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EXTENDED CONVERGENCE OF A TWO-STEP-SECANT-TYPE METHOD UNDER A RESTRICTED CONVERGENCE DOMAIN

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Abstract. We present a local as well as a semi-local convergence analysis of a two-step secant-type method for solving nonlinear equations involving Banach space valued operators. By using weakened Lipschitz and center Lipschitz conditions in combination with a more precise domain containing the iterates, we obtain tighter Lipschitz constants than in earlier studies. This technique lead to an extended convergence domain, more precise information on the location of the solution and tighter error bounds on the distances involved. These advantages are obtained under the same computational effort, since the new constants are special cases of the old ones used in earlier studies. The new technique can be used on other iterative methods. The numerical examples further illustrate the theoretical results.

1. Introduction

Let $F : D \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator, $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and D be a nonempty convex subset of $\mathcal{B}_1$. One of the most important problems in mathematics and computational sciences is finding a locally unique solution x^* of the equation

$$F(x) = 0. \tag{1.1}$$

Many problems in the aforementioned disciplines can be written in a form like (1.1) using mathematical modeling. The solution x^* is sought in closed form but this can be achieved only in special cases. This is the reason why most solution methods for equation (1.1) are iterative. The most popular methods for generating a sequence

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approximating x^* are one-step Newton or Secant-type or two step Newton or Secant-type methods [1–18].

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

In the present paper we study the local as well as the semi-local convergence of two-step secant-type method defined for each $n = 0, 1, 2, \ldots$, $A_n = [x_n, y_n; F]$ by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n), \\ y_{n+1} &= x_{n+1} - A_n^{-1} F(x_{n+1}), \end{aligned} \tag{1.2}$$

where $x_0, y_0 \in D$ are initial points and $[\cdot, \cdot; F] : D^2 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference of order one for F on D satisfying

$$[x, y; F](x - y) = F(x) - F(y), \quad \text{for each } x, y \in D \text{ with } x \neq y,$$

and

$$[x, x; F] = F'(x), \quad \text{for each } x \in D$$

(if F is Fréchet differentiable on D). Notice that in the case of the secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n)$$

or

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n),$$

we presented in [13] a convergence analysis under center Lipschitz and weak Lipschitz conditions (see (a_4) and (a_5)) leading to the following advantages (A) over other approaches (using only Lipschitz conditions), (see (a_4) and (c_4)).

(a) Extended convergence domain.
(b) Tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$, $\|y_n - x^*\|$.
(c) At least as precise information on the location of the solution.

Our semi-local convergence analysis also improves the corresponding one in [11], since in our article we use the center-Lipschitz condition to locate a subset D_0 of D containing the iterates. This way the Lipschitz constants are tighter than in [11], resulting to the advantages (a)-(c). It is worth noticing that these advantages are obtained under the same computational effort, since the new constants are tighter and special cases of the constants in [11]. Hence, we have extended the applicability of method (1.2). Moreover, we have provided the local convergence analysis of method (1.2) not given in [11].

Notice that extending the semi-local convergence domain is important, especially since the convergence domain of such methods is small in general. Tighter error bounds implies that fewer iterates must be computed to obtain prespecified error tolerance.

The local, semi-local convergence analysis for method (1.2) is given in Section 2, Section 3, respectively, whereas Section 4 contains the numerical examples.

2. Local Convergence

We shall define some scalar functions and parameters to be used in the local convergence analysis of method (1.2). Let $\ell_0, \ell, \ell_1, \ell_2, \ell_3$ and ℓ_4 be nonnegative parameters. Let $r_0 = \frac{1}{\ell_0+\ell}$ and $r_1 = \frac{1}{\ell_0+\ell+\ell_1+\ell_2}$. Define functions g_1, g_2, h_1 and h_2 on the interval $[0, r_0)$ by

$$\begin{aligned} g_1(t) =& \frac{(\ell_1+\ell_2)t}{1-(\ell_0+\ell)t}, \\ g_2(t) =& \frac{\ell_3(g_1(t)t+t)+\ell_4 t}{1-(\ell_0+\ell)t}, \\ h_1(t) =& g_1(t)-1 \end{aligned}$$

and

$$h_2(t) = g_2(t) - 1.$$

We have $h_1(r_1) = 0$ and for each $t \in [0, r_1),\ 0 \le g_1(t) < 1$. Moreover, $h_1(0) = -1$ and $h_2(t) \to +\infty$ as $t \to r_0^-$. Hence, function h_2 has zeros in the interval $(0, r_0)$. Denote by r_2 the smallest such zero. Define functions g_0 and h_0 on the interval $[0, r_0)$ by

$$g_0(t) = \ell_0 g_1(t) + \ell g_2(t)$$

and

$$h_0(t) = g_0(t) - 1.$$

We get that $h_0(t) = -1$ and $h_0(t) \to +\infty$ as $t \to r_0^-$. Denote by ρ the smallest zero of function h_0 on the interval $(0, r_0)$. Then, define functions g_3 and h_3 on the interval $[0, \rho)$ by

$$g_3(t) = \frac{\ell_1 g_1(t) + \ell_2 g_2(t)}{1-(\ell_0 g_1(t) + \ell g_2(t))}$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain that $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to \rho^-$. Denote by r_3 the smallest zero of function h_3. Define the radius of convergence r by

$$r = \min\{r_i : i = 1, 2, 3\}. \tag{2.1}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g_i(t) < 1. \tag{2.2}$$

Let $B(x, \lambda) = \{y \in X : \|x - y\| < \lambda\}$ and $\bar{B}(x, \lambda)$ be the closure of $B(x, \lambda)$.

Definition 2.1. Set $D_0 = D \cap B(x^*, \frac{1}{\ell})$. The set $T^* = (F, x_0, y_0, x^*)$ belongs to the class $K^* = K^*(\ell_0, \ell, \ell_1, \ell_2, \ell_3, \ell_4)$, if

(a_1) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a Fréchet differentiable operator and $[.,.;F] : D^2 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference for F of order one on D^2;
(a_2) there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$;
(a_3) there exist $\ell_0 \geq 0$, $\ell \geq 0$ with ℓ_0, ℓ not both zero such that for each $x, y \in D$

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \ell_0\|x - x^*\| + \ell\|y - x^*\|;$$

(a_4) there exist $\ell_i \geq 0$, $i = 1, 2, 3, 4$, such that for each $x, y, z \in D_0$

$$\begin{aligned}\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \leq &\ell_1\|x - x^*\| + \ell_2\|y - x^*\|,\\ \|F'(x^*)^{-1}([x, y; F] - [z, x^*; F])\| \leq &\ell_3\|x - x^*\| + \ell_4\|y - x^*\|;\end{aligned}$$

(a_5) $\bar{B}(x^*, r) \subseteq D$, where r is defined in (2.1).

The local convergence analysis of method (1.2) follows in the class K^*.

Theorem 2.1. *Suppose that $T^* \subseteq K^*$ holds. Then, sequence $\{x_n\}$ generated for $x_0, y_0 \in B(x^*, r) - \{x^*\}$ is well defined in $B(x^*, r)$, remains in $B(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^*. Moreover, the following estimates hold*

$$\|x_{n+1} - x^*\| \leq g_1(r)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.3}$$
$$\|y_{n+1} - x^*\| \leq g_2(r)\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < r \tag{2.4}$$

and

$$\|x_{n+2} - x^*\| \leq g_3(r)\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\|, \tag{2.5}$$

where the functions g_i, $i = 1, 2, 3$, are defined previously. Furthermore, the solution x^* of equation $F(x) = 0$ is unique in $D_1 = D \cap \bar{B}(x^*, R)$ for $R \in [r, \frac{1}{\ell_0+\ell})$.

Proof. We shall use mathematical induction to show estimates (2.3)-(2.5). By hypothesis $x_0, y_0 \in B(x^*, r) - \{x^*\}$, (2.1), (a_2) and (a_3), we have in turn that

$$\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \leq \ell_0\|x_0 - x^*\| + \ell\|y_0 - x^*\| \leq (\ell_0 + \ell)r < 1. \tag{2.6}$$

By (2.6) and the Banach lemma on invertible operators [1,4,5,10,15], we deduce that $A_0^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - (\ell_0\|x_0 - x^*\| + \ell\|y_0 - x^*\|)}.$$

Hence, x_1, y_1 are well defined by method (1.2) for $n = 0$. Then, using (a_2), (2.1), (2.2) and (a_4) we get in turn that

$$\begin{aligned}
\|x_1 - x^*\| =&\|x_0 - x^* - A_0^{-1}F(x_0)\| \\
\leq&\|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0 - [x_0, x^*; F])\|\|x_0 - x^*\| \\
\leq&\frac{\ell_1\|x_0 - x^*\| + \ell_2\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq&g_1(r)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.7)\\
\|y_1 - x^*\| \leq&\|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0 - [x_1, x^*; F])\|\|x_1 - x^*\| \\
\leq&\frac{\ell_3\|x_1 - x_0\| + \ell_4\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq&\frac{\ell_3(\|x_1 - x^*\| + \|x_0 - x^*\|) + \ell_4\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq&g_2(r)\|x_1 - x^*\| \leq \|x_1 - x^*\| < r, \qquad (2.8)
\end{aligned}$$

and similarly to (2.7)

$$\begin{aligned}
\|x_2 - x^*\| \leq&\frac{\ell_1\|x_1 - x^*\| + \ell_2\|y_0 - x^*\|}{1 - (\ell_0\|x_1 - x^*\| + \ell\|y_1 - x^*\|)}\|x_1 - x^*\| \qquad (2.9)\\
\leq&g_3(r)\|x_1 - x^*\| \leq \|x_1 - x^*\|.
\end{aligned}$$

That is estimates (2.7)–(2.9) show (2.3)–(2.5), respectively for $k = 0$. By simply replacing $x_0, y_0, x_1, y_1, x_2, y_2$ by $x_k, y_k, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}$ in the preceding estimates, we complete the induction for (2.3)–(2.5). Then, it follows from the estimate

$$\|x_{k+2} - x^*\| \leq c\|x_{k+1} - x^*\| < r,$$

where $c = g_3(r) \in [0, 1)$ that $\lim_{k\to\infty} x_k = x^*$. Finally, to show the uniqueness part, let $y^* \in D_0$ with $F(y^*) = 0$. Set $E = [x^*, y^*; F]$. Then, by (a_3), we get

$$\|F'(x^*)^{-1}(E - F'(x^*))\| \leq \ell\|y^* - x^*\| \leq \ell R < 1,$$

so $E^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Using the identity

$$0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*),$$

we conclude that $x^* = y^*$. □

Let $\rho = \min\left\{\frac{1}{\ell_0+\ell+\ell_1+\ell_2}, \frac{1}{\ell_0+\ell+2\ell_3+\ell_4}\right\}$. Define parameters $a_1 = \frac{\ell_1}{1-(\ell_0+\ell)\rho}$, $a_2 = \frac{\ell_2}{1-(\ell_0+\ell)\rho}$, $a_3 = a_4 = \frac{\ell_3}{1-(\ell_0+\ell)\rho}$ and $a_5 = \frac{\ell_4}{1-(\ell_0+\ell)\rho}$. Then, for $x_0, y_0 \in B(x^*, \rho)$, we have by the proof of Theorem 2.1, that

$$\|x_{n+1} - x^*\| \leq (a_1\|x_n - x^*\| + a_2\|y_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,$$

$$\begin{aligned}
\|y_{n+1}-x^*\| \leq &(a_3\|x_{n+1}-x^*\|+a_4\|x_n-x^*\|+a_5\|y_n-x^*\|)\|x_{n+1}-x^*\|\\
\leq &\|x_{n+1}-x^*\|<\rho,\\
\|x_{n+2}-x^*\| \leq &(a_1\|x_{n+1}-x^*\|+a_2\|y_{n+1}-x^*\|)\|x_{n+1}-x^*\|\\
\leq &\|x_{n+1}-x^*\|,\\
\|y_{n+2}-x^*\| \leq &(a_3\|x_{n+1}-x^*\|+a_4\|x_{n+1}-x^*\|+a_5\|y_{n+1}-x^*\|)\|x_{n+2}-x^*\|\\
\leq &\|x_{n+2}-x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+3}-x^*\| \leq &(a_1\|x_{n+2}-x^*\|+a_2\|y_{n+2}-x^*\|)\|x_{n+2}-x^*\|\\
\leq &(a_1+a_2)\|x_{n+2}-x^*\|^2.
\end{aligned}$$

Hence, we arrive at following proposition.

Proposition 2.1. *Let $T^*\subset K^*$ with r replaced by ρ. Then, sequence $\{x_n\}$ converges quadratically to x^* provided that $x_0,y_0\in B(x^*,\rho)-\{x^*\}$. Moreover, the solution x^* of equation $F(x)=0$ is unique in D_1 for $R\in\left[\rho,\frac{1}{\ell_0+\ell}\right)$.*

3. Semi-Local Convergence Analysis

Let $L_0,L,L_1,L_2>0$, $\eta\geq 0$ and $\eta_0\geq 0$ be given parameters. As in Section 2, we define a set.

Definition 3.1. Set $D_0=D\cap B(x^*,\frac{1}{L_0+L})$. The set $T=T(F,x_0,y_0)$ belongs to class $K=K(L_0,L,L_1,L_2,\eta_0,\eta)$, if

- (c_1) $F:D\subset\mathcal{B}_1\to\mathcal{B}_2$ is a Fréchet differentiable operator and $[\cdot,\cdot;F]:D^2\to\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ is a divided difference for F of order one on D^2;
- (c_2) there exists $x_0,y_0\in D$ and $\eta\geq 0$, $\eta\geq 0$ such that $A_0^{-1}\in\mathcal{L}(\mathcal{B}_2,\mathcal{B}_1)$, $\|x_0-y_0\|\leq\eta_0$ and $\|A_0^{-1}F(x_0)\|\leq\eta$;
- (c_3) there exist $L_0\geq 0$, $L\geq 0$ such that for each $x,y\in D$
$$\|A_0^{-1}([x,y;F]-A_0)\|\leq L_0\|x-x_0\|+L\|y-y_0\|;$$
- (c_4) there exist $L_i\geq 0$, $i=1,2$, such that for each $x,y,z\in D_0$
$$\|A_0^{-1}([x,y;F]-[y,z;F])\|\leq L_1\|x-y\|+L_2\|y-z\|;$$
- (c_5) $\bar{B}(x^*,t^*)\subseteq D$, where t^* is given in Lemma 3.1 that follows.

We need to define majorizing sequence $\{t_n\},\{u_n\}$ by

$$\begin{aligned}
t_0=&0,\quad u_0=\eta_0,\quad t_1=\eta,\quad u_1=L_1(1+L_0t_1+Lu_0),\\
t_2=&t_1\left(1+\frac{L_0t_1+Lu_0}{1-(L_0t_1+L(u_1+u_0))}\right),\\
u_{n+1}=&t_{n+1}+\frac{L_1(t_{n+1}-t_n)+L_2(u_n-t_n)}{1-(L_0t_n+L(u_n+u_0))}(t_{n+1}-t_n)
\end{aligned}$$

and

$$t_{n+2} = t_{n+1} + \frac{L_1(t_{n+1} - t_n) + L_2(u_n - t_n)}{1 - (L_0 t_{n+1} + L(u_{n+1} + u_0))}(t_{n+1} - t_n).$$

We also need the convergence result for the aforementioned majorizing sequences.

Lemma 3.1. ([12, Lemma 1, Page 734]). *Let $\alpha \in (0,1)$ be the unique solution of equation $q(t) = 0$, where*

$$q(t) = Lt^3 + L_0 t^2 + (L_1 + L_2)t - (L_1 + L_2).$$

Suppose that

$$0 < \frac{L_0(t_1 - t_0) + Lu_0}{1 - (L_0(t_1 - t_0) + L(u_1 + u_0))} \leq \alpha < 1 - \frac{(L_0 + L)t_1}{1 - Lu_0}.$$

Then, sequences $\{t_n\}, \{u_n\}$ are non-decreasing, bounded from above by $t^{**} = \frac{t_1}{1-\alpha}$ and converge to their unique least upper bound t^* such that $t^* \in [t_1, t^{**}]$. Moreover, for each $n = 1, 2, \ldots$ the following estimates hold:

$$\begin{aligned} 0 \leq & u_{n+1} - t_{n+1} \leq \alpha(t_{n+1} - t_n), \\ 0 \leq & t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n) \end{aligned}$$

and

$$t_n \leq u_n.$$

Based on Definition 3.1 and Lemma 3.1, we obtain the following semi-local convergence result for method (1.2).

Theorem 3.1. *Suppose $T \subseteq K$ and conditions of Lemma* 3.1 *hold. Then, sequences $\{x_n\}$ and $\{y_n\}$ generated by method* (1.2)*, starting at $x_0, y_0 \in D$ are well defined in $B(x_0, t^*)$, remain in $B(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$ and converges to the unique solution x^* of equation $F(x) = 0$ in $D_1 = D \cap \bar{B}\left(t^*, \frac{1}{L_0 + L}\right)$.*

Proof. It follows from the corresponding proof in [12, Theorem 1, Page 735] but see also the remark that follows. □

Remark 3.1. The semi-local convergence of method (1.2) was also established in [12] but there is a major difference effecting the convergence domain, error bounds on the distances $\|x_{n+1} - x_n\|, \|y_n - x_n\|$ and the uniqueness domain. Indeed, the condition used in [12] instead of (c_4) is

(c_4') $\|A_0^{-1}([x, y; F] - [u, v; F])\| \leq M_1\|x - u\| + M_2\|y - v\|$ for each $x, y, u, v \in D$ and some $M_1 \geq 0$ and $M_2 \geq 0$.

But (c_4) is weaker than $(\bar{c}_4)$ even, if $D_0 = D$. Therefore, $L_1 \leq M_1$ and $L_2 \leq M_2$, hold in general (see [1, 4, 5]). The iterates remain in D_0 which is a more accurate location than D, since $D_0 \subseteq D$ leading to tighter Lipschitz constants and the advantages (A). Define sequences $\{\bar{t}_n\}, \{\bar{u}_n\}$ as $\{t_n\}, \{u_n\}$, respectively but with M_1 replacing L_1 and

M_2 replacing L_2. Then, assuming that the rest of the hypotheses of Theorem 3.1 hold with these changes, a simple inductive argument shows that

$$\begin{aligned} 0 \leq & u_{n+1} - t_{n+1} \leq \bar{u}_{n+1} - \bar{t}_{n+1} \leq \bar{\alpha}(\bar{t}_{n+1} - \bar{t}_n), \\ 0 \leq & t_{n+2} - t_{n+1} \leq \bar{t}_{n+2} - \bar{t}_{n+1} \leq \bar{\alpha}(\bar{t}_{n+1} - \bar{t}_n), \\ t_n \leq & \bar{t}_n, \\ u_n \leq & \bar{u}_n \end{aligned}$$

and

$$t^* \leq \bar{t}^* = \lim_{n\to\infty} \bar{t}_n,$$

where $\bar{\alpha} \in (0,1)$ is the unique solution of equation $\bar{q}(t) = 0$, with

$$\bar{q}(t) = Lt^3 + L_0 t^2 + (M_1 + M_2)t - (M_1 + M_2).$$

Notice that

$$\begin{aligned} \bar{q}(\alpha) =& L\alpha^3 + L_0\alpha^2 + (M_1 + M_2)\alpha - (M_1 + M_2) \\ =& q(\alpha) + [(M_1 - L_1) + (M_2 - L_2)](\alpha - 1) < 0, \end{aligned}$$

since $q(\alpha) = 0$, $\alpha \in (0,1)$, $L_1 \leq M_1$ and $L_2 \leq M_2$. Therefore, we have $\alpha \leq \bar{\alpha}$. Hence, we justified the claim made in the introduction (see also the numerical examples).

4. Numerical Examples

We present the following examples to test the convergence criteria. Define the divided difference by

$$[x, y; F] = \int_0^1 F'(\tau x + (1-\tau)y) d\tau.$$

Example 4.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$ be the space of continuous functions defined in $[0,1]$ equipped with the max norm. Let $D = \{z \in C[0,1] : \|z\| \leq 1\}$. Define F on D by [1, 13]:

$$F(x)(s) = x(s) - f(s) - \frac{1}{8}\int_0^1 G(s,t)x(t)^3\, dt, \quad x \in C[0,1],\ s \in [0,1],$$

where $f \in C[0,1]$ is a given function and the kernel G is the Green's function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

Notice that nonlinear integral equation $F(x)(s) = 0$ is of Chandrasekhar type [1, 4, 5, 10]. Then $F'(x)$ is a linear operator given for each $x \in D$, by

$$[F'(x)(v)](s) = v(s) - \frac{3}{8}\int_0^1 G(s,t)x(t)^2 v(t)\, dt, \quad v \in C[0,1],\ s \in [0,1].$$

If we choose $x_0(s) = f(s) = s$, then we obtain $\|F'(x_0)\| \leq \frac{1}{64}$.

Choose $x_{-1} = 2s$, we see [13] that $L_1 = 0.08125\ldots$, $L_2 = 0.040625\ldots$, $L = 0.0359375\ldots$, $L_0 = 0.071875\ldots$, $t_1 = \eta = 0.0298507$ and $u_1 = \eta_1 = 1$. Notice that hypothesis $0 < \frac{L_0(t_1-t_0)+Lu_0}{1-(L_0(t_1-t_0)+L(u_1+u_0))} \leq \alpha < 1 - \frac{(L_0+L)t_1}{1-Lu_0}$ is satisfied. So, we can guarantee the convergence of the Secant method (1.2) from Theorem 2.1.

Example 4.2. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $D = B(0,1)$, $x^* = (0,0,0)^T$ and define F on D by

$$F(x) = F(x_1, x_2, x_3) = \left(e^{x_1} - 1, \frac{e-1}{2}{x_2}^2 + x_2, x_3\right)^T.$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and (a_3)-(a_4) and since $F'(x^*) = \mathrm{diag}(1,1,1)$, we can define parameters for method (1.2) by $\ell_1 = 0$, $\ell_0 = \ell = \ell_2 = \frac{e-1}{2}$, $\ell_3 = \frac{e^{\frac{1}{e-1}}}{2}$, $\ell_4 = \frac{e-1}{2}$. Then, the radius of convergence using (2.1) is given by $r = 0.2607$. Local results were not given in [12] but if they were, $\bar{\ell}_0 = \bar{\ell} = \frac{e-1}{2}$, $\bar{\ell}_1 = 0$, $\bar{\ell}_2 = \frac{e}{2}$, then $\bar{\ell}_3 = \bar{\ell}_4 = \frac{e}{2}$. Therefore, by (2.1) with ℓ_4 replacing $\bar{\ell}_4$, we get $\bar{r} = 0.2340$.

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