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VERTEX-EDGE ROMAN DOMINATION

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Abstract. A vertex-edge Roman dominating function (or just ve-RDF) of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge $e = uv$ either $\max\{f(u), f(v)\} \neq 0$ or there exists a vertex w such that either $wu \in E$ or $wv \in E$ and $f(w) = 2$. The weight of a ve-RDF is the sum of its function values over all vertices. The vertex-edge Roman domination number of a graph G, denoted by $\gamma_{veR}(G)$, is the minimum weight of a ve-RDF G. In this paper, we initiate a study of vertex-edge Roman dominaton. We first show that determining the number $\gamma_{veR}(G)$ is NP-complete even for bipartite graphs. Then we show that if T is a tree different from a star with order n, l leaves and s support vertices, then $\gamma_{veR}(T) \geq (n - l - s + 3)/2$, and we characterize the trees attaining this lower bound. Finally, we provide a characterization of all trees with $\gamma_{veR}(T) = 2\gamma'(T)$, where $\gamma'(T)$ is the edge domination number of T.

1. Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is the cardinality of its open neighborhood, denoted $d_G(v) = |N(v)|$. By $\delta(G) = \delta$ we denote the *minimum degree* of a graph G. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A support vertex is *strong* (*weak*, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). An edge incident with a leaf is called a *pendant edge*. A *star* of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with at least $n - 1$ leaves. A *double star* is a tree that contains exactly two vertices that are not leaves. A double star with respectively r and s leaves attached to each support vertex is denoted by $D_{r,s}$.

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Let D be a nonempty subset of E. The *subgraph* of G whose vertex set is the set of ends of edges in D and whose edge set is D is called the subgraph of G induced by D and is denoted by $\langle D\rangle$. The subgraph $\langle D\rangle$ is called *edge induced subgraph* of G. The *distance* between two vertices u and v in a connected graph G is the number of edges in a shortest between u and v. The *diameter,* $\operatorname{diam}(G)$, of a graph G is the greatest distance between any pair of vertices.

A set S of vertices is a *dominating set* of G if every vertex not in S is adjacent to some vertex in S. A subset X of E is an *edge dominating* set (or just EDS) of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set. An edge dominating set of G of minimum cardinality is called a $\gamma'(G)$-set. Edge domination was introduced by Mitchell and Hedetniemi [7].

A vertex v *ve-dominates* every edge incident to v, as well as, every edge adjacent to these incident edges, that is, a vertex v *ve*-dominates every edge incident to a vertex in $N[v]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The minimum cardinality of a *ve*-dominating set of G is called the *ve-domination number* $\gamma_{ve}(G)$. The concept of vertex-edge domination was introduced by Peters [8] in 1986 and studied further in [1, 5, 6].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. For more information on Roman domination, see [3, 4].

A variation of Roman dominating function, say, vertex-edge Roman dominating function was defined in [9]. A vertex-edge Roman dominating function (ve-RDF) is a function $f : V(G) \to \{0, 1, 2\}$ such that each edge $e = vu$ is either incident with a vertex having function value at least one or uv is *ve*-dominated by some vertex w with $f(w) = 2$. The *vertex-edge Roman domination number* $\gamma_{veR}(G)$ equals the minimum weight of all ve-RDF on G.

2. Complexity

We show that the Vertex-edge Roman domination problem (VERD-Dom) is NP-complete for bipartite graphs by proposing a polynomial reduction from the well-known NP-complete problem, Exact cover by 3-sets (X3C).

Vertex-Edge Roman Domination (VERD)
INSTANCE. Graph $G = (V, E)$, positive integer $k \leq |V|$.
QUESTION. Does G have an vertex-edge Roman dominating function of weight at most k?

Exact cover by 3-sets(X3C)
INSTANCE. A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION. Does C contain an exact cover for X, that is, a sub collection $C' \subseteq C$ such that for every element in X belongs to exactly one member of C'?

Theorem 2.1. *VERD problem in NP-complete for bipartite graphs.*

Proof. VERD problem is a member of *NP*, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has a weight at most k and that is a vertex-edge Roman dominating function. Now let us show how to transform any instance of X3C into an instance G of VERD, so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we create a path $P_6^i = x_i y_i z_i a_i b_i p_i$ and for each C_j we create a single vertex c_j. To obtain the graph G, we add edges $c_j x_i$ if $x_i \in C_j$. Clearly, G is bipartite graph. Let $Y = \{c_1, c_2, \ldots, c_t\}$ and $W = \{x_1, x_2, \ldots, x_{3q}\}$. Let H be the subgraph of G induced by all paths P_6^i's. Set $k = 8q$. Observe that for any vertex-edge Roman dominating function f on G, $f(V(P_6^i)) \geq 2$.

Suppose that the instance X, C of $X3C$ has a solution C'. We construct a vertex-edge Roman dominating function of G with weight k as follows. For each $i \in \{1, 2, \ldots, 3q\}$, we assign a 0 to every vertex of $\{x_i, y_i, z_i, b_i, p_i\}$ and we assign a 2 to every a_i. For every $j \in \{1, 2, \ldots, t\}$, we assign a 2 to c_j if $C_j \in C'$ and a 0 if $C_j \notin C'$. Note that since C' exists, its cardinality is precisely q and so the number of c_j's with weight 2 is q, having disjoint neighborhoods in W. Since C' is a solution for X3C, the edges incident with W are *ve*-Roman dominated by the c_j's. Hence it is straightforward to see that f is a vertex-edge Roman dominating set of G with cardinality $8q = k$.

Conversely, suppose that G has a vertex-edge Roman dominating function $f = (V_0, V_1, V_2)$ with weight at most k. As seen above we may assume, without loss of generality, that $a_i \in V_2$ and every vertex of $\{p_i, b_i, z_i, y_i\}$ is in V_0. Since $\sum_{i=1}^{3q} f(a_i) = 6q$, we deduce that $f(W \cup Y) \leq 2q$. If some x_i belongs to V_2, then we can substitue it by a vertex of $N(x_i) \cap Y$. Hence $W \cap V_2 = \emptyset$. Now if there are two vertices x_i and x_r assigned a 1 and have a common neighbor, say c_j, then we can reassign a 0 to each of x_i and x_r and a 2 to c_j. So all vertices of $V_1 \cap W$ have no common neighbors. Suppose x_i and x_j are assigned a 1. The vertices adjacent to $(N(x_i) \cap Y) \setminus \{x_i\}$ are assigned 0. To dominates the edges incident with these vertices, the vertex in $N(x_i) \cap Y$ are assigned weight 2. Since $|W| = 3q$, we must have $W \cap V_0 = \emptyset$, implying that $C \cap V_2 \neq \emptyset$. Let $y = |C \cap V_2|$. Clearly $y \leq 2q$ and using the fact that every c_j has exactly three neighbors in W, we deduce that $f(C) \geq 2q$. Now, combining all these facts with $f(V(G)) \leq k = 8q$, we obtain $y \geq q$ and hence $y = q$. Hence, $C' = \{C_j \mid f(c_j) = 2\}$ is an exact cover for C. $\square$

3. Bounds

We present in this section some sharp bounds on the vertex-edge Roman domination number. We begin with the following observation.

Observation 3.1. Let $f = (V_0, V_1, V_2)$ be an minimum vertex-edge Roman dominating function of a graph G. Then

(a) $|V_0| \geq 1$;
(b) no edge of G joins V_1 and V_2;
(c) $V_1 \cup V_2$ is a vertex edge dominating set of G.

In the following, we give a lower bound on the vertex-edge Roman domination for every graph in terms of the order and maximum degree.

Proposition 3.1. *If G is a connected graph of order $n \geq 2$, then $\gamma_{veR}(G) \geq \left\lceil \frac{2n}{(\Delta+1)^2} \right\rceil$, and the bound is sharp.*

Proof. Let $f = (V_0, V_1, V_2)$ be an $\gamma_{veR}(G)$-function. From the Observation 3.1, we have $|V_0| \geq 1$. The edge of G are ve-dominated by the vertices in $V_1 \cup V_2$. Therefore $|V_0| \leq \Delta^2|V_2| + \Delta|V_1|$. From $n = |V_0| + |V_1| + |V_2| \leq \Delta^2|V_2| + \Delta|V_1| + |V_1| + |V_2|$, we obtain $\frac{2n}{(\Delta+1)^2} \leq 2|V_2| + \frac{2|V_1|}{\Delta+1} \leq 2|V_2| + |V_1| = \gamma_{veR}(G)$. Since $\gamma_{veR}(G)$ is an integer, we get $\gamma_{veR}(G) \geq \left\lceil \frac{2n}{(\Delta+1)^2} \right\rceil$. The bound is sharp as it is attained for stars $K_{1,n}$. $\square$

Every Roman dominating function is a vertex-edge roman dominating function, we have the following.

Proposition 3.2. *If G is connected graph of order $n \geq 2$ with maximum degree Δ, then $\gamma_{veR}(G) \leq n - \Delta + 1$ and the bound is sharp.*

We now present an upper bound of vertex-edge Roman domination in terms of edge domination number.

Proposition 3.3. *For any graph G, $\gamma_{veR}(G) \leq 2\gamma'(G)$.*

Proof. Let D be a $\gamma'(G)$-set. Define a function f on $V(G)$ by assigning a 1 to the vertices incident with the edges in D and a 0 to the remaining vertices. It is easy to see that f is a veR-dominating function of G, and thus, $\gamma_{veR}(G) \leq 2\gamma'(G)$. $\square$

3.1. **Trees.** In this section we provide a lower bound of the vertex-edge Roman domination number for trees with diameter at least three in terms of order n, number of leaves l and support vertices s. We shall show that vertex-edge Roman domination number of a tree with diameter at least three of order n with l leaves and s support vertices bounded below by $(n - l - s + 3)/2$. Let T^* be the tree obtained from $K_{1,3}$ by subdividing two edges and α be the leaf which is incident to the edge which is not subdivided. Moreover, for the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let

$T_1 = P_5$ or P_7. If k is a positive integer, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_i.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_i adjacent to mP_2 where $m \geq 2$.
- Operation $\mathcal{O}_3$: Attach a tree T^* by joining the vertex α to a leaf of T_i.
- Operation $\mathcal{O}_4$: Attach a path P_4 by joining one of its leaves to a vertex of T_i is a leaf or adjacent to P_2 or P_4

Lemma 3.1. *If $T \in \mathcal{T}$, then $\gamma_{veR}(T) = (n - \ell - s + 3)/2$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If T is P_5, then obviously $\gamma_{veR}(T) = 2 = (n - \ell - s + 3)/2$. Let k be a positive integer. Assume the result is true for $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let v be a support vertex and x be a leaf adjacent to v in T'. Let the tree T is obtained from T' by attaching a vertex y to v. We have $n = n' + 1$, $l = l' + 1$ and $s' = s$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of T'. If $f_1(x) = 1$ then $f_1(v) = 0$. Replacing the weight of x and v, we get f_1 is a veR-dominating function of tree T. If $f_1(x) = 2$ or 0 then the vertex which dominates the edge vx dominates vy. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Let f be a γ_{veR}-dominating function of tree T. If $f(y) = 0$ then $f|_{V(T')}$ is a veR-dominating function of T'. Let $f(y) = 1$ then $f(x) = 1$. The function $f|_{V(T')}$ is a veR-dominating function of T'. Assume $f(y) = 2$ then $f(x) = 0$. Replacing the weight of x and y, we get $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - l - s + 3)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. Let u be the vertex in T' which is adjacent to many P_2. Let the tree T is obtained from T' by attaching the path $P_2 = xy$ by joining x to u. We have $n' = n - 2, l' = l - 1$ and $s' = s - 1$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. To dominate the edges incident to vertices in $V(T_u)$, the vertex u is assigned weight two. The function

$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 0, & \text{otherwise,} \end{cases}$$

is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges incident to vertices in $V(T_u)$, to the vertex u is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 2 - l + 1 - s + 1 + 3)/2 = (n - l - s + 3)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. Let d be the leaf in T'. Let the tree T is obtained from T' by attaching a tree T^* by the vertex α. We have $n = n' + 6$, $l = l' + 1$ and $s = s' + 1$. Let f_1 a $\gamma_{veR}(T')$-dominating function of tree T'.

The function

$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 2, & \text{if Child of } \alpha, \\ 0, & \text{otherwise }, \end{cases}$$

is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges incident to the vertices in $V(T_\alpha)$, to the child of α is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \le \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 2 = (n - 6 - l + 1 - s + 1 + 3)/2 + 2 = (n - l - s + 3)/2$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_4$. Let d be the leaf in T'. Let the tree T is obtained from T' by attaching a path $P_4 = wuvt$ by joining w to d. We have $n = n' + 4, l' = l$ and $s' = s$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. The function

$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = u, \\ 0, & \text{otherwise}, \end{cases}$$

is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges tv, vu, uw and wd, to the vertex u is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \le \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 2 = (n - 4 - l - s + 3)/2 + 2 = (n - l - s + 3)/2$.

Now, d is adjacent to a path P_2 or P_4. Let the tree T is obtained from T' by attaching a path $P_4 = wuvt$ by joining w to d. We have $n = n' + 4$, $l = l' + 1$ and $s = s' + 1$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. Thus, the weight of d is two in T'. Then the

$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 1, & \text{if } a = u, \\ 0, & \text{otherwise}, \end{cases}$$

is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 1$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges tv, vu, uw and wd, the vertex d is assigned the weight two and v is assigned the weight one. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \le \gamma_{veR}(T) - 1$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 1 = (n - 4 - l + 1 - s + 1 + 3)/2 + 1 = (n - l - s + 3)/2$. □

We now ready to establish the lower bound.

Theorem 3.1. *If T is a tree with* $\operatorname{diam}(T) \ge 3$ *of order n with l leaves and s support vertices, then $\gamma_{veR}(T) \ge (n - l - s + 3)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then by Lemma 3.1, $\gamma_{veR}(T) = (n - l - s + 3)/2$. If $\operatorname{diam}(T) = 3$, then T is a double star. We have $l = n - 2$ and $s = 2$. Consequently, $(n - l - s + 3)/2 = (n - n + 2 - 2 + 3)/4 = 3/2 < 2 = \gamma_{veR}(T)$. Now, assume that $\operatorname{diam}(T) \ge 4$. Thus, the order n of the tree is at least five. We obtain the result by induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

Assume any support vertex of T, say y, is strong. Let x and t be the leaves adjacent to y. Let $T' = T - x$. We have $n' = n - 1$ and $l' = l - 1$. Let f be a $\gamma_{veR}(T)$-dominating function of a tree T. If $f(x) = 0$ then $f|_{V(T')}$ is a veR-dominating function of T'. If $f(t) = 1$ then $f(x) = 1$. The function $f|_{V(T')}$ is a veR-dominating function of T'. Assume $f(x) = 2$ then $f(t) = 0$. Replacing the weight of x and t, we get $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') \geq (n'-l'-s'+3)/2 = (n-l-s+3)/2$. If $\gamma_{veR}(T) = (n-l-s+3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_1$. Therefore, $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

Let $x_0x_1x_2 \ldots x_{d-1}x_d$ be the longest path in tree T. We now root the tree at a vertex x_d. Clearly $d_T(x_0) = d_T(x_d) = 1$. From the previous paragraph, we can assume $d_T(x_1) = d_T(x_{d-1}) = 2$.

Now, assume that x_2 is adjacent to a leaf y_1. Let $T' = T - y_1$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edge x_0x_1 and x_1x_2, to the vertex x_2 is assigned the weight two. Clearly $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 1 - l + 1 - s + 1 + 3)/2 > (n - l - s + 3)/2$.

Now, assume that x_2 is adjacent to paths $P_i = y_{1_i}y_{2_i}$ where $i = 1, 2, \ldots, m$ $(m \geq 2)$ other than x_1x_0. Let $T' = T - T_{x_1}$. We have $n' = n-2$, $l' = l-1$ and $s' = s-1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_2x_1, x_1x_0, $x_2y_{1_i}$ and $y_{1_i}y_{2_i}$, to the vertex x_2 is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_2$. Therefore, $T \in \mathcal{T}$.

Assume that x_2 is adjacent to a path $P_2 = y_1y_2$ other than x_1x_0. If $d_T(x_2) = 2$, then $T = P_5$, we have $\gamma_{veR}(P_5) = 2 = (n - l - s + 3)/2$. Thus, $T \in \mathcal{T}$. Assume $\deg(x_2) = 3$. Let us consider some child of x_3 say t is not a leaf. It suffices to consider x_3 is adjacent to isomorphic copy of T_{x_2}. Let $T' = T - T_{x_2}$. We have $n' = n - 5$, $l' = l - 2$ and $s' = s - 2$. To dominate the edges incident to vertices in $V(T_t)$, to the vertex t is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 5 - l + 2 - s + 2 + 3)/2 + 2 > (n - l - s + 3)/2$.

Assume x_3 is adjacent to path $P_3 : tuv$. Let $T' = T - T_t$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. To dominate the edge x_0x_1, x_1x_2, to the vertex x_2 is assigned the weight two. It is easy to see that the vertex x_2 dominates the edge x_3t. To dominate the edge tu and uv, to the vertex u is assigned the weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 \geq (n' - l' - s' + 3)/2 + 1 \geq (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume x_3 is adjacent to path $P_2 : tu$. Let $T' = T - T_t$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. To dominate the edge x_0x_1, x_1x_2, to the vertex x_2 is assigned the weight two. It is clear that the vertex x_2 dominates the edge x_3t. To dominate the edge tu, to the vertex u is assigned the weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 \geq (n' - l' - s' + 3)/2 + 1 \geq (n - 2 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume x_3 is a support vertex. Let t be a child of x_3 other than x_2. From operation $\mathcal{O}_1$, it suffices to consider $d_T(x_3) = 3$. Let $T' = T - T_t$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. To dominate the edge x_3t, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') \geq (n'-l'-s'+3)/2 \geq (n-1-l+1-s+1+3)/2 > (n-l-s+3)/2$.

Suppose $\deg(x_3) = 2$. Now assume that $d_T(x_4) \geq 3$. Let $T' = T - T_{x_3}$. We have $n' = n - 6, l' = l - 2$ and $s' = s - 2$. To dominate the edges incident to $V(T_{x_3})$, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 6 - l + 2 - s + 2 + 3)/2 + 2 > (n - l - s + 3)/2$.

Now $\deg(x_4) = 2$. Let $T' = T - T_{x_3}$. We have $n' = n - 6$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges incident to the vertices in $V(T_{x_3})$, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 6 - l + 1 - s + 1 + 3)/2 + 2 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_3$. Therefore, $T \in \mathcal{T}$.

Now, assume $d_T(x_2) = 2$. Suppose that x_3 is adjacent to a path $P_3 = y_2y_1y_0$ other than $x_0x_1x_2$. Let x_3 be adjacent to y_2. Let $d_T(x_3) = 2$. We have $T = P_7$. It is easy to see that $\gamma_{veR}(P_7) = (n - l - s + 3)/2$. Thus, $T \in \mathcal{T}$. Now assume that $d_T(x_3) \geq 3$. Let $T' = T - T_{x_2}$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges y_0y_1, y_1y_2, y_2x_3 and x_3x_2, to the vertex y_2 is assigned the weight two. To dominate the edges x_2x_1 and x_1x_0, to the vertex x_1 is assigned weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 = (n' - l' - s' + 3)/2 + 1 = (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume that x_3 is adjacent to a path $P_2 = y_2y_1$ with x_3 adjacent to y_2. Let $T' = T - T_{x_2}$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges y_1y_2, y_2x_3, x_2x_1 and x_3x_2, to the vertex x_3 is assigned the weight two. To dominate the edge x_1x_0, either x_1 or x_0 is assigned weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T')+1 = (n'-l'-s'+3)/2+1 = (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Now, assume that x_3 is a support vertex. Let x be the leaf adjacent to x_3. Let $T' = T - T_x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges x_0x_1, x_2x_1, x_2x_3 and x_3x, to the vertex x_2 is assigned the weight two. It is clear that the function $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 1 - l + 1 - s + 1 + 3)/2 > (n - l - s + 3)/2$.

Assume that some child of x_4, say y_1 other than x_3 such that distance of d to the most distance vertex of T_{y_1} is 2 or 4. It suffices to consider the case when T_x is $P_2 = y_1y_2$ or $P_4 = y_1y_2y_3y_4$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_4x_3, x_3x_2, x_2x_1, x_1x_0, x_4y_1 and y_1y_2, to the vertices x_4 and x_1 are assigned the weights 2 and 1 respectively. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 = (n' - l' - s' + 3)/2 + 1 = (n - 4 - l + 1 - s + 1 + 3)/2 + 1 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Therefore, $T \in \mathcal{T}$.

Assume that some child of x_4, say x other than x_3 such that distance of d to the most distance vertex of T_x is one or three. It suffices to consider the case when T_x is $P_1 = y_1$ or $P_3 = y_1y_2y_3$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_4x_3 ,x_3x_2, x_2x_1 and x_1x_0, to the vertex x_2 is assigned the weight two. Thus, $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T')+2 = (n'-l'-s'+3)/2+2 = (n - 4 - l + 1 - s + 1 + 3)/2 + 2 > (n - l - s + 3)/2$.

Now, $d_T(x_4) = 2$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l$ and $s' = s$. To dominate the edges x_4x_3 ,x_3x_2, x_2x_1 and x_1x_0, to the vertex x_2 is assigned the weight two. Thus, $f|_{V(T')}$ is a veR-dominating function of T'. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T')+2 = (n'-l'-s'+3)/2+1 = (n - 4 - l - s + 3)/2 + 1 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Therefore, $T \in \mathcal{T}$. $\square$

4. Trees T with $\gamma_{veR}(T) = 2\gamma'(T)$

In this section we provide a constructive characterization of trees with equal vertex-edge Roman domination number and twice edge domination number. For the purpose of characterizing the trees with equal vertex-edge Roman domination number and twice edge domination number, we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_4$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

- Operation $\mathcal{O}_5$: Attach a vertex by joining it to any support vertex of T_i.
- Operation $\mathcal{O}_6$: Attach a path $P_4 = pqrs$ by joining the vertex q of a vertex w of T_i adjacent to path $P_4 = xuvt$ with w adjacent to u.
- Operation $\mathcal{O}_7$: Attach a double star $D_{r,s}(r, s \geq 2)$ by joining one of its leaf to a vertex of T_i adjacent to a path P_4 or P_3 or P_2 or P_1 or double star.

Lemma 4.1. *If $T \in \mathcal{F}$, then $\gamma_{veR}(T) = 2\gamma'(T)$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If T is P_5, then obviously $\gamma_{veR}(T) = 2 = 2\gamma'(T)$. Let k be a positive integer.

Assume the result is true for $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_5$. Let u be a support vertex and x be a leaf adjacent to u in the graph T'. The graph T is obtained from T' by adding a vertex y to u. Let D be a $\gamma'(T)$-set. To dominate the edges ux and uy, an edge incident with u other than ux and uy is in D. It is obvious that D is an EDS of T'. Thus, $\gamma'(T') \le \gamma'(T)$. Let D' be a $\gamma'(T')$-set. The edge which dominates ux dominates the edge uy in graph T. Thus, $\gamma'(T) \le \gamma'(T')$. We have $\gamma'(T) = \gamma'(T')$. Let f_1 be a $veR(T')$-dominating function of T'. If the vertex x has weight one, then the vertex u has weight zero. Replace the weight of these two vertices. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T')$. Let f be a γ_{veR}-dominating function of T. To dominate the edges ux and yu, the vertex u is assigned with weight one or a vertex in $N(u)$ is assigned with weight two. If the leaf y is assigned weight two, then the vertex x has weight zero. Replace the weight of x from zero to two. The function f is a veR-dominating function of T'. If the vertex u is assigned with weight one then f is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \le \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = 2\gamma'(T') = 2\gamma'(T)$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_6$. Let the vertex $w \in T'$ be adjacent to path $P_4 = xuvt$ with u adjacent to w. The graph T is obtained from T' by adding another path $P_4 = pqrs$ with q adjacent to w. Let D be a $\gamma'(T')$-set of T'. It is clear that $D \cup \{qr\}$ is an EDS of T. Thus, $\gamma'(T) \le \gamma'(T) + 1$. Let D' be a $\gamma'(T)$-set. To dominate the edges rs and vt, the edges $qr, uv \in D'$. It is easy to verify that $D' \setminus \{qr\}$ is an EDS of the graph T'. Thus, $\gamma'(T') \le \gamma'(T) - 1$. We have $\gamma'(T) = \gamma'(T') + 1$. Let f be a γ_{veR}-function of T'. To dominate the edges vt, uv and ux, the vertex u is assigned with weight two. Define a function f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = r, \\ 0, & \text{if } a = p, q, s. \end{cases}$$

Clearly, f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2$. Let f_1 be a $\gamma_{veR}(T)$-dominating function. As in the previous case, the vertex r and u are assigned a weight two. The function $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \le \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T') = \gamma_{veR}(T) - 2$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = 2\gamma'(T') + 2 = 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_7$. Let d be a vertex of T' with $d_{T'}(d) \ge 3$. Let d be adjacent to P_4 or P_3 or P_2 or P_1 or $D_{r,s}$, $r, s \ge 2$. The graph T is obtained from T' by joining a leaf of $D_{r,s}$, $r, s \ge 2$, to d. Let the support vertices of $D_{r,s}$ be u and v. Let the leaves of u be w and w_1 and the leaves of v be t and t_1. Let w be adjacent to d. Let D be a $\gamma'(T')$-set. The vertex d is adjacent to P_4 or P_3 or P_2 or P_1 or $D_{r,s}(r, s \ge 2)$, an edge incident with d is in D. It is easy to see that $D \cup \{uv\}$ is an EDS of the graph T. Thus, $\gamma'(T) \le \gamma'(T') + 1$. Let D' be a $\gamma'(T)$-set. To dominate the edges vt, uw and uw_1, the edge uv is in D'. It is obvious that $D' \setminus \{uv\}$

is an EDS of graph T'. Thus, $\gamma'(T') \leq \gamma'(T) - 1$. We have $\gamma'(T') = \gamma'(T) - 1$. Let f_1 be a γ_{veR}-dominating function of T. To dominate the edges vt and uv, the vertex u is assigned with weight two. It is obvious that $f_1|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 2$. Let f be a $\gamma_{veR}(G)$-dominating function of T'. Define f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = u, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = 2\gamma'(T') + 2 = 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. ☐

The following theorem gives a characterization of trees for which $\gamma_{veR}(T) = 2\gamma'(T)$.

Theorem 4.1. *Let T be a nontrivial tree. Then $\gamma_{veR}(T) = 2\gamma'(T)$ with equality if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then by Lemma 4.1, $\gamma_{veR}(T) = 2\gamma'(T)$. If $\operatorname{diam}(T) = 1$ or 2, then T is P_2 or star. We have $\gamma_{veR}(T) = 1 < 2 = 2\gamma'(T)$. Assume $\operatorname{diam}(T) = 3$. If T is P_4. We have $\gamma_{veR}(T) = 2\gamma'(T)$. If T is a double star other than P_4, then T can be obtained from P_4 by applying operation $\mathcal{O}_1$. The result is proved by induction on order n. Assume that the result is true for all tree T' of order $n' < n$.

Let u be a strong support vertex. Let u be adjacent to two leaves x and y. Let $T' = T - x$. Let D be a any $\gamma'(T')$-set. To dominate the edges ux and uy, an edge incident with u other than ux and uy is in D. It is easy to see that D is an EDS of T'. Thus, $\gamma'(T') \leq \gamma'(T)$. Let f_1 be a $veR(T')$-dominating function of G. If the vertex x has weight one, then the vertex u has weight zero. Replace the weight of these two vertices. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') \leq 2\gamma'(T') \leq 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By the inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_5$. Thus, $T \in \mathcal{F}$. Henceforth, we can assume that every support vertex of T is weak.

Let $u_1u_2u_3 \ldots u_k$ be the longest path in the tree T. Then $k \geq 4$ and $d_T(u_1) = d_T(u_k) = 1$. The vertices u_2 and u_{k-1} are support vertices, we can assume $d_T(u_2) = d_T(u_{k-1}) = 2$.

Assume that u_3 is adjacent to a path $P_2 = pq$ other than u_2u_1. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and pq, the edges u_2u_3, pu_3 is in D. Define a function f on $V(T)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_2, u_3, p\}$, assigning weight two to u_3 and zero to all other vertices. It is clear that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 2 < 2\gamma'(T)$. Hence, the vertex u_3 is a support vertex. By operation $\mathcal{O}_5$, it suffices to consider $d_T(u_3) = 3$. Let x be a leaf adjacent to u_3.

Assume that u_4 is adjacent to a path $P_3 = pqr$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rq, the edges u_2u_3, pq is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_2, p\}$, assigning weight two to u and zero to all other vertices. It is easy to observe that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is adjacent to a path $P_2 = pq$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and pq, the edges u_2u_3, pu_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_4, u_3, p\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is a support vertex. Let y be the leaf adjacent to u_4. Let $d_T(u_4) = 2$. We have T is G_1, where G_1 is obtained from P_5 by attaching a leaf adjacent to vertex of P_5 with minimum eccentricity. We have $\gamma_{veR}(G_1) = 2 < 4 = 2\gamma'(G_1)$. Assume $d_T(u_4) \geq 3$. Let d be a vertex adjacent to u_4 other than u_3 and y. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and u_4y, the edges u_3u_2, du_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, d\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is adjacent to $P_4 = pqrs$ with q adjacent to u_4. Let $T' = T - T_q$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rs, the edges $u_3u_2, qr \in D'$. It is easy to verify that $D \setminus \{qr\}$ is an EDS of the graph T'. Thus, $\gamma'(T') \leq \gamma'(T) - 1$. Let f be a γ_{veR}-function of T. To dominate the edges u_1u_2, u_2u_3 and u_3x, the vertex u is assigned with weight two. Define a function f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = q, \\ 0, & \text{if } a = p, r, s. \end{cases}$$

Clearly, f_1 is a veR-dominating function of H. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2 \leq 2\gamma'(T') + 2 \leq 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_6$. Thus, $T \in \mathcal{F}$.

Assume $d_T(u_4) = 2$. Let $d_T(u_5) \geq 3$. Let $T' = T - T_{u_4}$. Let D be a $\gamma'(T)$-set. To dominate the edges u_4u_3, u_3x and u_2u_1, the edge u_3u_2 is in D. It is obvious that $D \setminus \{u_3v_2\}$ is an EDS of graph G. Thus, $\gamma'(T') \leq \gamma'(T) - 1$. Let f be a $\gamma_{veR}(T')$-dominating function. Define f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = u_3, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f_1 is a veR-dominating function of H. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2 \leq 2\gamma'(T') + 2 \leq 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_7$. Thus, $T \in \mathcal{F}$.

Assume $d_T(u_5) = 2$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and u_5u_4, the edges u_2u_3, u_5u_6 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_2, u_3, u_5\}$, assigning weight two to u_3 and zero to all other vertices. It is clear that f is a veR-dominating function of T. Thus, $\gamma_{veR}(G) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Now, assume $d_T(u_3) = 2$. Assume the vertex u_4 is adjacent to path $P_3 = pqr$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rq, the edges u_2u_3, pq is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_2, p\}$, assigning weight two to u_3 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume the vertex u_4 is adjacent to path $P_2 = pq$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and pq, the edges u_3u_2, pu_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, p\}$, assigning weight two to u_4 and zero to all other vertices. It is clear that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume the vertex u_4 is a support vertex. Let x be the leaf adjacent to u_4. Assume that $d_T(u_4) = 2$. We have $T = P_5$ and $\gamma_{veR}(T) = 2 < 4 = 2\gamma'(T)$. Now assume $d_T(u_4) \geq 3$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and xu_4, the edges u_3u_2 and an edge incident with u_4, say u_4d, other than u_4u_3 and u_4x is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, d\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to see that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Now, $d_T(u_4) = 2$. Let $d_G(u_5) = 1$. Then T is P_5. We have $\gamma_{veR}(T) = 2 < 4 = 2\gamma'(T)$. Assume $d_T(u_5) \geq 2$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and u_4u_5, the edges u_3u_2, u_4u_6 is in D. Define a function f on $V(T)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_5, u_6\}$, assigning weight two to the vertex u_5 and zero to all other vertices. It is obvious that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$. □

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References

[1] R. Boutrig, M. Chellali, T. W. Haynes and S. T. Hedetniemi, *Vertex-edge domination in graphs*, Aequationes Math. **90** (2016), 355–366.

[2] M. Chellali, T. W. Haynes and S. T. Hedetniemi, *Bounds on weak Roman and 2-rainbow domination numbers*, Discrete Appl. Math. **178** (2014), 27–32.

[3] M. Chellali and N. Jafari Rad, *Trees with unique Roman dominating functions of minimum weight*, Discrete Math. Algorithms Appl. **6** (2014), Paper ID 1450038.

[4] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **78** (2004), 11–22.
[5] B. Krishnakumari, Y. B. Venkatakrishnan and M. Krzywkowski, *Bounds on the vertex-edge domination number of a tree*, C. R. Math. Acad. Sci. Paris **352** (2014), 363–366.
[6] J. R. Lewis, S. T. Hedetniemi, T. W. Haynes and G. H. Fricke, *Vertex-edge domination*, Util. Math. **81** (2010), 193–213.
[7] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Congr. Numer. **19** (1977), 489–509.
[8] K. W. Peters, *Theoretical and algorithmic results on domination and connectivity*, Ph.D. Thesis, Clemson University, 1986.
[9] E. N. Satheesh, *Some variations of domination and applications*, Ph.D. Thesis, Mahatma Gandhi University, 2014.

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