Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 7–19.

(ω, c)-ALMOST PERIODIC DISTRIBUTIONS

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Abstract. The aim of this work is the introduction of (w, c)-almost periodicity (resp. asymptotic (w, c)-almost periodicity) in distributions spaces. The characterizations and main properties of these distributions are given. We also study the existence of distributional (w, c)-almost periodic solutions of linear differential systems.

1. Introduction

The theory of almost periodicity was introduced by H. Bohr around 1925 and generalized by many other authors, see [3, 5].

The (ω, c)-almost periodicity of continuous functions and their Stepanov generalizations is introduced and studied recently by M. T. Khalladi, M. Kostić, A. Rahmani and D. Velinov.

Almost periodic distributions extending the classical Bohr and Stepanoff almost periodic functions are due to L. Schwartz, see [9]. Asymptotic almost periodicity of Schwartz distributions was introduced by I. Cioransescu [4].

This work is aimed to introduce and investigate (ω, c)-almost periodicity (resp. asymptotic (w, c)-almost periodicity) in the setting of Schwartz-Sobolev distributions.

The paper is organized as follows. In the second section, we recall the concept of (w, c)-almost periodicity which is a generalization of the classical notion of almost periodicity and give some of their fundamental properties. Next, we introduce the

Key words and phrases. (w, c)-Almost periodic functions, almost periodic Schwartz distributions, (w, c)-almost periodic distributions, asymptotically (w, c)-almost periodic distributions, linear differential systems.

2010 *Mathematics Subject Classification.* Primary: 46F05, 46E30, 28A20. Secondary: 34C25, 42A75.

Received: May 18, 2020.

Accepted: June 10, 2020.

space $L^p_{w,c}$ of (w,c)-Lebesgue functions with exponent p, and then, in a similar way to L. Schwartz's work [9], we define the functional space $\mathcal{D}_{L^p_{w,c}}$ of all infinitely differentiable functions belonging to the space $L^p_{w,c}$ as well as each of their derivatives. Some properties of these spaces of (w,c)-functions are given. At the end of this section, we introduce the space of (w,c)-smooth almost periodic functions and analyze their basic properties. The third section is devoted to the study of (w,c)-almost periodic distributions (resp. asymptotically (w,c)-almost periodic distributions) by first defining the space $\mathcal{D}'_{L^p_{w,c}}$ as topological dual of $\mathcal{D}_{L^q_{w,c}}$, $\frac{1}{p}+\frac{1}{q}=1$. In particular, we study the space $\mathcal{B}'_{w,c}$ of (w,c)-bounded distributions. This space provides a general framework for our investigation of generalized (ω,c)-almost periodicity. We also give some characterizations of (w,c)-almost periodic distributions and their main properties. Finally, we apply our abstract theoretical results in the study of the existence of distributional (w,c)-almost periodic solutions of linear differential systems. Throughout the paper, we consider functions and distributions defined on the whole space of real numbers $\mathbb{R}$.

2. Smooth (w,c)-Almost Periodic Functions

In this section, we introduce the space of smooth (w,c)-almost periodic functions and investigate some of their basic properties. Denote by AP the well-known space of Bohr almost periodic functions on $\mathbb{R}$. We recall the definition and some properties of the space $AP_{w,c}$ of (ω,c)- almost periodic functions.

In the sequel we will use the following notations:

$$\varphi_{w,c}(\cdot)=c^{-\frac{(\cdot)}{w}}\varphi(\cdot),\ \varphi\in\mathcal{C}^\infty \text{ or } L^p, 1\leq p\leq+\infty, \text{ and } T_{w,c}=c^{-\frac{(\cdot)}{w}}T,\ T\in\mathcal{D}', \tag{2.1}$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

Definition 2.1. Let $c\in\mathbb{C}\backslash\{0\}$ and $w>0$. A complex-valued function f defined and continuous on $\mathbb{R}$ is called (w,c)-almost periodic if and only if $f_{\omega,c}\in AP$. Denote by $AP_{w,c}$ the set of all such functions.

When $c=1$ and $w>0$ arbitrary, $AP_{w,c}=AP$, the space of Bohr almost periodic functions.

The space $AP_{\omega,c}$ is a vector space together with the usual operations of addition and pointwise multiplication with scalars.

Some properties of (w,c)-almost periodic functions are summarized in the following proposition.

Proposition 2.1. (i) *The space $AP_{w,c}$ endowed with the (w,c)-norm*

$$\|f\|_{w,c}=\sup_{t\in\mathbb{R}}|f_{w,c}(t)|$$

is a Banach space.

(ii) *If $f\in AP_{w,c}$, then $\widetilde{f}(\cdot)=f(-\cdot)\in AP_{w,1/c}$.*

(iii) *If $w>0, c\in\mathbb{C}\backslash\{0\}$ such that $|c|=1$ and if $f\in AP_{w,c}$ such that $\inf_{x\in\mathbb{R}}|f(x)|>0$, then $1/f\in AP_{w,1/c}$.*

(iv) If $f \in AP_{w,c}$ *and* $g_{w,c} \in L^1$, *then* $f * g \in AP_{w,c}$.

To construct the (w, c)-smooth almost periodic functions, we need to introduce some new functional spaces. Let $p \in [1, +\infty]$ and f a complex valued measurable function on $\mathbb{R}$.

We say that f is a (w, c)-Lebesgue function with exponent p, if

$$\left(\int_{\mathbb{R}} |f_{w,c}(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \quad \text{for } 1 \leq p < +\infty,$$

and

$$\sup_{t \in \mathbb{R}} |f_{w,c}(t)| < \infty, \quad \text{for } p = +\infty.$$

We denote by $L^p_{w,c}$ the set of (w, c)-Lebesgue functions with exponent p, i.e.,

$$L^p_{w,c} := \{ f : \mathbb{R} \to \mathbb{C} \text{ measurable}, \ f_{w,c} \in L^p \}.$$

When $c = 1$, $L^p_{w,c} := L^p$ is the classical Lebesgue space over $\mathbb{R}$.

Proposition 2.2. *The space* $L^p_{w,c}$ *endowed with the* (w, c)*-norm*

$$\|f\|_{L^p_{w,c}} := \|f_{w,c}\|_{L^p}, \quad \textit{for } 1 \leq p < +\infty,$$

and

$$\|f\|_{L^\infty_{w,c}} := \|f\|_{w,c}, \quad \textit{for } p = +\infty,$$

is a Banach space.

Proposition 2.3. $\mathcal{D}$ *is dense in* $L^p_{w,c}$, $1 \leq p < \infty$.

Proof. Since $\mathcal{D}$ is dense in the space $\mathcal{C}_c$ of continuous functions with compact support it suffices to show that $\mathcal{C}_c$ is dense in $L^p_{w,c}$ for $1 \leq p < \infty$.

Let S be the set of all simple measurable functions s, with complex values, defined on $\mathbb{R}$ and such that

$$\text{mes}\{t : s(t) \neq 0\} < \infty.$$

First, it is clear that S is dense in $L^p_{w,c}$ for $1 \leq p < \infty$. Indeed, as $c^{-\frac{t}{w}} s \in L^p$, then $S \subset L^p_{w,c}$. Suppose $f \in L^p_{w,c}$ is positive and define the sequence $(s_n)_n$ such that $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, and for each $t \in \mathbb{R}$, $s_n(t) \to f(t)$ when $n \to +\infty$. Then $(f - s_n)_{w,c} = c^{-\frac{t}{w}} (f - s_n) \in L^p$, hence $s_n \in S$. Furthermore, since

$$\left| c^{-\frac{t}{w}} (f - s_n) \right|^p \leq f^p,$$

Lebesgue's dominated convergence theorem shows that

$$\left\| (f - s_n)_{w,c} \right\|_{L^p} = \left\| c^{-\frac{t}{w}} (f - s_n) \right\|_{L^p} \to 0,$$

when $n \to +\infty$. Hence, $\|f - s_n\|_{L^p_{w,c}} \to 0$ when $n \to +\infty$. On the other hand, by Lusin's theorem, for $s \in S$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}_c$ such that $g(t) = s(t)$, except on a set of measure less than ε and $|g| \leq \|s\|_\infty$ and since s takes only a finite number of values, there exists a constant $C > 0$ which depends on c and w such that

$$\left\|(g-s)_{w,c}\right\|_{L^p} = \left(\int_{\mathbb{R}} \left|c^{-\frac{t}{w}}\left(g\left(t\right)-s\left(t\right)\right)\right|^p dt\right)^{\frac{1}{p}} \leq 2C\varepsilon^{\frac{1}{p}} \left\|s\right\|_\infty .$$

The density of S in $L^p_{w,c}$ completes the proof. □

We define

$$\mathcal{D}_{L^p_{w,c}} := \left\{\varphi \in \mathcal{C}^\infty : \varphi_{w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{Z}_+\right\}.$$

When $c = 1$, we get $\mathcal{D}_{L^p_{w,c}} := \mathcal{D}_{L^p}$. Moreover, it is easy to show that the space $\mathcal{D}_{L^p_{w,c}}$, $1 \leq p \leq \infty$, endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,p;w,c} := \sum_{j\leq k} \left\|\left(\varphi_{w,c}\right)^{(j)}\right\|_{L^p}, \quad k \in \mathbb{Z}_+,$$

is a Fréchet subspace of $\mathcal{C}^\infty$.

Proposition 2.4. *Let $1 \leq p \leq \infty$. If $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$.*

Proof. Let $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$. Then $\varphi_{2w,c} \in \mathcal{D}_{L^p}$ and $\psi_{2w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{Z}_+$. So, $\varphi^{(j)}_{2w,c} \in L^p$ and $\psi^{(j)}_{2w,c} \in L^p$. By Leibniz's rule, we obtain

$$\left(\left(\varphi\psi\right)_{w,c}\right)^{(j)} = \left(c^{-\frac{t}{2w}}\varphi c^{-\frac{t}{2w}}\psi\right)^{(j)} = \left(\varphi_{2w,c}\psi_{2w,c}\right)^{(j)} = \sum_{i=1}^{j}\binom{i}{j}\varphi^{(i)}_{2w,c}\psi^{(j-i)}_{2w,c} \in L^p. \tag{2.2}$$

This shows that $\left(\varphi\psi\right)_{w,c} \in \mathcal{D}_{L^p}$. Hence, $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$. □

The following result shows that the family of norms $|\cdot|_{k,p;w,c}$ is submultiplicative.

Proposition 2.5. *Let $1 \leq p \leq \infty$. If $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then for all $k \in \mathbb{Z}_+$, there exists $C_k > 0$ such that*

$$|\varphi\psi|_{k,p;w,c} \leq C_k\,|\varphi|_{k,p;2w,c} \cdot |\psi|_{k,p;2w,c}\,.$$

Proof. Let $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$. By Proposition 2.2-Proposition 2.4, we have

$$\begin{aligned}
\sum_{j\leq k}\left\|\left(\left(\varphi\psi\right)_{w,c}\right)^{(j)}\right\|_{L^p} &= \sum_{j\leq k}\left\|\sum_{i=1}^{j}\binom{i}{j}\left(\varphi_{2w,c}\right)^{(i)}\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p} \\
&\leq \sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\varphi_{2w,c}\right)^{(i)}\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p} \\
&\leq \sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\varphi_{2w,c}\right)^{(i)}\right\|_{L^p}\sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p}.
\end{aligned}$$

So, there exists $C_k = \left(\sum\limits_{j\leq k}\sum\limits_{i=1}^{j}\binom{i}{j}\right)^2 > 0$ such that

$$|\varphi\psi|_{k,p;w,c} \leq C_k\,|\varphi|_{k,p;2w,c} \cdot |\psi|_{k,p;2w,c}\,.$$

□

For $1 \leq p < \infty$, we have $\mathcal{D} \subset \mathcal{D}_{L^p_{w,c}} \subset \mathcal{D}_{L^\infty_{w,c}}$. Moreover, we have the following result.

Proposition 2.6. *For $1 \leq p < \infty$, the space $\mathcal{D}$ is dense in $\mathcal{D}_{L^p_{w,c}}$.*

Proof. It follows from the fact that $\mathcal{D}_{L^p_{w,c}} \subset L^p_{w,c}$ and the density of $\mathcal{D}$ in $L^p_{w,c}$, see Proposition 2.3. □

The space $\mathcal{D}$ is not dense in $\mathcal{D}_{L^\infty_{w,c}}$, we then define $\dot{\mathcal{D}}_{L^\infty_{w,c}}$as the subspace of all functions in $\mathcal{D}_{L^\infty_{w,c}}$ which vanish at infinity with all their derivatives. This space is the closure of the space $\mathcal{D}_{L^\infty_{w,c}}$ in $\mathcal{D}$. It is clear that $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$, hence it is a Fréchet space. Moreover, it is easy to check the following properties on the structure of $\mathcal{D}_{L^p_{w,c}}$.

Proposition 2.7. *For $1 \leq p < \infty$, we have $\mathcal{D}_{L^p_{w,c}} \hookrightarrow \dot{\mathcal{D}}_{L^\infty_{w,c}} \hookrightarrow \mathcal{D}_{L^\infty_{w,c}}$, with continuous embedding.*

Recall also the following space of smooth almost periodic functions introduced by L. Schwartz

$$\mathcal{B}_{ap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP,\ j \in \mathbb{Z}_+ \right\}.$$

We have the following properties of $\mathcal{B}_{ap}$.

Proposition 2.8. (i) $\mathcal{B}_{ap} = AP \cap \mathcal{D}_{L^\infty}$.

(ii) $\mathcal{B}_{ap}$ is a closed differential subalgebra of $\mathcal{D}_{L^\infty}$.

(iii) If $f \in L^1$ and $\varphi \in \mathcal{B}_{ap}$, then $f * \varphi \in \mathcal{B}_{ap}$.

Proof. See [9]. □

Now, we can introduce the space of smooth (w, c)-almost periodic functions.

Definition 2.2. The space of smooth (w, c)-almost periodic functions on $\mathbb{R}$, is defined by

$$\mathcal{B}_{AP_{w,c}} := \left\{ \varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi_{w,c} \in \mathcal{B}_{ap},\ j \in \mathbb{Z}_+ \right\}.$$

We endow $\mathcal{B}_{AP_{w,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{w,c}}$. Some properties of $\mathcal{B}_{AP_{w,c}}$ are given in the following.

Proposition 2.9. (i) $\mathcal{B}_{AP_{w,c}} = AP_{w,c} \cap \mathcal{D}_{L^\infty_{w,c}}$.

(ii) $\mathcal{B}_{AP_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$.

(iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $c^{\frac{t}{w}} (f_{w,c} * \varphi_{w,c}) \in \mathcal{B}_{AP_{w,c}}$.

Proof. (i) Obvious.

(ii) It follows from (i) and the completeness of $(AP, \|\cdot\|_\infty)$.

(iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $f_{w,c} \in L^1$ and $\varphi_{w,c} \in \mathcal{B}_{ap}$. From Proposition 2.8, we have $f_{w,c} * \varphi_{w,c} \in \mathcal{B}_{ap}$, hence $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (f_{w,c} * \varphi_{w,c}) \right) \in \mathcal{B}_{ap}$, which shows that $c^{\frac{t}{w}} (f_{w,c} * \varphi_{w,c}) \in \mathcal{B}_{AP_{w,c}}$. □

Corollary 2.1. *If* $f \in \mathcal{D}_{L^{\infty}_{w,c}}$ *and* $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c}\right) \in AP_{w,c}$, $\varphi \in \mathcal{D}$, *then* $f \in \mathcal{B}_{AP_{w,c}}$.

Remark 2.1. It is clear that $\mathcal{B}_{AP_{w,c}} \subset AP_{w,c} \cap \mathcal{C}^{\infty}$, whereas the converse inclusion is not true. Indeed, the function

$$f\left(t\right) = 2^{-t}\sqrt{2+\cos t + \cos\sqrt{2}t}$$

is an element of $AP_{w,c} \cap \mathcal{C}^{\infty}$, with $c = 2$ and $w = 1$. However,

$$f'\left(t\right) = 2^{-t}\left(\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} - \ln 2\sqrt{2+\cos t+\cos\sqrt{2}t}\right)$$

is not bounded, because $\inf\limits_{t\in\mathbb{R}}\left(2+\cos t+\cos\sqrt{2}t\right) = 0$ and therefore

$$\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} \notin AP,$$

hence $f \notin \mathcal{B}_{AP_{w,c}}$.

3. (w,c)-Almost Periodic Distributions

This section deals with the concept of (w,c)-almost periodicity in the setting of Sobolev-Schwartz distributions. For this we need to introduce the so-called space of $L^p_{w,c}$-distributions, $1 \le p \le \infty$. We first recall the space of L^p-distributions, $1 \le p \le \infty$, which has been introduced for the first time by L. Schwartz in [9]. L. Schwartz has introduced the space $\mathcal{D}'_{L^p}$ as topological dual of $\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$. These spaces is related to Sobolev spaces. For more details, see [1] and [9].

Definition 3.1. Let $1 < p \le \infty$, the space $\mathcal{D}'_{L^p}$ is the topological dual of $\mathcal{D}_{L^q}$, where $\frac{1}{p}+\frac{1}{q}=1$. An element of $\mathcal{D}'_{L^\infty}$ is called a bounded distribution.

Theorem 3.1. *Let* $T \in \mathcal{D}'$. *Then the following statements are equivalent.*
(*i*) $T \in \mathcal{D}'_{L^p}$.
(*ii*) $T * \varphi \in L^p$, $\varphi \in \mathcal{D}$.
(*iii*) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0\le j\le k} \subset L^p : T = \sum\limits_{j=0}^{k} f_j^{(j)}$.

Proof. See [1] or [9]. □

Thanks to the density of the space $\mathcal{D}$ in $\mathcal{D}_{L^p_{w,c}}$, $1 \le p < \infty$, (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$), we have that the space $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) is a normal space of distributions, i.e., the elements of topological dual of $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) can be identified with continuous linear forms on $\mathcal{D}$.

Definition 3.2. For $1 < p \le \infty$, we denote by $\mathcal{D}'_{L^p_{w,c}}$ the topological dual of $\mathcal{D}_{L^q_{w,c}}$, where $\frac{1}{p}+\frac{1}{q}=1$.

The following spaces of $L^p_{w,c}$-distributions are needed to define and study the (w,c)-almost periodicity of distributions.

Definition 3.3. (i) The topological dual of $\mathcal{D}_{L^1_{w,c}}$, denoted by $\mathcal{B}'_{w,c}$, is called the space of (w,c)-bounded distributions.

(ii) The topological dual of $\dot{\mathcal{D}}_{L^\infty_{w,c}}$, denoted by $\mathcal{D}'_{L^1_{w,c}}$, is called the space of (w,c)-integrable distributions.

By applying Theorem 3.1, we can easily show the following characterizations of $L^p_{w,c}$-distributions.

Theorem 3.2. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent.*

(i) $T \in \mathcal{D}'_{L^p_{w,c}}$.

(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in L^p_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) *There exist $k \in \mathbb{Z}_+$ and $(f_j)_{0\le j\le k} \subset L^p_{w,c} : T = c^{\frac{t}{w}} \sum\limits_{j=0}^{k} (f_{w,c})^{(j)}_j$, where $\left((f_{w,c})_j\right)_{0\le j\le k} = \left(c^{-\frac{t}{w}} f_j\right)_{0\le j\le k}$.*

Remark 3.1. As a consequence of Theorem 3.2, we have that $T \in \mathcal{D}'_{L^p_{w,c}}$ if and only if $T_{w,c} \in \mathcal{D}'_{L^p}$.

Returning to the notation (2.1), we recall that a distribution $T \in \mathcal{D}'$ is zero on an open subset V of $\mathbb{R}$ if

$$\langle T, \varphi\rangle = 0, \quad \varphi \in \mathcal{D}(V),$$

and that two distributions $T, S \in \mathcal{D}'$ coincide on V if $T - S = 0$ on V.

Lemma 3.1. *Let $f \in \mathcal{C}^\infty$ and $T \in \mathcal{D}'$. If $fT = 0$, then $T = 0$ on the set $G = \{x \in \mathbb{R} : f(x) \neq 0\}$.*

Proof. Let $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subset G$. Then we have

$$\langle T, \varphi\rangle = \left\langle T, f\frac{\varphi}{f}\right\rangle = \left\langle fT, \frac{\varphi}{f}\right\rangle = 0,$$

because $\frac{\varphi}{f} \in \mathcal{D}$ and by hypothesis $fT = 0$. □

Proposition 3.1. *Let $T \in \mathcal{D}'$. Then $T \in \mathcal{D}'_{L^p_{w,c}}$, $1 \le p \le \infty$, if and only if there exists $S \in \mathcal{D}'_{L^p}$, $1 \le p \le \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.*

Proof. $(\Longrightarrow)$ If $T \in \mathcal{D}'_{L^p_{w,c}}$, then we have (see Remark 3.1) $T_{w,c} = c^{-\frac{t}{w}} T \in \mathcal{D}'_{L^p}$. So, there exists $S \in \mathcal{D}'_{L^p}$ such that $c^{-\frac{t}{w}} T - S = 0$ in $\mathcal{D}'_{L^p}$, i.e., $c^{-\frac{t}{w}}\left(T - c^{\frac{t}{w}} S\right) = 0$ in $\mathcal{D}'_{L^p}$. By applying Lemma 3.1, it follows that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.

$(\Longleftarrow)$ Suppose that $T \in \mathcal{D}'$ and there exists $S \in \mathcal{D}'_{L^p}$, $1 \le p \le \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$, then $c^{-\frac{t}{w}} T = S \in \mathcal{D}'_{L^p}$, hence $T \in \mathcal{D}'_{L^p_{w,c}}$. □

Recall that the space $\mathcal{B}'_{ap}$ of almost periodic distributions which was introduced and studied by L. Schwartz is based on the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h, denoted by $\tau_h T$, is defined as:

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle, \quad \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) = \varphi(x+h)$.

The following result gives the basic characterizations of Schwartz almost periodic distributions.

Theorem 3.3. *For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent.*

(i) *The set $\{\tau_h T : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.*

(ii) $T * \varphi \in AP$, $\varphi \in \mathcal{D}$.

(iii) *There exist $k \in \mathbb{Z}_+$ and $(f_j)_{0 \le j \le k} \subset AP : T = \sum_{j=0}^{k} f_j^{(j)}$.*

Proof. See [9]. □

The following proposition summarizes the main properties of $\mathcal{B}'_{ap}$.

Proposition 3.2. (i) *If $T \in \mathcal{B}'_{ap}$, then $T^{(j)} \in \mathcal{B}'_{ap}$, $j \in \mathbb{Z}_+$.*

(ii) $\mathcal{B}_{ap} \times \mathcal{B}'_{ap} \subset \mathcal{B}'_{ap}$.

(iii) $\mathcal{B}'_{ap} * \mathcal{D}'_{L^1} \subset \mathcal{B}'_{ap}$.

Proof. See [9]. □

Now we will introduce the following concept.

Definition 3.4. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w,c)-almost periodic if and only if $T_{w,c} \in \mathcal{B}'_{ap}$, i.e., the set $\{\tau_h T_{w,c} : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of (w,c)-almost periodic distributions is denoted by $\mathcal{B}'_{AP_{w,c}}$.

Example 3.1. (i) The associated distribution of a (w,c)-almost periodic function (resp. Stepanov (p,w,c)-almost periodic function) is a (w,c)-almost periodic distribution, i.e.,

$$AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}} \quad (\text{resp. } S^p AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}}).$$

(ii) When $c = 1$ it follows that $\mathcal{B}'_{AP_{w,c}} := \mathcal{B}'_{ap}$.

Characterizations of (w,c)-almost periodic distributions are given in the following theorem.

Theorem 3.4. *Let $T \in \mathcal{B}'_{w,c}$. Then the following statements are equivalent.*

(i) $T \in \mathcal{B}'_{AP_{w,c}}$.

(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in AP_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) *There exist $k \in \mathbb{Z}_+$ and $(f_j)_{0 \le j \le k} \subset AP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$, where $\left((f_{w,c})_j\right)_{0 \le j \le k} = \left(c^{-\frac{t}{w}} f_j\right)_{0 \le j \le k}$.*

Proof. Since for every $T \in \mathcal{B}'_{AP_{w,c}}$, we have $T_{w,c} \in \mathcal{B}'_{ap}$. Hence, the result follows immediately from Theorem 3.3. □

The main propreties of $\mathcal{B}'_{AP_{w,c}}$ are given in the following proposition.

Proposition 3.3. *(i) If $T \in \mathcal{B}'_{AP_{w,c}}$, then $c^{\frac{t}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{AP_{w,c}}$, $j \in \mathbb{Z}_+$.*
(ii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.
(iii) If $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$, then $c^{\frac{t}{w}} (T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$.

Proof. (i) Obvious.
(ii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c} \in \mathcal{B}_{ap}$ and $T_{w,c} \in \mathcal{B}'_{ap}$. From Proposition 3.2(ii), we get $\varphi_{w,c}T_{w,c} \in \mathcal{B}'_{ap}$ and therefore $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (\varphi_{w,c}T_{w,c})\right) \in \mathcal{B}'_{ap}$, which gives $c^{\frac{t}{w}} (\varphi_{w,c}T_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$. Hence $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.
(iii) Let $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$. Then $T_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{D}'_{L^1}$. According to Proposition 3.2 (iii), we have $T_{w,c} * S_{w,c} \in \mathcal{B}'_{ap}$ and $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (T_{w,c} * S_{w,c})\right) \in \mathcal{B}'_{ap}$. Hence, $c^{\frac{t}{w}} (T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$. □

The following result shows that $\mathcal{B}_{AP_{w,c}}$ is dense in $\mathcal{B}'_{AP_{w,c}}$.

Proposition 3.4. *Let $T \in \mathcal{B}'_{w,c}$. Then $T \in \mathcal{B}'_{AP_{w,c}}$ if and only if there exists $(\varphi_n)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{AP_{w,c}}$ such that $\lim_{n\to+\infty} \varphi_n = T$ in $\mathcal{B}'_{w,c}$.*

Proof. If $T \in \mathcal{B}'_{AP_{w,c}}$, then $T_{w,c} \in \mathcal{B}'_{ap}$ and from the density of $\mathcal{B}_{ap}$ in $\mathcal{B}'_{ap}$ there exists $(\psi_n)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{ap}$ such that

$$\lim_{n\to+\infty} \psi_n = T_{w,c} \text{ in } \mathcal{D}'_{L^\infty}.$$

This is equivalent to

$$c^{\frac{t}{w}} \lim_{n\to+\infty} \psi_n = \lim_{n\to+\infty} \left(c^{\frac{t}{w}} \psi_n\right) = c^{\frac{t}{w}} T_{w,c} = T \text{ in } \mathcal{B}'_{w,c}.$$

Hence, there exists $(\varphi_n)_{n\in\mathbb{Z}_+} = \left(c^{\frac{t}{w}} \psi_n\right)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{AP_{w,c}}$ such that

$$\lim_{n\to+\infty} \varphi_n = T \text{ in } \mathcal{B}'_{w,c}.$$ □

Now we will introduce the concept of asymptotic (w, c)-almost periodicity of distributions. M. Fréchet introduced the space $AAP(\mathbb{R}_+)$ of classical asymptotically almost periodic functions in [6] and proved the main properties of these functions. The space $AAP_{w,c}(\mathbb{R}_+)$ of asymptotically (w, c)-almost periodic functions were introduced recently by M. T. Khalladi, M. Kostić, A. Rahmani and D. Velinov. Asymptotically almost periodic Schwartz distributions have been introduced and studied by I. Cioranescu in [4]. We recall the definition and some properties of asymptotically almost periodic Schwartz distributions.

Definition 3.5. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called vanishing at infinity if

$$\lim_{h\to+\infty} \langle \tau_{-h}T, \varphi\rangle = 0 \quad \text{in } \mathbb{C},\ \varphi \in \mathcal{D}.$$

Denote by $\mathcal{B}'_{0+}$ the space of bounded distributions vanishing at infinity.

Definition 3.6. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called asymptotically almost periodic if there exist $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$. The space of asymptotically almost periodic Schwartz distributions is denoted by $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

Proposition 3.5. *If $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, then the decomposition $T = R + S$ on $\mathbb{R}_+$ is unique in $\mathcal{D}'_{L^\infty}$.*

Proof. See [4]. ☐

Set $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset \mathbb{R}_+\}$. Then we have the following characterization of space $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

Theorem 3.5. *Let $T \in \mathcal{D}'_{L^\infty}$. Then the following assertions are equivalent.*

(i) $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) $T * \check{\varphi} \in AAP(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) = \varphi(-x)$.

(iii) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0\le j\le k} \subset AAP(\mathbb{R}_+) : T = \sum_{j=0}^{k} f_j^{(j)}$ *on* $\mathbb{R}_+$.

Proof. See [4]. ☐

Asymptotic (w, c)-almost periodicity of distributions is introduced in the following.

Definition 3.7. Let $c \in \mathbb{C}$, $|c| \ge 1$ and $w > 0$. Then a distribution $T \in \mathcal{B}'_{w,c}$ is said asymptotically (w, c)-almost periodic if and only if $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. The space of asymptotically (w, c)-almost periodic distributions is denoted by $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

Remark 3.2. (i) When $c = 1$ it follows that $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+) := \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) The associated distribution of an asymptotically (w, c)-almost periodic function (resp. asymptotically Stepanov (p, w, c)-almost periodic function) is asymptotically (w, c)-almost periodic distribution.

Now let us define the space $\left(\mathcal{B}'_{w,c}\right)_{0+}$ of (w, c)-bounded distributions vanishing at infinity as follows.

Definition 3.8. Let $c \in \mathbb{C}$, $|c| \ge 1$ and $w > 0$. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c)-bounded distribution vanishing at infinity if and only if $T_{w,c} \in \mathcal{B}'_{0+}$.

We have the following result.

Theorem 3.6. *Let $c \in \mathbb{C}$, $|c| \ge 1$, $w > 0$ and $T \in \mathcal{B}'_{w,c}$. Then $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ if and only if there exist $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that*

$$T = R + S \quad \text{on } \mathbb{R}_+. \tag{3.1}$$

Proof. ($\Longrightarrow$) Let $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. Then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$ and by Definition 3.6, there exist $P \in \mathcal{B}'_{ap}$ and $Q \in \mathcal{B}'_{0+}$ such that $T_{w,c} = P + Q$ on $\mathbb{R}_+$. On the other hand, we have

$$T_{w,c} = c^{-\frac{t}{w}}T = P + Q \Longrightarrow \left\langle c^{-\frac{t}{w}}T, \varphi \right\rangle = \langle P, \varphi \rangle + \langle Q, \varphi \rangle, \quad \varphi \in \mathcal{D}$$
$$\Longrightarrow \langle T, \psi \rangle = \left\langle c^{\frac{t}{w}}P, \psi \right\rangle + \left\langle c^{\frac{t}{w}}Q, \psi \right\rangle, \quad \psi = c^{-\frac{t}{w}}\varphi \in \mathcal{D}.$$

Thus, there exist $R = c^{\frac{t}{w}}P \in \mathcal{B}'_{AP_{w,c}}$ and $S = c^{\frac{t}{w}}Q \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$.

($\Longleftarrow$) If there exist $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$, then $c^{-\frac{t}{w}}T = c^{-\frac{t}{w}}R + c^{-\frac{t}{w}}S$ on $\mathbb{R}_+$, i.e., $T_{w,c} = R_{w,c} + S_{w,c}$ on $\mathbb{R}_+$, where $R_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{B}'_{0+}$. Hence, $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, which shows that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. ☐

Proposition 3.6. *The decomposition* (3.1) *is unique in* $\mathcal{B}'_{w,c}$.

Proof. Suppose that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ is such that $T = R + S$ on $\mathbb{R}_+$, where $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$. Then the result follows from the proof of the implication ($\Longleftarrow$) of Theorem 3.6 and the uniqueness of the decomposition of asymptotically almost periodic distributions. ☐

Some characterizations of asymptotically (w, c)-almost periodic distributions are given in the following result.

Theorem 3.7. *Let* $c \in \mathbb{C}$, $|c| \geq 1$, $w > 0$ *and* $T \in \mathcal{B}'_{w,c}$. *The following assertions are equivalent.*

(*i*) $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

(*ii*) $c^{\frac{t}{w}}\left(T_{w,c} * \check{\varphi}\right) \in AAP_{w,c}(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) = \varphi(-x)$.

(*iii*) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0\leq j\leq k} \subset AAP_{w,c}(\mathbb{R}_+) : T = c^{\frac{t}{w}}\sum_{j=0}^{k}(f_{w,c})_j^{(j)}$ *on* $\mathbb{R}_+$, *where* $\left((f_{w,c})_j\right)_{0\leq j\leq k} = \left(c^{-\frac{t}{w}}f_j\right)_{0\leq j\leq k}$.

Proof. It is clear that if $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$, then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. Thus, by applying Theorem 3.5 we obtain the result. ☐

4. Linear Differential Equations in $\mathcal{B}'_{AP_{w,c}}$

In this section we will study the existence of distributional (w, c)-almost periodic solutions of the following system of linear ordinary differential equations

$$T' = AT + S, \tag{4.1}$$

where $A = (a_{ij})_{1\leq i,j\leq k}$, $k \in \mathbb{N}$, is a given square matrix of complex numbers, $S = (S_i)_{1\leq i\leq k} \in (\mathcal{D}')^k$ is a vector distribution and $T = (T_i)_{1\leq i\leq k}$ is the unknown vector distribution.

First, consider the system (4.1) with $S \in (AP)^k$ and let us recall the following result.

Theorem 4.1. *If the matrix A has no eigenvalues with real part zero, then for any $S \in (AP)^k$, there exists a unique solution $T \in (AP)^k$ of the system* (4.1).

Proof. See [5]. □

Let I_k be the unit matrix of order k. The following result gives the (w,c)-almost periodicity of the solution (if it exists) of the system (4.1).

Theorem 4.2. *Let $S \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. If the matrix $A - \frac{\log c}{w} I_k$ has no eigenvalues with real part zero, then the system* (4.1) *admits a unique solution $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ which is an (w,c)-almost periodic vector distribution.*

Proof. Let $\varphi \in \mathcal{D}$. We have

$$c^{-\frac{t}{w}} T' * \varphi = \left(c^{-\frac{t}{w}} T * \varphi\right)' + \frac{\log c}{w} c^{-\frac{t}{w}} T * \varphi. \tag{4.2}$$

On the other hand, if $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ satisfies system (4.1), then

$$c^{-\frac{t}{w}} T' * \varphi = A c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi.$$

So from (4.2), we have

$$\left(c^{-\frac{t}{w}} T * \varphi\right)' = \left(A - \frac{\log c}{w} I_k\right) c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi,$$

i.e.,

$$(T_{w,c} * \varphi)' = \left(A - \frac{\log c}{w} I_k\right)(T_{w,c} * \varphi) + S_{w,c} * \varphi, \tag{4.3}$$

where

$$T_{w,c} * \varphi = \left((T_{w,c})_i * \varphi\right)_{1\le i\le k} = \left(\left(c^{-\frac{t}{w}} T_i\right) * \varphi\right)_{1\le i\le k}$$

and

$$S_{w,c} * \varphi = \left((S_{w,c})_i * \varphi\right)_{1\le i\le k} = \left(\left(c^{-\frac{t}{w}} S_i\right) * \varphi\right)_{1\le i\le k}.$$

Then the system (4.3) is equivalent in $(\mathcal{C}^\infty)^k$ to the following system of differential equations

$$P' = BP + Q,$$

with $B = A - \frac{\log c}{w} I_k$, $P = T_{w,c} * \varphi \in (\mathcal{C}^\infty)^k$ and $Q = S_{w,c} * \varphi \in (AP)^k$. According to Theorem 4.1, it follows that there exists a unique bounded solution P which is almost periodic. Therefore, $(T_{w,c})_i * \varphi \in AP$, $1 \le i \le k$, $\varphi \in \mathcal{D}$, hence $c^{\frac{t}{w}}\left((T_{w,c})_i * \varphi\right) \in AP_{w,c}$, $1 \le i \le k$, $\varphi \in \mathcal{D}$. Thus, according to Theorem 3.4, we get $(T_i)_{1\le i\le k} \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. □

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