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ITERATIVE CONTINUOUS COLLOCATION METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. This paper is concerned with the numerical solution of nonlinear Volterra integral equations. The main purpose of this work is to provide a new numerical approach based on the use of continuous collocation Lagrange polynomials for the numerical solution of nonlinear Volterra integral equations. It is shown that this method is convergent. The results are compared with the results obtained by other well-known numerical methods to prove the effectiveness of the presented algorithm.

1. Introduction

In this paper, we study a numerical method based on iterative continuous collocation method for the solution of nonlinear Volterra integral equations of the form,

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds, \quad t \in I = [0, T], \tag{1.1}$$

where the functions f, K are sufficiently smooth.

The integral equations are often involved in various fields such as physics and biology (see, for example [5, 14, 15]), and they also occur as reformulations of other mathematical problems, such as ordinary differential equations and partial differential equations (see [14]).

There has been a growing interest in the numerical solution of Equation (1.1) (see, for example, [2, 3, 7, 8, 10, 11, 13, 15–17, 19, 21]) such as, Chebyshev approximation [2], Adomian's method [3, 15], Taylor polynomial approximations [21], homotopy perturbation method [10], the series expansion method [11], fixed point method [16],

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Haar wavelet method [17], rationalized Haar functions method [19]. Moreover, many collocation methods for approximating the solutions for Equation (1.1) have been developed recently (see, [5, 9, 18, 20, 22]) such as Lagrange spline collocation method [5], cubic B-spline collocation method [9], quintic B-spline collocation method [18], Taylor collocation method [20], and sinc-collocation method for Volterra integral equations is used in [22].

The numerical solution of these equations has a high computational cost due to the nonlinearity and most of the collocation methods for nonlinear Volterra integral equations transform (1.1) into a system of nonlinear algebraic equations.

This paper is concerned with the continuous piecewise polynomial collocation method based on the use of Lagrange polynomials. Our goal is to develop an iterative explicit solution to approximate the solution of nonlinear Volterra integral equation (1.1).

The main advantages of the current collocation method are that it is direct and there is no algebraic system to be solved, which makes the proposed algorithm very effective, easy to implement and the calculation cost low.

This paper is organized as follows. In Section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1.1) in each interval by using iterative Lagrange polynomials. Global convergence is established in Section 3. Finally, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in Section 4.

2. Description of the Collocation Method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh$, $n = 0, \dots, N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 \le c_1 < \cdots < c_m \le 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1, \dots, m$, $n = 0, \dots, N-1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}]$.

Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree m as follows

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, \dots, N-1\}.$$

It holds for any $y \in C^{m+1}([0, T])$ that

$$y(t_n + sh) = L_0(s)y(t_n) + \sum_{j=1}^{m} L_j(s)y(t_{n,j}) + h^{m+1}\frac{y^{(m+1)}(\zeta_n(s))}{(m+1)!} s \prod_{j=1}^{m}(s - c_j), \tag{2.1}$$

where $s \in [0, 1]$, $L_0(v) = (-1)^m \prod_{l=1}^{m} \frac{v-c_l}{c_l}$ and $L_j(v) = \frac{v}{c_j} \prod_{l\neq j}^{m} \frac{v-c_l}{c_j-c_l}$, $j = 1, \dots, m$, are the Lagrange polynomials associated with the parameters c_j, $j = 1, \dots, m$.

Inserting (2.1) for the function $s \mapsto K(t,s,x(s))ds$ into (1.1), we obtain for each $j = 1,\dots,m$, $n = 0,\dots,N-1$,

$$\begin{aligned} x(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j},t_p,x(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j},t_{p,v},x(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j},t_n,x(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j},t_{n,v},x(t_{n,v})) + o(h^{m+1}), \end{aligned} \tag{2.2}$$

such that $a_{j,v} = \int_0^{c_j} L_v(\eta)d\eta$ and $b_v = \int_0^1 L_v(\eta)d\eta$, $v = 0,\dots,m$.

It holds for any $u \in S_m^0(I,\Pi_N)$ that

$$u_n(t_n + sh) = L_0(s)u_{n-1}(t_n) + \sum_{j=1}^{m} L_j(s)u_n(t_{n,j}), \quad s \in [0,1]. \tag{2.3}$$

Now, we approximate the exact solution x by $u \in S_m^0(I,\Pi_N)$ such that $u(t_{n,j})$ satisfies the following nonlinear system,

$$\begin{aligned} u_n(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j},t_p,u_p(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j},t_{p,v},u_p(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j},t_n,u_{n-1}(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j},t_{n,v},u_n(t_{n,v})), \end{aligned} \tag{2.4}$$

for $j = 1,\dots,m$, $n = 0,\dots,N-1$, where $u_{-1}(t_0) = x(0) = f(0)$.

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_m^0(I,\Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1.1) such that

$$u_n^q(t_n + sh) = L_0(s)u_{n-1}^q(t_n) + \sum_{j=1}^{m} L_j(s)u_n^q(t_{n,j}), \quad s \in [0,1], \tag{2.5}$$

where the coefficients $u_n^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} u_n^q(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j},t_p,u_p^q(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j},t_{p,v},u_p^q(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j},t_n,u_{n-1}^q(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j},t_{n,v},u_n^{q-1}(t_{n,v})), \end{aligned} \tag{2.6}$$

such that $u_{-1}^q(t_0) = f(0)$ for all $q \in \mathbb{N}$ and the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).

The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system. The complexity of the proposed algorithm can be measured in terms of how many times the function K must be evaluated at each collocation point.

From (2.5) it follows that the number of such evaluations is $O(mn)$ for each iteration. Since the optimal number of iterations is $q = m+1$ (as it will be shown in the next section), we conclude that the total number of evaluations is $O(m^2 n)$, which makes this method competitive, in comparison with other methods where a nonlinear system of equations is solved by an iterative algorithm.

In the next section, we prove the convergence of the approximate solution u^q to the exact solution x of (1.1) is of order m for all $q \geq m$.

3. Convergence Analysis

In this section, we assume that the function K satisfies the Lipschitz condition with respect to the third variable: there exists $L \geq 0$ such that

$$|K(t,s,y_1) - K(t,s,y_2)| \leq L|y_1 - y_2|.$$

The following lemmas will be used in this section.

Lemma 3.1 ([6]). *Assume that $\{\alpha_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ are given non-negative sequences and the sequence $(\varepsilon_n)_{n\geq 1}$ satisfies $\varepsilon_1 \leq \beta$ and*

$$\varepsilon_n \leq \beta + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \varepsilon_j, \quad n \geq 2,$$

then

$$\varepsilon_n \leq \left(\beta + \sum_{j=1}^{n-1} q_j\right) \exp\left(\sum_{j=1}^{n-1} \alpha_j\right), \quad n \geq 2.$$

Lemma 3.2 ([1]). *If $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ are nonnegative sequences and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0,$$

then

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

Lemma 3.3 ([12]). *Assume that the sequence $\{\varepsilon_n\}_{n\geq 0}$ of nonnegative numbers satisfies*

$$\varepsilon_n \leq A\varepsilon_{n-1} + B\sum_{i=0}^{n-1} \varepsilon_i + K, \quad n \geq 0,$$

where A, B and K are nonnegative constants, then

$$\varepsilon_n \leq \frac{\varepsilon_0}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{K}{R_2 - R_1}[R_2^n - R_1^n],$$

where

$$R_1 = \frac{1 + A + B - \sqrt{(1-A)^2 + B^2 + 2AB + 2B}}{2},$$

$$R_2 = \frac{1 + A + B + \sqrt{(1-A)^2 + B^2 + 2AB + 2B}}{2}. \quad (3.1)$$

Therefore, $0 \leq R_1 \leq 1 \leq R_2$.

The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.4).

Lemma 3.4. *The nonlinear system* (2.4) *has a unique solution* $u \in S_m^0(I, \Pi_N)$ *for sufficiently small* h.

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (2.4) becomes

$$u_0(t_{0,j}) = f(t_{0,j}) + ha_{j,0}K(t_{0,j}, t_0, f(0)) + h\sum_{v=1}^{m} a_{j,v}K(t_{0,j}, t_{0,v}, u_0(t_{0,v})), \quad j = 1, \dots, m.$$

We consider the operator Ψ defined by

$$\begin{aligned} \Psi &: \mathbb{R}^m \to \mathbb{R}^m, \\ x = (x_1, \dots, x_m) &\mapsto \Psi(x) = (\Psi_1(x), \dots, \Psi_m(x)), \end{aligned}$$

such that for $j = 1, \dots, m$, we have

$$\Psi_j(x) = f(t_{0,j}) + ha_{j,0}K(t_{0,j}, t_0, f(0)) + h\sum_{v=1}^{m} a_{j,v}K(t_{0,j}, t_{0,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|,$$

where $a = \max\{|a_{j,v}|, j = 1, \dots, m, v = 0, \dots, m\}$.

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (2.4) has a unique solution u_0 on σ_0.

(ii) Suppose that u_i exists and is unique on the intervals σ_i, $i = 0, \dots, n-1$, for $n \geq 1$. We show that u_n exists and is unique on the interval σ_n.

On the interval σ_n, the nonlinear system (2.4) becomes

$$u_n(t_{n,j}) = F(t_{n,j}) + h\sum_{v=1}^{m} a_{j,v}K(t_{n,j}, t_{n,v}, u_n(t_{n,v})), \tag{3.2}$$

where

$$\begin{aligned} F(t_{n,j}) = &f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0K(t_{n,j}, t_p, u_p(t_p)) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_vK(t_{n,j}, t_{p,v}, u_p(t_{p,v})) + ha_{j,0}K(t_{n,j}, t_n, u_{n-1}(t_n)). \end{aligned}$$

We consider the operator Ψ defined by

$$\begin{aligned} \Psi &: \mathbb{R}^m \to \mathbb{R}^m, \\ x = (x_1, \dots, x_m) &\mapsto \Psi(x) = (\Psi_1(x), \dots, \Psi_m(x)), \end{aligned}$$

such that for $j = 1, \dots, m$,

$$\Psi_j(x) = F(t_{n,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|.$$

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (3.2) has a unique solution u_n on σ_n. □

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.1. *Let f, K be $m+1$ times continuously differentiable on their respective domains. If $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, then, for sufficiently small h, the collocation solution u converges to the exact solution x and the resulting error function $e := x - u$ satisfies*

$$\|e\| \leq Ch^{m+1},$$

where C is a finite constant independent of h.

Proof. Define the error e on σ_n by $e(t) = e_n(t) = x(t) - u_n(t)$ for all $n \in \{0, 1, \dots, N-1\}$.

We have, from (2.4) and (2.2), for all $n = 0, \dots, N-1$, and $j = 1, \dots, m$,

$$\begin{aligned} |e_n(t_{n,j})| \leq & hbL \sum_{p=0}^{n-1} |e_p(t_p)| + hbL \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e_p(t_{p,v})| + haL|e_{n-1}(t_n)| \\ & + haL \sum_{v=1}^{m} |e_n(t_{n,v})| + \alpha h^{m+1}, \end{aligned} \tag{3.3}$$

where α is a positive number and $e_{-1}(t_0) = 0$.

We consider the sequence $\varepsilon_n = \sum_{v=1}^{m} |e_n(t_{n,v})|$ for $n = 0, \dots, N-1$. Then, from (3.3), ε_n satisfies for $n = 0, \dots, N-1$,

$$\begin{aligned} \varepsilon_n &\leq hbLm \sum_{p=0}^{n-1} |e_p(t_p)| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm|e_{n-1}(t_n)| + haLm\varepsilon_n + \alpha m h^{m+1} \\ &\leq 2hbLm \sum_{p=0}^{n-1} \|e_p\| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm\varepsilon_n + \alpha m h^{m+1}. \end{aligned}$$

Hence, for $\overline{h} < \frac{1}{Lam}$ and $h \in (0, \overline{h}]$, we have

$$\varepsilon_n \leq \underbrace{\frac{2bLm}{1 - Lam\overline{h}}}_{\alpha_1} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\frac{bLm}{1 - Lam\overline{h}}}_{\alpha_2} h \sum_{p=0}^{n-1} \varepsilon_p + \underbrace{\frac{\alpha m}{1 - Lam\overline{h}}}_{\alpha_3} h^{m+1}.$$

Then, by Lemma 3.1, for all $n = 0, \dots, N-1$,

$$\varepsilon_n \le \underbrace{\alpha_1 \exp(T\alpha_2)}_{\alpha_4} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\alpha_3 \exp(T\alpha_2)}_{\alpha_5} h^{m+1}.$$

Therefore, by using (2.1) and (2.3), we obtain

$$\begin{aligned}\|e_n\| &\le |R(\infty)|\|e_{n-1}\| + \rho\varepsilon_n + \beta h^{m+1} \\ &\le |R(\infty)|\|e_{n-1}\| + \underbrace{\rho\alpha_4}_{\alpha_6} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{(\rho\alpha_5 + \beta)}_{\alpha_7} h^{m+1},\end{aligned}$$

where $\rho = \max\{|L_j(t)| : t \in [0,1], j = 1, \dots, m\}$.

Hence, by Lemma 3.3, we obtain for all $n = 0, \dots, N-1$,

$$\begin{aligned}\|e_n\| &\le \frac{\|e_0\|}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\le \frac{\|e_0\|}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \\ &\le \left(\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right)\alpha_7 h^{m+1},\end{aligned}$$

where R_1, R_2 are defined by (3.1) such that $A = |R(\infty)|$, $B = \alpha_6 h$, $K = \alpha_7 h^{m+1}$.

Since

$$\begin{aligned}&\lim_{h \longrightarrow 0}\left(\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right) \\ =&\frac{1}{1 - |R(\infty)|}\exp\left(\frac{2T\alpha_6}{1 - |R(\infty)|}\right) < +\infty.\end{aligned}$$

Then there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$

$$\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \le \gamma.$$

Thus, the proof is completed by taking $C = \alpha_7\gamma$. □

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.2. *Consider the iterative collocation solution u^q, $q \ge 1$, defined by (2.5) and (2.6). If $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, then for any initial condition $u^0(t_{n,j}) \in J$, the iterative collocation solution u^q, $q \ge 1$, converges to the exact solution x for sufficiently small h. Moreover, the following error estimate holds*

$$\|u^q - x\| \le d\beta^q h^q + Ch^{m+1},$$

where d, β and C are finite constants independent of h.

Proof. We define the errors e^q and ξ^q by $e^q(t) = e_n^q(t) = u_n^q(t) - x(t)$ and $\xi^q = \xi_n^q = u_n^q(t) - u_n(t)$ on σ_n, $n = 0, \dots, N-1$, where u is defined by Lemma 3.4.

We have, from (2.4) and (2.6), for all $n = 0, \dots, N-1$ and $j = 1, \dots, m$,

$$\begin{aligned}|\xi_n^q(t_{n,j})| \leq & hbL \sum_{p=0}^{n-1} |\xi_p^q(t_p)| + hbL \sum_{p=0}^{n-1}\sum_{v=1}^{m} |\xi_p^q(t_{p,v})| + haL|\xi_{n-1}^q(t_n)| \\ & + haL \sum_{v=1}^{m} |\xi_n^{q-1}(t_{n,v})|.\end{aligned}$$

Now, for each fixed $q \geq 1$, we consider the sequence $\eta_n^q = \max\{|\xi_n^q(t_{n,v})| : v = 1, \dots, m\}$ for $n = 0, \dots, N-1$, it follows that

$$\begin{aligned}\eta_n^q &\leq hbL \sum_{p=0}^{n-1} |\xi_p^q(t_p)| + hbLm \sum_{p=0}^{n-1} \eta_p^q + haL|\xi_{n-1}^q(t_n)| + haLm\eta_n^{q-1} \\ &\leq 2hbL \sum_{p=0}^{n-1} \|\xi_p^q\| + hbLm \sum_{p=0}^{n-1} \eta_p^q + haLm\eta_n^{q-1}.\end{aligned}$$

Hence, by Lemma 3.2, for all $n = 0, \dots, N-1$,

$$\begin{aligned}\eta_n^q \leq & 2hbL \sum_{p=0}^{n-1} \|\xi_p^q\| + haLm\eta_n^{q-1} + \exp(TLbm)ab(hLm)^2 \sum_{p=0}^{n-1} \eta_p^{q-1} \\ & + 2\exp(TLbm)Tm(bL)^2 h \sum_{p=0}^{n-1} \|\xi_p^q\|. \qquad (3.4)\end{aligned}$$

We consider the sequence $\eta^q = \max\{\eta_n^q, n = 0, \dots, N-1\}$ for $q \geq 1$. Then, from (3.4), η^q satisfies

$$\eta_n^q \leq \underbrace{2(bL + \exp(TLbm)Tm(bL)^2)}_{\alpha_1} h \sum_{p=0}^{n-1} \|\xi_p^q\| + \alpha_2 h\eta^{q-1},$$

where $\alpha_2 = (aLm + \exp(TLbm)abT(Lm)^2)$.

Therefore, by using (2.3) and (2.5), we obtain

$$\|\xi_n^q\| \leq |R(\infty)|\|\xi_{n-1}^q\| + \rho m\eta_n^q \leq |R(\infty)|\|\xi_{n-1}^q\| + \rho m\alpha_1 h \sum_{p=0}^{n-1} \|\xi_p^q\| + \rho m\alpha_2 h\eta^{q-1}.$$

Hence, by Lemma 3.3, we obtain for all $n = 0, \dots, N-1$,

$$\begin{aligned}\|\xi_n^q\| &\leq \frac{\|\xi_0^q\|}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{\rho m\alpha_2 h\eta^{q-1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\leq \frac{\|\xi_0^q\|}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{\rho m\alpha_2 h\eta^{q-1}}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \\ &\leq \left(\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right)\rho m\alpha_2 h\eta^{q-1}, \qquad (3.5)\end{aligned}$$

where R_1 and R_2 are defined by (3.1) such that $A = |R(\infty)|$, $B = \rho m\alpha_1 h$, $K = \rho m\alpha_2 h\eta^{q-1}$. Since

$$\lim_{h\to 0}\left(\frac{1}{R_2-R_1}\left[(R_2-1)R_2^{\frac{T}{h}}+1\right]+\frac{1}{R_2-R_1}\left[R_2^{\frac{T}{h}}\right]\right)=\frac{\exp(\frac{2T\rho m\alpha_1}{1-|R(\infty)|})}{1-|R(\infty)|}<+\infty.$$

Then there exists $\gamma > 0$ such that for all $h \in (0,\overline{h}]$

$$\frac{1}{R_2-R_1}\left[(R_2-1)R_2^{\frac{T}{h}}+1\right]+\frac{1}{R_2-R_1}\left[R_2^{\frac{T}{h}}\right]\le\gamma.$$

It follows, from (3.5), that for all $n = 0,\dots,N-1$,

$$\|\xi_n^q\|\le\gamma\rho m\alpha_2 h\eta^{q-1}\le\gamma\rho m\alpha_2 h\|\xi^{q-1}\|,$$

which implies, for all $q \ge 1$, that

$$\|\xi^q\|\le\gamma\rho m\alpha_2 h\|\xi^{q-1}\|\le\cdots\le(\gamma\rho m\alpha_2)^q h^q\|\xi^0\|.$$

Since, $u_{-1}^0(t_0) = f(0), u^0(t_{n,j}) \in J$ (bounded interval), then by (2.3) the function u^0 is bounded. Hence, there exists $d > 0$ such that $\|\xi^0\| = \|u^0-u\| \le \|u^0-x\|+\|x-u\| < d$. Which implies that for all $q \ge 1$

$$\|\xi^q\|\le d(\underbrace{\gamma\rho m\alpha_2}_{\beta})^q h^q.$$

Hence, by Theorem 3.1, we deduce that

$$\|e^q\|\le\|\xi^q\|+\|u-x\|\le d\beta^q h^q+Ch^{m+1}.$$

Thus, the proof is completed. ☐

Remark 3.1. From the error estimate in Theorem 3.2 it follows that the optimal number of iterations is $q = m+1$. Actually, with $m+1$ iterations the total error has the order of $O(h^{m+1})$, which will not be improved if more iterations are performed.

4. Numerical Examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and q. In each example, we calculate the error between x and the iterative collocation solution u^q.

The absolute errors at some particular points are given to compare our solutions with the solutions obtained by [3, 9, 13, 16, 18].

These results of these numerical examples are in agreement with the theory presented in Section 3 and they confirm the advantages of our method in comparison with those described in [3, 9, 13, 16, 18].

Example 4.1 ([9, 13]). Consider the following nonlinear Volterra integral equation

$$x(t)=1+(\sin(t))^2-\int_0^t 3\sin(t-s)(x(s))^2ds,\quad t\in[0,1],$$

where $u(x) = \cos(x)$ is the exact solution.

The absolute errors for $N = 10, 20$ and $m = q = 3$ at $t = 0, 0.1, \ldots, 1$, are displayed in Table 1. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = -0.02$. The numerical results obtained by the present method are considerably more accurate in comparison with the numerical results obtained in [9,13].

TABLE 1. Comparison of the absolute errors of Example 4.1

t	Method in [9]		Method in [13]		Our method	
	$N=10$	$N=20$	$N=10$	$N=20$	$N=10$	$N=20$
0.1	1.01E−5	1.59E−6	1.24E−5	2.54E−8	3.32E−8	7.92E−9
0.2	2.48E−5	3.26E−6	1.62E−6	3.44E−7	1.84E−9	5.15E−9
0.3	3.65E−5	4.72E−6	2.03E−4	9.19E−7	3.58E−8	3.87E−9
0.4	4.61E−5	5.87E−6	2.07E−5	1.44E−6	5.29E−8	8.00E−9
0.5	5.26E−5	6.63E−6	3.84E−5	1.88E−6	9.91E−8	8.90E−10
0.6	5.59E−5	6.98E−6	5.11E−5	2.18E−6	1.48E−7	5.90E−9
0.7	5.58E−5	6.92E−6	7.22E−5	1.83E−6	1.77E−7	9.71E−9
0.8	5.28E−5	6.47E−6	6.43E−5	6.41E−6	2.00E−7	3.34E−9
0.9	4.65E−5	5.70E−6	1.96E−5	1.00E−4	2.04E−7	2.07E−8
1	3.97E−5	4.71E−6	6.36E−4	9.25E−4	1.95E−7	5.13E−9

Example 4.2 ([3, 18]). Consider the following linear Volterra integral equation with exact solution $x(t) = 1 - \sinh(t)$:

$$x(t) = 1 - t - \frac{t^2}{2} + \int_0^t (t-s)x(s)ds, \quad t \in [0,1].$$

The absolute errors for $m = q = 3$ and $N = 20$ at $t = 0, 0.1, \ldots, 1$, are displayed in Table 2. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = -0.02$. The numerical results obtained here are compared in Table 2 with the numerical results obtained by using the methods in [3, 18].

It is seen from Table 2 that the results obtained by the present method are much more accurate than those obtained in [3, 18].

The absolute errors for $N = 5$ and $(q, m) \in \{(2,2), (3,2), (3,3), (3,5), (4,5)\}$ at $t = 0, 0.1, \ldots, 1$, are presented in Table 3, we note that the absolute error reduces as q or m increases.

We calculate the experimental order of convergence (EOC) at $t = 1$ for $N = 2^l$, $l = 1, 2, 3, 4, 5$, $m = 1, 2, 3$ and $q = m+1$ in Table 4, the result confirms the theoretical result and suggests that the order of convergence with $q = m + 1$ is $m + 1$. As we have remarked (see Remark 3.1) this is the maximal convergence order that can be obtained with the present method.

Moreover, we calculate the run time to solve the approximate solution u for $N = 6, \ldots, 10$, $m = 7, \ldots, 10$, and $q = m + 1$, the numerical results are solved by using Maple version 16.

The computations were performed in a PC with a 2.16 GHz processor, running with 2.00 Go RAM. As it could be expected, the computing time increases with m and N. However, we cannot see a simple relationship between the computing time and the complexity of the algorithm, probably because this time depends on other factors than the number of evaluations of the function K. This table shows that accurate results can be obtained by our method in a small computer with a low computational cost.

TABLE 2. Comparison of the absolute errors of Example 4.2

t	Our method		Method in [3]	Method in [18]
	$N = 10$	$N = 20$		
0.0	0	0	0	1.98E−14
0.1	1.30E−8	1.98E−9	5.38E−6	1.21E−7
0.2	3.35E−8	2.54E−9	2.20E−5	2.35E−7
0.3	3.14E−8	6.55E−9	4.82E−5	3.54E−7
0.4	5.98E−8	5.80E−9	8.33E−5	4.77E−7
0.5	6.94E−8	3.50E−9	1.26E−4	6.05E−7
0.6	8.01E−8	8.51E−10	1.77E−4	7.39E−7
0.7	1.00E−7	5.83E−9	2.34E−4	8.80E−7
0.8	1.15E−7	7.38E−9	2.97E−4	1.03E−6
0.9	1.37E−7	8.90E−9	3.65E−4	1.19E−6
1	1.62E−7	9.38E−9	4.38E−4	1.36E−6

TABLE 3. Absolute errors for Example 4.2

t	$q = 2$ $m = 2$	$q = 3$ $m = 2$	$q = 3$ $m = 3$	$q = 3$ $m = 5$	$q = 4$ $m = 5$
0	0.0	0.0	0.0	0.0	0.0
0.1	8.231E−6	7.282E−6	3.015E−7	7.451E−8	3.701E−8
0.2	8.563E−5	8.373E−5	4.115E−7	1.147E−6	8.474E−7
0.3	1.053E−5	7.583E−6	6.394E−7	5.824E−8	4.007E−8
0.4	1.027E−4	9.863E−5	8.478E−7	8.031E−7	4.328E−7
0.5	1.064E−5	5.410E−6	1.017E−6	2.316E−8	1.897E−8
0.6	1.143E−4	1.070E−4	1.324E−6	1.058E−7	4.785E−8
0.7	1.033E−5	2.283E−6	1.470E−6	1.309E−8	3.040E−8
0.8	1.297E−4	1.175E−4	1.909E−6	1.114E−7	7.258E−8
0.9	9.861E−6	1.815E−6	2.021E−6	8.470E−9	8.137E−10
1	1.514E−4	1.314E−4	2.620E−6	1.156E−7	4.245E−8

TABLE 4. EOC and the run-time/sec of Example 4.2

N	$m=1$	$m=2$	$m=3$
2			
4	2.04	2.91	4.00
8	2.05	2.96	4.01
16	2.04	2.91	4.00
32	2.04	2.91	4.00

N	$m=7$	$m=8$	$m=9$	$m=10$
6	3.9	5.6	9.9	14.8
7	5.8	9.9	17.9	31.1
8	8.9	16.7	24.3	56.6
9	13.3	32.7	43.9	118.4
10	17.4	35.9	126.3	232.3

Example 4.3 ([16]). We consider the following nonlinear Volterra integral equation

$$x(t) = \frac{t}{e^{t^2}} + \int_0^t 2tse^{-x^2(s)}ds, \quad t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 20$ and $m = 3$, $q = 5$ at $t = 0, 0.2, \ldots, 1$, are compared with the absolute error of the method in [16] in Table 5, where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = -0.64$.

TABLE 5. Comparison of the absolute errors of Example 4.3

t	Method in [16] $N=20$	Our method $N=20$
0	0	0
0.2	1.49E−8	8.9E−9
0.4	7.74E−7	2.69E−8
0.6	9.36E−6	8.90E−9
0.8	4.58E−5	3.39E−8
1	1.29E−4	2.49E−8

Example 4.4 ([16]). We consider the following nonlinear Volterra integral equation

$$x(t) = t\cos(t) + \int_0^t t\sin(x(s))ds, \quad t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 25$ and $m = q = 4$ at $t = 0.001, 0.2, 0.4, 0.6, 0.8, 1$ are compared with the absolute error of the method in [16] in Table 6.

Where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = 0.35$.

It is seen from Table 6 that the results obtained by the present method is very superior to that obtained by the method in [16].

5. CONCLUSION

In this paper, we have used an iterative collocation method based on the Lagrange polynomials for the numerical solution of nonlinear Volterra integral equations (1.1)

Table 6. Comparison of the absolute errors of Example 4.4

t	Method in [16] $N = 25$	Our method $N = 25$
0.001	1.25E−12	1.75E−11
0.2	3.53E−6	6.30E−8
0.4	5.81E−6	5.40E−8
0.6	7.74E−7	9.60E−8
0.8	1.20E−5	6.00E−9
1	3.98E−5	7.20E−8

in the spline space $S_m^{(0)}(\Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples showing that the method is convergent with a good accuracy and the comparison of the results obtained by the present method with the other methods reveals that the method is very effective and convenient.

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