hadamard fractional derivative** of order ϑ is defined as

$$^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)=\frac{1}{\Gamma(n-\vartheta)}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\vartheta-1}\delta^n\frac{g(s)}{s}ds.$$

Lemma 2.1 (Hadamard fractional derivative)**.** *Let* $v\in\mathcal{C}([b,\mathcal{T}],\mathbb{R})$ *and* $x\in\mathcal{C}^2_{\delta}([b,\mathcal{T}],\mathbb{R})$*. Then*

$$\begin{aligned}&^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t)=v(t),\quad t\in\mathcal{J}:=[b,\mathcal{T}],\\ &x(b)=0,x(\mathcal{T})=\lambda\int_0^{\sigma}x(s)ds,\quad b<\sigma<\mathcal{T},\lambda\in\mathbb{R},\end{aligned}\tag{2.1}$$

is equivalent to the integral equation given by

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds+\frac{\left(\ln\frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}-\lambda\left[\int_b^{\sigma}\left(\ln\frac{s}{b}\right)^{\vartheta-1}ds\right]\right]}\\ &\times\left[\lambda\int_b^{\sigma}\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{v(r)}{r}drds-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds\right]\end{aligned}\tag{2.2}$$

Lemma 2.2 (Caputo-Hadamard fractional derivative)**.** *Let* $v\in\mathcal{C}([b,\mathcal{T}],\mathbb{R})$ *and* $x\in\mathcal{C}^2_{\delta}([b,\mathcal{T}],\mathbb{R})$*.*

$$\begin{aligned}&^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)=v(t),\quad t\in\mathcal{J}:=[b,\mathcal{T}],\\ &x(b)=0,\ x(\mathcal{T})=\lambda\int_0^{\sigma}x(s)ds,\quad b<\sigma<\mathcal{T},\lambda\in\mathbb{R},\end{aligned}\tag{2.3}$$

is equivalent to the integral equation given by

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds+\frac{\left(\ln\frac{t}{s}\right)}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{s}\right)-\lambda\left[\sigma\left(\ln\frac{\sigma}{b}-1\right)+b\right]\right]}\\ &\times\left[\lambda\int_b^{\sigma}\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{v(r)}{r}drds-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds\right].\end{aligned}\tag{2.4}$$

Lemma 2.3 (Nonlinear alternative of Lerary-Schauder type, [7])**.** *Let* $\mathcal{B}$ *be a Banach space,* $\mathcal{C}$ *a closed, convex subset of* $\mathcal{B}$*,* $\mathcal{U}$ *an open subset of* $\mathcal{C}$ *and* $0\in\mathcal{U}$*. Suppose that* $F:\overline{\mathcal{U}}\to\mathcal{C}$ *is a continuous, compact map. Then either* (i) F *has a fixed point in* $\overline{\mathcal{U}}$*, or* (ii) *there is a* $u\in\partial\mathcal{U}$ *and* $\lambda\in(0,1)$*, with* $u=\lambda F(u)$*.*

3. Main Results

To prove the existence and uniqueness results we need the following assumptions.

Assumption 3.1. The function $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Assumption 3.2. There exists constants $K_g > 0$ and $0 < L_g < 1$ such that

$$|g(t,u,v) - g(t,u_1,v_1)| \leq K_g|u-u_1| + L_g|v-v_1|, \quad \text{for any } u,v,u_1,v_1 \in \mathbb{R}.$$

Assumption 3.3. There exist a continuous nondecreasing function φ on $[0,\infty) \to (0,\infty)$ and a function $p(t) \in \mathcal{C}^1([b,\mathcal{T}],\mathbb{R}^+)$ such that

$$||g(t,u,v)|| \leq p(t)\varphi(||u|| + ||v||).$$

The integral boundary conditions for implicit fractional differential equations with Hadamard fractional derivative (1.1)–(1.2) is equivalent to the integral equation

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(s))}{s} ds + \frac{\left(\ln \frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right]} \\ &\times \left[\lambda \int_b^{\sigma} \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r,x(r),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(r))}{r} dr ds \right. \\ &\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(s))}{s} ds\right], \end{aligned}$$

where $N_1 = \int_b^{\sigma} \left(\ln \frac{s}{b}\right)^{\vartheta-1} ds$.

The integral boundary conditions for implicit fractional differential equations with Caputo-Hadamard fractional derivative (1.3)–(1.4) is equivalent to the integral equation

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(s))}{s} ds + \frac{\left(\ln \frac{t}{b}\right)}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{b}\right) - \lambda N_2\right]} \\ &\times \left[\lambda \int_b^{\sigma} \int_r^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r,x(r),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(r))}{r} dr ds \right. \\ &\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(s))}{s} ds\right] \end{aligned}$$

or

$$x(t) = I^{\vartheta}f(s) + \left(\frac{(\ln \frac{t}{s})}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{s}\right) - \lambda N_2\right]}\right)\left[\lambda \int_b^{\sigma} I^{\vartheta}f_1(r)ds - I^{\vartheta}f_2(s)\right],$$

where $N_2 = \sigma\left(\ln \frac{\sigma}{b} - 1\right) + b$ and $f, f_1, f_2 \in \mathcal{C}(\mathcal{J},\mathbb{R})$ satisfies the functional equations

$$\begin{aligned} f(s) &= g(s, I^{\vartheta}f(s), f(s)), \\ f_1(r) &= g(r, I^{\vartheta}f_1(r), f_1(r)), \\ f_2(s) &= g(s, I^{\vartheta}f_2(s), f_2(r)), \\ I^{\vartheta}f(s) &= \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(s))}{s} ds, \end{aligned}$$

$$I^{\vartheta}f_1(r) = \int_b^s \left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{g(r,x(r),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(r))}{r}dr,$$
$$I^{\vartheta}f_2(s) = \int_b^{\mathcal{T}} \left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(s))}{s}ds.$$

Theorem 3.1. *Assume that assumptions* 3.1 *and* 3.2 *hold. If*

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}(|\lambda|(\sigma-b)-1)\right]\frac{K_g}{(1-L_g)} < 1,$$

then there exists a unique solution for (1.1)–(1.2) *on* $\mathcal{J} := [b,\mathcal{T}]$.

Proof. Let $B_r = \{x \in \mathcal{C}([b,\mathcal{T}],\mathbb{R}) : \|x\| \le r\}$. Consider the operator $\mathcal{H} : \mathcal{C}([b,\mathcal{T}],\mathbb{R}) \to \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ defined by

$$\mathcal{H}(x)(t) = I^{\vartheta}f(s) + \left(\frac{\left(\ln\frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right]}\right)\left(\lambda\int_b^{\sigma} I^{\vartheta}f_1(r)ds - I^{\vartheta}f_2(s)\right), \tag{3.1}$$

where $f, f_1, f_2 \in \mathcal{C}(\mathcal{J},\mathbb{R})$ satisfies the functional equations

$$f(s) = f(s, I^{\vartheta}f(s), f(s)),$$
$$f_1(r) = f(r, I^{\vartheta}f_1(r), f_1(r)),$$
$$f_2(s) = f(s, I^{\vartheta}f_2(s), f_2(s)),$$

where $N_1 = \int_b^{\sigma}\left(\ln\frac{s}{b}\right)^{\vartheta-1}ds$ and

$$I^{\vartheta}f(s) = \frac{1}{\Gamma(\vartheta)}\int_b^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{g(s,x(s)),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(s))}{s}ds,$$
$$I^{\vartheta}f_1(r) = \int_b^s \left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{g(r,x(r),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(r))}{r}dr,$$
$$I^{\vartheta}f_2(s) = \int_b^{\mathcal{T}} \left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{g(s,x(s)),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(s))}{s}ds.$$

Clearly, the fixed point of operator $\mathcal{H}$ is solution of problem (1.1)–(1.2). Let $x_1, x_2 \in \mathcal{C}([b,\mathcal{T}],\mathbb{R})$. Then

$$\begin{aligned}(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t) =& \frac{1}{\Gamma(\vartheta)}\int_b^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{f(s)-h(s)}{s}ds \\ &+ \left(\frac{\left(\ln\frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}-\lambda N_1\right]}\right) \\ &\times \left[\lambda\int_b^{\sigma}\int_b^s \left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{f(r)-h(r)}{r}drds\right.\end{aligned}$$

$$-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}\frac{f(s)-h(s)}{s}ds\Big],$$

where $f(s), h(s), f(r), h(r) \in \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ are such that

$$\begin{aligned} f(s) &= f(s, x_1(s), f(s)), \quad f(r) = f(r, x_2(r), f(r)),\\ h(s) &= h(s, x_1(s), h(s)), \quad h(r) = h(r, x_2(r), h(r)). \end{aligned}$$

Now,

$$\begin{aligned} |(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)| \leq & \frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{|f(s)-h(s)|}{s}ds\\ &+\left(\frac{\left(\ln\frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}-\lambda N_1\right|}\right)\\ &\times\Big[|\lambda|\int_b^{\sigma}\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{|f(r)-h(r)|}{r}drds\\ &-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}\frac{|f(s)-h(s)|}{s}ds\Big], \end{aligned} \tag{3.2}$$

and, by Assumption 3.2, we have

$$|(f(s)-h(s))| = |g(s,x_1(s),f(s)) - g(s,x_2(s),h(s))|,$$
$$|(f(s)-h(s))| \leq K_g|x_1(s)-x_2(s)| + L_g|x_1(s)-x_2(s)| \leq \frac{K_g}{1-L_g}|x_1(s)-x_2(s)|,$$
$$|(f(s)-h(s))| \leq \frac{K_g}{1-L_g}|x_1(s)-x_2(s)|.$$

Similary,

$$|(f(r)-h(r))| \leq \frac{K_g}{1-L_g}|x_1(r)-x_2(r)|.$$

The equation (3.2) implies

$$\begin{aligned} |(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)| \leq & \frac{1}{\Gamma(\vartheta+1)}\left(\frac{K_g}{1-L_g}\right)||x_1-x_2||\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta}\\ &+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}\\ &\times\left((|\lambda|(\sigma-b)-1)\left(\frac{K_g}{1-L_g}\right)\right)||x_1-x_2|| \end{aligned}$$

$$\leq \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b} \right)^{2\vartheta-1}}{\Gamma(\vartheta+1) \left| \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta-1} - \lambda N_1 \right|} \times (|\lambda|(\sigma-b)-1) \Bigg) \left(\frac{K_g}{1-L_g} \right) ||x_1 - x_2||_{\infty}.$$

Thus,

$$|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)| \leq \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b} \right)^{2\vartheta-1}}{\Gamma(\vartheta+1) \left| \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta-1} - \lambda N_1 \right|} \times (|\lambda|(\sigma-b)-1) \left(\frac{K_g}{1-L_g} \right) ||x_1 - x_2||_{\infty}.$$

By (3.1), the operator $\mathcal{H}$ is continuous. Hence, by Banach's contraction principle, $\mathcal{H}$ has a unique fixed point which is a unique solution of the problem (1.1)–(1.2) on $\mathcal{J} := [b, \mathcal{T}]$. □

Theorem 3.2. *Assume that assumptions* 3.1 *and* 3.2 *hold. If*

$$\left[\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta+1}}{\Gamma(\vartheta+1) \left| \left(\ln \frac{\mathcal{T}}{b} \right) - \lambda N_2 \right|} (|\lambda|(\sigma-b)-1) \right] \left(\frac{K_g}{1-L_g} \right) < 1,$$

then there exists a unique solution for (1.3)–(1.4) *on* $\mathcal{J} := [b, \mathcal{T}]$.

The proof of Theorem 3.2 is similar to the Theorem 3.1.

Theorem 3.3. *Assume that assumptions* 3.1 *and* 3.3 *hold. Then there is at least one solution for the problem* (1.1)–(1.2) *on* $\mathcal{J} =: [b, \mathcal{T}]$.

Proof. **Step 1.** Show that $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}([b, \mathcal{T}], \mathbb{R})$.

For a positive number r_1, let $B_{r_1} = \{x \in \mathcal{C}([b, \mathcal{T}], \mathbb{R}) : ||\mathcal{Z}^*|| \leq r_1\}$ be a bounded ball in $\mathcal{C}([b, \mathcal{T}], \mathbb{R})$, where

$$||\mathcal{Z}^*|| = \sup_{t \in [b, \mathcal{T}]} (||x|| + ||g||).$$

Then

$$|\mathcal{H}(x)(t)| \leq \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s} \right)^{\vartheta-1} \frac{|g(s, x(s)), {}^c\mathcal{D}^{\vartheta} x(s))|}{s} ds + \frac{\left(\ln \frac{t}{b} \right)^{\vartheta-1}}{\Gamma(\vartheta) \left| \left(\ln \frac{\mathcal{T}}{b} \right)^{\vartheta-1} - \lambda N_1 \right|}$$

$$\times \left[|\lambda| \int_b^{\sigma} \int_b^s \left(\ln \frac{s}{r} \right)^{\vartheta-1} \frac{|g(r, x(r), {}^c\mathcal{D}^{\vartheta} x(r))|}{r} dr ds - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s} \right)^{\vartheta-1} \frac{|g(s, x(s)), {}^c\mathcal{D}^{\vartheta} x(s))|}{s} ds \right]$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds + \frac{\left(\ln \frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|} \\
&\times \left[|\lambda| \int_b^\sigma \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{r} drds \right. \\
&\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds \right],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
|\mathcal{H}(x)(t)| \leq & \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|} \\
& \times (|\lambda|(\sigma-b)-1) \Bigg) \varphi(r)||p||.
\end{aligned}
$$

Step 2. Show that $\mathcal{H}$ maps bounded sets (balls) into equicontinuous sets in $\mathcal{C}([b,\mathcal{T}],\mathbb{R})$. Let $\mu_1, \mu_2 \in [b,\mathcal{T}]$, $\mu_1 < \mu_2$. Then, we have

$$
\begin{aligned}
||\mathcal{H}(x)(\mu_1) - \mathcal{H}(x)(\mu_2)|| \leq & \frac{1}{\Gamma(\vartheta)} \Bigg[\int_b^{\mu_1} \left[\left(\ln \frac{\mu_2}{r}\right)^{\vartheta-1} - \left(\ln \frac{\mu_1}{r}\right)^{\vartheta-1} \right] \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds \\
& + \int_{\mu_1}^{\mu_2} \left(\ln \frac{\mu_2}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds \Bigg] \\
& + \frac{\left(\ln \frac{\mu_2}{b}\right)^{\vartheta-1} - \left(\ln \frac{\mu_1}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|} \\
& \times \Bigg[|\lambda| \int_b^\sigma \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{r} drds \\
& - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds \Bigg].
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u, v \in B_{r_1}$ as $\mu_2 - \mu_1 \to 0$. As $\mathcal{H}$ satisfies the above assumptions, therefore, by the Arzela-Ascoli theorem, it follows that $\mathcal{H} : \mathcal{C}([b,\mathcal{T}],\mathbb{R}) \to \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ is completely continuous. Let x be a solution. Then, for $t \in [b,\mathcal{T}]$ and following the similar computations as in the first step, we have

$$
\begin{aligned}
|x(t)| = & \lambda |\mathcal{H}(x)(t)| \\
\leq & \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta} + \frac{\left(ln \frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|} (|\lambda|(\sigma-b)-1) \Bigg)
\end{aligned}
$$

$$\times \varphi(||x||)||p||.$$

Consequently, we have

$$\frac{||x(t)||}{\left(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right)\varphi(||x||)||p||}\leq 1.$$

There exists $\mathcal{M}^*$ such that $||x|| \neq \mathcal{M}^*$. Let us set

$$U = \{x \in \mathcal{C}([b,\mathcal{T}],\mathbb{R}) : ||x|| < \mathcal{M}^*\}.$$

Note that the operator $\mathcal{H} : \overline{\mathcal{U}} \to \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x = \lambda\mathcal{H}x$ for some $0 \leq \lambda \leq 1$. Consequently, by the nonlinear alternative of Lerary-Schauder type (Lemma 2.3), we deduce that $\mathcal{H}$ has fixed point $x \in \overline{\mathcal{U}}$ which is a solution of the problem (1.1)–(1.2). □

Theorem 3.4. *Assume that assumptions* 3.1 *and* 3.3 *hold and there exists a constant* $\mathcal{M}^* > 0$*, such that*

$$\mathcal{M}^* > \left(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta+1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)-\lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right)||p||\varphi(||x||).$$

Then, there is at least one solution for the problem (1.3)–(1.4) *on* $\mathcal{J} =: [b,\mathcal{T}]$.

The proof of Theorem 3.4 is similar to the Theorem 3.3.

4. Examples

In this section, some examples are introduced for Hadamard and Caputo-Hadamard fractional derivatives of implicit fractional differential equations with integral boundary conditions.

Example 4.1. Consider the implicit Hadamard FDE's with three point integral boundary conditions of the form

$$(4.1)\qquad {}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t)|)}, \quad 1<\vartheta\leq 2,$$

$$(4.2)\qquad x(1)=0,\ x(b)=\lambda\int_b^{\sigma}x(s)ds.$$

Here $\vartheta = \frac{10}{7}$,

$$g(t,x(t),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t)) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t)|)},$$

$\sigma = 3$, $\lambda = 5$. Hence, the Assumption 3.2 holds, with $K_g = L_g = \frac{1}{49}$ and we will check that

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right]\frac{K_g}{(1-L_g)} < 1.$$

Thus, the Theorem 3.1 is satisfied and shows that the problem (4.1)–(4.2) has a unique solution on $\mathcal{J} =: [b, \mathcal{T}]$.

Example 4.2. Consider the implicit Caputo-Hadamard FDE's with three point integral boundary conditions of the form

$$ {}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t)|)}, \quad 1 < \vartheta \leq 2, \tag{4.3}$$

$$ x(1) = 0,\ x(b) = \lambda\int_b^{\sigma} x(s)ds. \tag{4.4}$$

Here $\vartheta = \frac{10}{7}$,

$$g(t, x(t), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t)|)},$$

$\sigma = 3$, $\lambda = 5$. Hence, the Assumption 3.2 holds, with $K_g = L_g = \frac{1}{49}$ and we will check that

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta+1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right) - \lambda N_2\right|}\left(|\lambda|(\sigma-b)-1\right)\right]\frac{K_g}{(1-L_g)} < 1.$$

Thus, the Theorem 3.2 is satisfied and shows that the problem (4.3)–(4.4) has a unique solution on $\mathcal{J} =: [b, \mathcal{T}]$.

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