EXISTENCE AND STABILITY OF SOLUTIONS FOR NABLA FRACTIONAL DIFFERENCE SYSTEMS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, we propose sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions for coupled systems of fractional nabla difference equations with anti-periodic boundary conditions, by using fixed point theorems. We also support these results through a couple of examples.

1. Introduction

The study of anti-periodic boundary value problems garnered significant interest due to their occurrence in the mathematical modelling of a variety of real-world problems in engineering and science. For example, we refer [19,31,32,40] and the references therein.

The boundary value problems (BVPs) connected with nabla fractional difference equations can be tackled with almost similar methods as their continuous counterparts. Peterson et al. [15,24] have initiated the study of BVPs for linear and nonlinear nabla fractional difference equations with conjugate boundary conditions. Gholami et al. [20] studied the existence of solutions for a coupled system of two-point nabla fractional difference BVPs. Recently, the author [26,27] obtained sufficient conditions on existence and uniqueness of solutions for nonlinear nabla fractional difference equations associated with different classes of boundary conditions. In spite of the

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existence of a substantial mathematical theory of the continuous fractional anti-periodic BVPs \([5–7,13,16,36,42]\), there has been no progress in developing the theory of discrete fractional anti-periodic BVPs in nabla perspective.

On the other hand, Hyers responses to Ulam’s questions have initiated the study of stability of functional equations \([23,38]\). Rassias \([35]\) generalized the Hyers result for linear mappings. Later, several mathematicians have extended Ulam’s problem in different directions \([28]\). There were significant contributions towards the study of Ulam-Hyers stability of ordinary as well as fractional differential equations \([33,41]\). The study of Ulam-Hyers stability enriched the qualitative theory of fractional difference equations \([17,18,25]\).

Motivated by these facts, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions:

\[
\begin{align*}
\left( \nabla_0^{\alpha_1-1} \left( \nabla u_1 \right) \right)(t) + f_1(t, u_1(t), u_2(t)) &= 0, \quad t \in \mathbb{N}_T^2, \\
\left( \nabla_0^{\alpha_2-1} \left( \nabla u_2 \right) \right)(t) + f_2(t, u_1(t), u_2(t)) &= 0, \quad t \in \mathbb{N}_T^2, \\
u_1(0) + u_1(T) &= 0, \quad \left( \nabla u_1 \right)(1) + \left( \nabla u_1 \right)(T) = 0, \\
u_2(0) + u_2(T) &= 0, \quad \left( \nabla u_2 \right)(1) + \left( \nabla u_2 \right)(T) = 0.
\end{align*}
\] (1.1)

Here \(T \in \mathbb{N}_2\), \(1 < \alpha_1, \alpha_2 < 2\), \(f_1, f_2 : \mathbb{N}_0^T \times \mathbb{R}^2 \rightarrow \mathbb{R}\) are continuous, \(\nabla_\nu\) denotes the \(\nu\)-th order Riemann-Liouville type backward (nabla) difference operator where \(\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}\) and \(\nabla\) denotes the first order nabla difference operator.

The present paper is organized as follows. Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions of the BVP (1.1). We present a few examples in Section 4.

2. PRELIMINARIES

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in \([21]\).

Denote by \(\mathbb{N}_a = \{a, a+1, a+2, \ldots\}\) and \(\mathbb{N}_b^a = \{a, a+1, a+2, \ldots, b\}\) for any \(a, b \in \mathbb{R}\) such that \(b - a \in \mathbb{N}\). The backward jump operator \(\rho : \mathbb{N}_a \rightarrow \mathbb{N}_a\) is defined by \(\rho(t) = \max\{a, t - 1\}\) for all \(t \in \mathbb{N}_a\).

**Definition 2.1** \([21]\). Define the \(\mu\)-th order nabla fractional Taylor monomial by

\[
H_\mu(t, a) = \frac{(t-a)^\mu}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}, \quad \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}.
\]

Here \(\Gamma(\cdot)\) denotes the Euler gamma function. Observe that

\[H_\mu(a, a) = 0\]

and

\[H_\mu(t, a) = 0, \quad \text{for all } \mu \in \{\ldots, -2, -1\} \text{ and } t \in \mathbb{N}_a.\]
The first order backward (nabla) difference of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by $\left( \nabla u \right)(t) = u(t) - u(t - 1)$ for $t \in \mathbb{N}_{a+1}$.

**Definition 2.2** ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The $\nu$th-order nabla sum of $u$ based at $a$ is given by

$$\left( \nabla_a^{-\nu} u \right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left( \nabla_a^{-\nu} u \right)(a) = 0$.

**Definition 2.3** ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The $\nu$th-order nabla difference of $u$ is given by

$$\left( \nabla_a^{\nu} u \right)(t) = \left( \nabla \left( \nabla_a^{-(1-\nu)} u \right) \right)(t), \quad t \in \mathbb{N}_{a+1}.$$

**Lemma 2.1** ([21]). We observe the following properties of nabla fractional Taylor monomials:

(a) $\nabla H_\mu(t, a) = H_{\mu-1}(t, a), \quad t \in \mathbb{N}_{a};$

(b) $\sum_{s=a+1}^{t} H_\mu(s, a) = H_{\mu+1}(t, a), \quad t \in \mathbb{N}_{a};$

(c) $\sum_{s=a+1}^{t} H_\mu(t, \rho(s)) = H_{\mu+1}(t, a), \quad t \in \mathbb{N}_{a}.$

**Proposition 2.1** ([24]). Let $s \in \mathbb{N}_a$ and $-1 < \mu$. The following properties hold.

(a) $H_\mu(t, \rho(s)) \geq 0$ for $t \in \mathbb{N}_{\rho(s)}$ and $H_\mu(t, \rho(s)) > 0$ for $t \in \mathbb{N}_s$.

(b) $H_\mu(t, \rho(s))$ is a decreasing function with respect to $s$ for $t \in \mathbb{N}_{\rho(s)}$ and $\mu \in (0, \infty)$.

(c) If $t \in \mathbb{N}_s$ and $\mu \in (-1, 0)$, then $H_\mu(t, \rho(s))$ is an increasing function of $s$.

(d) $H_\mu(t, \rho(s))$ is a non-decreasing function with respect to $t$ for $t \in \mathbb{N}_{\rho(s)}$ and $\mu \in [0, \infty)$.

(e) If $t \in \mathbb{N}_s$ and $\mu \in (0, \infty)$, then $H_\mu(t, \rho(s))$ is an increasing function of $t$.

(f) $H_\mu(t, \rho(s))$ is a decreasing function with respect to $t$ for $t \in \mathbb{N}_{s+1}$ and $\mu \in (-1, 0)$.

**Proposition 2.2** ([24]). Let $u$ and $v$ be two nonnegative real-valued functions defined on a set $S$. Further, assume $u$ and $v$ achieve their maximum values in $S$. Then,

$$|u(t) - v(t)| \leq \max\{u(t), v(t)\} \leq \max\{\max_{t \in S} u(t), \max_{t \in S} v(t)\},$$

for every fixed $t$ in $S$.

3. **Green’s Function and Its Property**

Assume $T \in \mathbb{N}_2$, $1 < \alpha < 2$ and $h : \mathbb{N}_2^T \to \mathbb{R}$. Consider the boundary value problem

$$\begin{align*}
&\left( \nabla_0^{\alpha-1} \nabla u \right)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T, \\
&u(0) + u(T) = 0, \quad \left( \nabla u \right)(1) + \left( \nabla u \right)(T) = 0.
\end{align*}$$

(3.1)
First, we construct the Green’s function, \( G(t,s) \) corresponding to (3.1), and obtain an expression for its unique solution. Denote by
\[
D_1 = \{ (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \geq s \}, \quad D_2 = \{ (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \leq \rho(s) \}
\]
and
\[
\xi_\alpha = 2 \left[ 1 + H_{\alpha-2}(T,0) \right].
\]

**Theorem 3.1.** The unique solution of the nabla fractional boundary value problem (3.1) is given by
\[
u(t) = \sum_{s=2}^{T} G_{\alpha}(t,s)h(s), \quad t \in \mathbb{N}_0^T,
\]
where
\[
G_{\alpha}(t,s) = \begin{cases} 
K_{\alpha}(t,s) - H_{\alpha-1}(t,\rho(s)), & (t,s) \in D_1, \\
K_{\alpha}(t,s), & (t,s) \in D_2.
\end{cases}
\]
Here
\[
K_{\alpha}(t,s) = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) - H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \right].
\]

**Proof.** Denote by
\[
(\nabla u)(t) = v(t), \quad t \in \mathbb{N}_1^T.
\]
Subsequently, the difference equation in (3.1) takes the form
\[
(\nabla_{0,1}^{-1} v)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T.
\]
Let \( v(1) = c_2 \). Then, by Lemma 5.1 of [4], the unique solution of (3.5) is given by
\[
v(t) = H_{\alpha-2}(t,0)c_2 - (\nabla_{1}^{-\alpha-1} h)(t), \quad t \in \mathbb{N}_1^T.
\]
That is,
\[
(\nabla u)(t) = H_{\alpha-2}(t,0)c_2 - (\nabla_{1}^{-\alpha-1} h)(t), \quad t \in \mathbb{N}_1^T.
\]
Applying the first order nabla sum operator, \( \nabla_{0,1}^{-1} \) on both sides of (3.6), we obtain
\[
u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - (\nabla_{1}^{-\alpha} h)(t), \quad t \in \mathbb{N}_0^T,
\]
where \( c_1 = u(0) \). We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants \( c_1 \) and \( c_2 \) in (3.7). It follows from the first boundary condition \( u(0) + u(T) = 0 \) that
\[
2c_1 + H_{\alpha-1}(T,0)c_2 = (\nabla_{1}^{-\alpha} h)(T).
\]
The second boundary condition \( (\nabla u)(1) + (\nabla u)(T) = 0 \) yields
\[
[1 + H_{\alpha-2}(T,0)]c_2 = (\nabla_{1}^{-\alpha-1} h)(T).
\]
Solving (3.8) and (3.9) for \( c_1 \) and \( c_2 \), we obtain

\[
(3.10) \quad c_1 = \frac{1}{2} \left[ \sum_{s=2}^{T} H_{\alpha-1}(T, \rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_{\alpha}} \sum_{s=2}^{T} H_{\alpha-2}(T, \rho(s))h(s) \right],
\]

\[
(3.11) \quad c_2 = \frac{2}{\xi_{\alpha}} \sum_{s=2}^{T} H_{\alpha-2}(T, \rho(s))h(s).
\]

Substituting these expressions in (3.7), we achieve (3.4).

\[\square\]

**Lemma 3.1.** Observe that

\[
(3.12) \quad |K_{\alpha}(t,s)| \leq \frac{1}{\xi_{\alpha}} \left[ H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1) \right],
\]

for all \((t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T\).

**Proof.** Denote by

\[
(3.13) \quad K'_{\alpha}(t,s) = \frac{1}{\xi_{\alpha}} \left[ H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T, \rho(s)) 
+ H_{\alpha-1}(T, \rho(s))H_{\alpha-2}(T,0) \right]
\]

and

\[
(3.14) \quad K''_{\alpha}(t,s) = \frac{1}{\xi_{\alpha}} \left[ H_{\alpha-1}(T,0)H_{\alpha-2}(T, \rho(s)) \right],
\]

so that

\( K_{\alpha}(t,s) = K'_{\alpha}(t,s) - K''_{\alpha}(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \)

Clearly, from Proposition 2.1,

\( K'_{\alpha}(t,s) \geq 0, \quad K''_{\alpha}(t,s) > 0, \quad \text{for all} \ (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \)

From Proposition 2.2, it is obvious that

\[
(3.15) \quad |K_{\alpha}(t,s)| \leq \left\{ \max_{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K'_{\alpha}(t,s), \max_{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K''_{\alpha}(t,s) \right\}.
\]

First, we evaluate the first backward difference of \( K'_{\alpha}(t,s) \) with respect to \( t \) for a fixed \( s \). Consider

\[
\nabla K'_{\alpha}(t,s) = \frac{1}{\xi_{\alpha}} \left[ 2H_{\alpha-2}(t,0)H_{\alpha-2}(T, \rho(s)) \right] > 0,
\]

for all \((t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T\), implying that \( K'_{\alpha}(t,s) \) is an increasing function of \( t \) for a fixed \( s \). Thus, we have

\[
(3.16) \quad K'_{\alpha}(t,s) \leq K'_{\alpha}(T,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.
\]
It follows from (3.13)–(3.16) that
\[
\left| K_\alpha(t, s) \right| \leq \left\{ \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_0^T} K'_\alpha(t, s), \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_0^T} K''_\alpha(t, s) \right\} \\
\leq \left\{ \max_{s \in \mathbb{N}_0^T} K'_\alpha(T, s), \max_{s \in \mathbb{N}_0^T} K''_\alpha(t, s) \right\}
\]
\[
= \max_{s \in \mathbb{N}_0^T} K'_\alpha(T, s)
\]
\[
= \frac{1}{\xi_\alpha} \max_{s \in \mathbb{N}_0^T} \left[ H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(T, 0)H_{\alpha-2}(T, \rho(s)) + H_{\alpha-2}(T, 0) \right]
\]
\[
\leq \frac{1}{\xi_\alpha} \left[ \max_{s \in \mathbb{N}_0^T} H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(T, 0) \max_{s \in \mathbb{N}_0^T} H_{\alpha-2}(T, \rho(s)) + H_{\alpha-2}(T, 0) \right]
\]
\[
= \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T, 2) + 2H_{\alpha-1}(T, 0)H_{\alpha-2}(T, \rho(T)) + H_{\alpha-2}(T, 0)H_{\alpha-1}(T, 1) \right].
\]

The proof is complete. \(\square\)

4. Existence and Uniqueness of Solutions of (1.1)

Let \(X = \mathbb{R}^{T+1}\) be the Banach space of all real \((T + 1)\)-tuples equipped with the maximum norm
\[
\|u\|_X = \max_{t \in \mathbb{N}_0^T} |u(t)|.
\]

Obviously, the product space \((X \times X, \| \cdot \|_{X \times X})\) is also a Banach space with the norm
\[
\|(u_1, u_2)\|_{X \times X} = \|u_1\|_X + \|u_2\|_X.
\]

A closed ball with radius \(R\) centred on the zero function in \(X \times X\) is defined by
\[
\mathcal{B}_R = \{(u_1, u_2) \in X \times X : \|(u_1, u_2)\|_{X \times X} \leq R\}.
\]

Define the operator \(T : X \times X \to X \times X\) by
\[
(4.1) \quad T(u_1, u_2)(t) = \begin{pmatrix} T_1(u_1, u_2)(t) \\ T_2(u_1, u_2)(t) \end{pmatrix}, \quad t \in \mathbb{N}_0^T.
\]
where
\[ T_1(u_1, u_2)(t) = \sum_{s=2}^{T} G_{\alpha_1}(t, s)f_1(s, u_1(s), u_2(s)) \]

(4.2) \[ = \sum_{s=2}^{T} K_{\alpha_1}(t, s)f_1(s, u_1(s), u_2(s)) - \sum_{s=2}^{T} H_{\alpha_1-1}(t, s)f_1(s, u_1(s), u_2(s)) \]

and
\[ T_2(u_1, u_2)(t) = \sum_{s=2}^{T} G_{\alpha_2}(t, s)f_2(s, u_1(s), u_2(s)) \]

(4.3) \[ = \sum_{s=2}^{T} K_{\alpha_2}(t, s)f_2(s, u_1(s), u_2(s)) - \sum_{s=2}^{T} H_{\alpha_2-1}(t, s)f_2(s, u_1(s), u_2(s)). \]

Clearly, \((u_1, u_2)\) is a fixed point of \(T\) if and only if \((u_1, u_2)\) is a solution of (1.1). For our convenience, denote by
\[ \Lambda_i = \frac{1}{\xi_{\alpha_i}} \left[ H_{\alpha_i-1}(T, 1) + 2H_{\alpha_i-1}(T, 0) + H_{\alpha_i-2}(T, 0)H_{\alpha_i-1}(T, 1) \right], \]

(4.4) \[ a_i = \xi_i \left[ \Lambda_i(T-1) + H_{\alpha_i}(T, 1) \right], \]

(4.5) \[ b_i = \xi_i \left[ \Lambda_i(T-1) + H_{\alpha_i}(T, 1) \right], \]

(4.6) \[ c_i = \xi_i \left[ \Lambda_i(T-1) + H_{\alpha_i}(T, 1) \right], \]

(4.7) \[ d_i = M_i \left[ \Lambda_i(T-1) + H_{\alpha_i}(T, 1) \right], \]

(4.8)  for \( i = 1, 2 \). Assume

(H1) for each \( i \in \{1, 2\} \), there exist nonnegative numbers \( l_i \) and \( m_i \) such that
\[ |f_i(t, u_1, u_2) - f_i(t, v_1, v_2)| \leq l_i\|u_1 - v_1\|_X + m_i\|u_2 - v_2\|_X, \]

for all \((t, u_1, u_2), (t, v_1, v_2) \in \mathbb{N}_0^T \times X \times X;\)

(H1) for each \( i \in \{1, 2\} \), there exist nonnegative numbers \( l_i \) and \( m_i \) such that
\[ |f_i(t, u_1, u_2) - f_i(t, v_1, v_2)| \leq l_i\|u_1 - v_1\|_X + m_i\|u_2 - v_2\|_X, \]

for all \((t, u_1, u_2), (t, v_1, v_2) \in \mathbb{N}_0^T \times X \times X;\)

(H2)' for each \( i \in \{1, 2\} \), there exist nonnegative numbers \( L_i \) such that
\[ |f_i(t, u_1, u_2)| \leq L_i, \]

for all \((t, u_1, u_2) \in \mathbb{N}_0^T \times X \times X;\)

(H2) for each \( i \in \{1, 2\} \), there exist nonnegative numbers \( l_i, m_i, \) and \( n_i \) such that
\[ |f_i(t, u_1, u_2)| \leq l_i\|u_1\|_X + m_i\|u_2\|_X + n_i, \]

for all \((t, u_1, u_2) \in \mathbb{N}_0^T \times \mathbb{B}_R;\)

(H3) for each \( i \in \{1, 2\} \),
\[ \max_{t \in \mathbb{N}_0^T} |f_i(t, 0, 0)| = M_i; \]

(H4) \( \lambda = (a_1 + a_2) + (b_1 + b_2) \in (0, 1). \)
We apply Banach’s fixed point theorem to establish existence and uniqueness of solutions of (1.1).

**Theorem 4.1 ([37]).** Let $S$ be a closed subset of a Banach space $X$. Then, any contraction mapping $T$ of $X$ into itself has a unique fixed point.

**Theorem 4.2.** Assume (H1), (H3) and (H4) hold. If we choose

$$R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]},$$

then the system (1.1) has a unique solution $(u_1, u_2) \in \mathcal{B}_R$.

**Proof.** Clearly, $T : \mathcal{B}_R \rightarrow X \times X$. First, we show that $T$ is a contraction mapping. To see this, let $(u_1, u_2), (v_1, v_2) \in \mathcal{B}_R$, and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$|T_i(u_1, u_2)(t) - T_i(v_1, v_2)(t)|$$

$$\leq \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))|$$

$$+ \sum_{s=2}^{t} H_{\alpha_i-1}(t, s) |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))|$$

$$\leq |u_1 - v_1|_X + m_i |u_2 - v_2|_X \left[ \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t, s) \right]$$

$$\leq |u_1 - v_1|_X + m_i |u_2 - v_2|_X \left[ |A_i(T - 1) + H_{\alpha_i}(t, 1)| \right]$$

$$\leq |u_1 - v_1|_X + m_i |u_2 - v_2|_X \left[ |A_i(T - 1) + H_{\alpha_i}(T, 1)| \right]$$

$$\leq a_i |u_1 - v_1|_X + b_i |u_2 - v_2|_X,$$

implying that, for each $i \in \{1, 2\}$,

$$\left\| T_i(u_1, u_2) - T_i(v_1, v_2) \right\|_X \leq \left[ a_i |u_1 - v_1|_X + b_i |u_2 - v_2|_X \right].$$

Thus, we have

$$\left\| T(u_1, u_2) - T(v_1, v_2) \right\|_{X \times X}$$

$$= \left\| T_1(u_1, u_2) - T_1(v_1, v_2) \right\|_X + \left\| T_2(u_1, u_2) - T_2(v_1, v_2) \right\|_X$$

$$\leq [(a_1 + a_2) |u_1 - v_1|_X + (b_1 + b_2) |u_2 - v_2|_X]$$

$$\leq \lambda \left( |u_1 - v_1|_X + |u_2 - v_2|_X \right)$$

$$= \lambda \left( (u_1, u_2) - (v_1, v_2) \right)_{X \times X}.$$

Since $\lambda < 1$, $T$ is a contraction mapping with contraction constant $\lambda$. Next, we show that

$$T : \mathcal{B}_R \rightarrow \mathcal{B}_R.$$
To see this, let \((u_1, u_2) \in B_R\), and \(t \in \mathbb{N}_0^T\). For each \(i \in \{1, 2\}\), consider

\[
|T_i(u_1, u_2)(t)| \leq \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| \left| f_i(s, u_1(s), u_2(s)) \right| + \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) \left| f_i(s, u_1(s), u_2(s)) \right| \\
\leq \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| \left| f_i(s, u_1(s), u_2(s)) - f_i(s, 0, 0) \right| + \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| \left| f_i(s, 0, 0) \right| \\
+ \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) \left| f_i(s, u_1(s), u_2(s)) - f_i(s, 0, 0) \right| + \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) \left| f_i(s, 0, 0) \right| \\
\leq \left[ l_i \|u_1\|_X + m_i \|u_2\|_X \right] \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| + M_i \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| \\
+ \left[ l_i \|u_1\|_X + m_i \|u_2\|_X \right] \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) + M_i \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) \\
\leq \left[ l_i \|u_1\|_X + m_i \|u_2\|_X + M_i \right] [\Lambda_i(T-1) + H_{\alpha_i}(1, t)] \\
\leq \left[ l_i \|u_1\|_X + m_i \|u_2\|_X + M_i \right] [\Lambda_i(T-1) + H_{\alpha_i}(T, 1)] \\
\leq a_i \|u_1\|_X + b_i \|u_2\|_X + d_i,
\]

implying that, for each \(i \in \{1, 2\}\),

\[(4.11) \quad \|T_i(u_1, u_2)\|_X \leq a_i \|u_1\|_X + b_i \|u_2\|_X + d_i.\]

Thus, we have

\[
\|T(u_1, u_2)\|_{X \times X} = \|T_1(u_1, u_2)\|_X + \|T_2(u_1, u_2)\|_X \\
\leq (a_1 + a_2)R + (b_1 + b_2)R + (d_1 + d_2) \leq R,
\]

implying that (4.10) holds. Therefore, by Theorem 4.1, \(T\) has a unique fixed point \((u_1, u_2) \in B_R\). The proof is complete. \(\square\)

**Corollary 4.1.** Assume (H1)' and (H4) hold. Then, the system (1.1) has a unique solution \((u_1, u_2) \in X \times X\).

We apply Brouwer’s fixed point theorem to establish existence of solutions of (1.1).

**Theorem 4.3** ([37]). Let \(C\) be a non-empty bounded closed convex subset of \(\mathbb{R}^n\) and \(T : C \to C\) be a continuous mapping. Then, \(T\) has a fixed point in \(C\).

**Theorem 4.4.** Assume (H2) and (H4) hold. If we choose

\[
R \geq \frac{(c_1 + c_2)}{1 - \left[(a_1 + a_2) + (b_1 + b_2)\right]},
\]

then the system (1.1) has at least one solution \((u_1, u_2) \in B_R\).
Thus, we have

\[
\varepsilon \text{ is said to be Ulam-Hyers stable if there exist }
\]

\[
\text{operators. Then, the operational equations system }
\]

\[
\text{Definition 4.1}
\]

\[
\text{of a pair of contractive type operators on complete metric spaces. We use Urs’s [39]}
\]

\[
\text{fixed point}
\]

\[
\text{implying that, for each } i \in \{1, 2\},
\]

\[
\text{Proof. We claim that } T : B_R \to B_R. \text{ To see this, let } (u_1, u_2) \in B_R \text{ and } t \in \mathbb{N}_0^T. \text{ For}
\]

\[
\text{each } i \in \{1, 2\}, \text{ consider}
\]

\[
\left| T_i(u_1, u_2)(t) \right| \leq \sum_{s=2}^{T} |K_{\alpha_i}(t, s)| |f_i(s, u_1(s), u_2(s))| + \sum_{s=2}^{T} H_{\alpha_i-1}(t, s) |f_i(s, u_1(s), u_2(s))|
\]

\[
\leq \left[ l_i \|u_1\|_X + m_i\|u_2\|_X + n_i \right] \sum_{s=2}^{T} |K_{\alpha_i}(t, s)|
\]

\[
+ \left[ l_i \|u_1\|_X + m_i\|u_2\|_X + n_i \right] \sum_{s=2}^{T} H_{\alpha_i-1}(t, s)
\]

\[
\leq \left[ l_i \|u_1\|_X + m_i\|u_2\|_X + n_i \right] [\Lambda_i(T - 1) + H_{\alpha_i}(t, 1)]
\]

\[
\leq \left[ l_i \|u_1\|_X + m_i\|u_2\|_X + n_i \right] [\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)]
\]

\[
\leq a_i \|u_1\|_X + b_i\|u_2\|_X + c_i,
\]

implying that, for each \( i \in \{1, 2\}, \)

\[
\|T_i(u_1, u_2)\|_X \leq a_i \|u_1\|_X + b_i\|u_2\|_X + c_i.
\]

Thus, we have

\[
\|T(u_1, u_2)\|_{X \times X} = \|T_1(u_1, u_2)\|_X + \|T_2(u_1, u_2)\|_X 
\leq (a_1 + a_2)R + (b_1 + b_2)R + (c_1 + c_2) \leq R,
\]

implying that \( T : B_R \to B_R. \) Therefore, by Brouwer’s fixed point theorem, \( T \) has a

\[
\text{fixed point } (u_1, u_2) \in B_R. \text{ The proof is complete.}\]

\[
\square
\]

\[
\text{Corollary 4.2. Assume (H2)’ hold. Then, the system (1.1) has at least one solution}
\]

\[
(u_1, u_2) \in X \times X.
\]

Urs [39] presented some Ulam-Hyers stability results for the coupled fixed point of a pair of contractive type operators on complete metric spaces. We use Urs’s [39] approach to establish Ulam-Hyers stability of solutions of (1.1).

\[
\text{Definition 4.1 ([39]). Let } X \text{ be a Banach space and } T_1, T_2 : X \times X \to X \text{ be two operators. Then, the operational equations system}
\]

\[
\left\{ \begin{array}{l}
  u_1 = T_1(u_1, u_2), \\
  u_2 = T_2(u_1, u_2),
\end{array} \right.
\]

is said to be Ulam-Hyers stable if there exist \( C_1, C_2, C_3, C_4 > 0 \) such that for each \( \varepsilon_1, \varepsilon_2 > 0 \) and each solution-pair \( (u_1^*, u_2^*) \in X \times X \) of the in-equations:

\[
\left\{ \begin{array}{l}
  \|u_1 - T_1(u_1, u_2)\|_X \leq \varepsilon_1, \\
  \|u_2 - T_2(u_1, u_2)\|_X \leq \varepsilon_2,
\end{array} \right.
\]
there exists a solution \((v^*_1, v^*_2) \in X \times X\) of (4.13) such that

\[
\begin{align*}
\|u^*_1 - v^*_1\|_X &\leq C_1\varepsilon_1 + C_2\varepsilon_2, \\
\|u^*_2 - v^*_2\|_X &\leq C_3\varepsilon_1 + C_4\varepsilon_2.
\end{align*}
\]

**Theorem 4.5** ([39]). Let \(X\) be a Banach space, \(T_1, T_2 : X \times X \to X\) be two operators such that

\[
\begin{align*}
\|T_1(u_1, u_2) - T_1(v_1, v_2)\|_X &\leq k_1\|u_1 - v_1\|_X + k_2\|u_2 - v_2\|_X, \\
\|T_2(u_1, u_2) - T_2(v_1, v_2)\|_X &\leq k_3\|u_1 - v_1\|_X + k_4\|u_2 - v_2\|_X,
\end{align*}
\]

for all \((u_1, u_2), (v_1, v_2) \in X \times X\). Suppose

\[
H = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}
\]

converges to zero. Then, the operational equations system (4.13) is Ulam-Hyers stable.

Set

\[
H = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.
\]

**Theorem 4.6.** Assume the hypothesis of Theorem 4.2 holds. Further, assume the spectral radius of \(H\) is less than one. Then, the unique solution of the system (1.1) is Ulam-Hyers stable.

**Proof.** In view of Theorem 4.2, we have

\[
\begin{align*}
\|T_1(u_1, u_2) - T_1(v_1, v_2)\|_X &\leq a_1\|u_1 - v_1\|_X + b_1\|u_2 - v_2\|_X, \\
\|T_2(u_1, u_2) - T_2(v_1, v_2)\|_X &\leq a_2\|u_1 - v_1\|_X + b_2\|u_2 - v_2\|_X,
\end{align*}
\]

which implies that

\[
\|T(u_1, u_2) - T(v_1, v_2)\|_{X \times X} \leq H \left( \frac{\|u_1 - v_1\|_X}{\|u_2 - v_2\|_X} \right).
\]

Since the spectral radius of \(H\) is less than one, the unique solution of (1.1) is Ulam-Hyers stable. The proof is complete.

5. **Examples**

In this section, we provide two examples to illustrate the applicability of Theorem 4.2, Theorem 4.4 and Theorem 4.6.
Example 5.1. Consider the following coupled system of two-point nabla fractional difference boundary value problems

\[(5.1)\]
\[
\begin{cases}
\left(\nabla_0^{0.5}(\nabla u_1)\right)(t) + (0.001)e^{-t}[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)] = 0, \quad t \in \mathbb{N}_2^0, \\
\left(\nabla_0^{0.5}(\nabla u_2)\right)(t) + (0.002)[e^{-t} + \sin u_1(t) + \sin u_2(t)] = 0, \quad t \in \mathbb{N}_2^0, \\
u_1(0) + u_1(9) = 0, \quad \left(\nabla u_1\right)(1) + \left(\nabla u_1\right)(9) = 0, \\
u_2(0) + u_2(9) = 0, \quad \left(\nabla u_2\right)(1) + \left(\nabla u_2\right)(9) = 0.
\end{cases}
\]

Comparing (1.1) and (5.1), we have $T = 9$, $\alpha_1 = \alpha_2 = 1.5$,

\[f_1(t, u_1, u_2) = (0.001)e^{-t}\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right],\]

and

\[f_2(t, u_1, u_2) = (0.002)\left[e^{-t} + \sin u_1 + \sin u_2\right],\]

for all $(t, u_1, u_2) \in \mathbb{N}_2^0 \times \mathbb{R}^2$. Clearly, $f_1$ and $f_2$ are continuous on $\mathbb{N}_2^0 \times \mathbb{R}^2$. Next, $f_1$ and $f_2$ satisfy assumption (H1) with $l_1 = 0.001$, $m_1 = 0.001$, $l_2 = 0.002$ and $m_2 = 0.002$. We have

\[M_1 = \max_{t \in \mathbb{N}_2^0} |f_1(t, 0, 0)| = 0.001,\]

\[M_2 = \max_{t \in \mathbb{N}_2^0} |f_2(t, 0, 0)| = 0.002,\]

\[a_1 = l_1 [A_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.0527,\]

\[a_2 = l_2 [A_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.1053,\]

\[b_1 = m_1 [A_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.0527,\]

\[b_2 = m_2 [A_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.1053,\]

\[d_1 = M_1 [A_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.0527,\]

\[d_2 = M_2 [A_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1053.\]

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.316 \in (0, 1)$, implying that assumptions (H3) and (H4) hold. Choose

\[R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]]} = 0.231.\]

Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1, u_2) \in \mathcal{B}_R$. Further,

\[M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.0527 & 0.0527 \\ 0.1053 & 0.1053 \end{pmatrix}.\]

The spectral radius of $M$ is 0.158, which is less than one, implying that $M$ converges to zero. Thus, by Theorem 4.6, the unique solution of (5.1) is Ulam-Hyers stable.
Example 5.2. Consider the following coupled system of two-point nabla fractional difference boundary value problems

\[
\begin{align*}
(5.2) \quad & \begin{cases}
(\nabla_0^{0.5} \nabla u_1)(t) + (0.01) & e^{-t} + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t) = 0, \quad t \in \mathbb{N}_2^4, \\
(\nabla_0^{0.5} \nabla u_2)(t) + (0.02) & e^{-t} + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}} = 0, \quad t \in \mathbb{N}_2^4, \\
u_1(0) + u_1(4) & = 0, \quad (\nabla u_1)(1) + (\nabla u_1)(4) = 0, \\
u_2(0) + u_2(4) & = 0, \quad (\nabla u_2)(1) + (\nabla u_2)(4) = 0.
\end{cases}
\end{align*}
\]

Comparing (1.1) and (5.2), we have $T = 4$, $\alpha_1 = \alpha_2 = 1.5$,

\[
f_1(t, u_1, u_2) = (0.01) \left[ e^{-t} + \frac{1}{\sqrt{1+u_1^2}} + u_2 \right],
\]

and

\[
f_2(t, u_1, u_2) = (0.02) \left[ e^{-t} + u_1 + \frac{1}{\sqrt{1+u_2^2}} \right],
\]

for all $(t, u_1, u_2) \in \mathbb{N}_0^4 \times \mathbb{R}^2$. Clearly, $f_1$ and $f_2$ are continuous on $\mathbb{N}_0^4 \times \mathbb{R}^2$. Next, $f_1$ and $f_2$ satisfy assumption (H2) with $l_1 = 0.01$, $m_1 = 0.01$, $l_2 = 0.02$, $m_2 = 0.02$, $n_1 = 0.01$ and $n_2 = 0.02$. We have

\[
a_1 = l_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
a_2 = l_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438, \\
b_1 = m_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
b_2 = m_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438, \\
c_1 = n_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
c_2 = n_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438.
\]

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.7314 \in (0, 1)$, implying that assumption (H4) hold. Choose

\[
R \geq \frac{(c_1 + c_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]} = 1.3615.
\]

Hence, by Theorem 4.2, the system (5.1) has at least one solution $(u_1, u_2) \in \mathcal{B}_R$.

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