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FRACTIONAL CALCULUS PERTAINING TO MULTIVARIABLE I-FUNCTION

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Abstract. In this paper, we study and investigate unified and extended fractional integral operator involving the multivariable I-function defined by Prasad, Raizada's generalized polynomial set and general class of multivariable polynomials. During the present study, we derive five theorems pertaining to Mellin transforms of these operators. Furthermore, on account of the general nature of the functions involved herein, many known and (presumably) new fractional integral operators involved simpler functions can be obtained. We also give the special case concerning the multivariable H-function.

1. Introduction and Preliminaries

Fractional calculus is a branch of mathematical analysis that deals with derivatives and integrals of arbitrary orders. Recently, it has been shown many phenomena in physics, mechanics, biology, chemistry and other sciences can be described successfully by models using mathematical tools from fractional calculus. Baleanu et al. [2] have given generalized fractional integrals for the product of two H-functions and a general class of polynomials. Certain unified fractional integrals and derivatives for a product of Aleph function and a general class of multivariable polynomials have been studied by Choi and Kumar [5]. Chaurasia and Srivastava [3], Daiya et al. [6], Kumar and Daiya [12], Kumar et al. [13] and others have studied the fractional calculus pertaining to the multivariable H-function [24]. Recently, Kumar and Ayant [9] have given formulas for fractional calculus pertaining to the multivariable I-function defined by

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Prathima. The eeader can also refer to recent work on multivariable special functions, for example, see [1, 10, 11, 18].

The explicit form of Raizada's generalized polynomial set [16, Eqn.(2.3.4), p.7] is defined and represented by

$$\begin{aligned} &S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] \\ =&B^{qn} x^{l(m+n)} (1-\tau x^r)\, l^{m+n} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \frac{(-1)^\delta (-\delta)_p (\alpha)_\delta}{p!\,\delta!\,e!\,v!} \\ &\times \frac{(-v)_e (-\alpha-qn)_p}{(1-\alpha-\delta)_p} \left(-\frac{\beta}{\tau} - sn\right)_v \left(\frac{p+k+re}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^v \left(\frac{Ax}{B}\right)^\delta . \end{aligned} \tag{1.1}$$

It may be pointed out here that the polynomial set defined by (1.1) is very general in nature and it unifies and extends a number of classical polynomials.

The generalized polynomials defined by Srivastava [23], are given in the following manner:

$$\begin{aligned} &S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}[y_1,\ldots,y_s] \\ =&\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \times \cdots \times \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1,K_1;\ldots;N_s,K_s]\, y_1^{K_1} \cdots y_s^{K_s}, \end{aligned}$$

where $M_1, \ldots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \ldots; N_s, K_s]$ are arbitrary constants, real or complex.

The multivariable I-function [15] is an extension of the multivariable H-function [24]. It is defined in term of multiple Mellin-Barnes type integral, given by

$$\begin{aligned} I(z_1,\ldots,z_r) =& I^{0,n_2;0,n_3;\ldots;0,n_r:m',n';\ldots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\ldots;p_r,q_r:p',q';\ldots;p^{(r)},q^{(r)}} \left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{l} \left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\ldots; \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\ldots; \end{array} \right. \\ & \left. \begin{array}{l} \left(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj}\right)_{1,p_r} : \left(a'_j,\alpha'_j\right)_{1,p'};\ldots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \left(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj}\right)_{1,q_r} : \left(b'_j,\beta'_j\right)_{1,q'};\ldots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array} \right), \\ =& \frac{1}{(2\pi\omega)^r} \int_{\mathfrak{L}_1} \cdots \int_{\mathfrak{L}_r} \phi(s_1,\ldots,s_r) \prod_{i=1}^r \theta_i(s_i)\, z_i^{s_i}\, \mathrm{d}s_1 \cdots \mathrm{d}s_r . \end{aligned} \tag{1.2}$$

For the existence and convergence conditions of defined integral of the above function, see Prasad [15].

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.2) can be obtained by extension of the corresponding conditions for multivariable H-function, given by

$$|\arg z_i| < \frac{1}{2}\Omega_i\,\pi,$$

where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right)$$
$$+ \cdots + \left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)} \right),$$

where $i = 1, \ldots, r$ and $z_i \in \mathbb{C} \setminus \{0\}$.

Throughout the present paper, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may express the asymptotic expansion in the following convenient form:

$$I(z_1, \ldots, z_r) = 0 \left(|z_1|^{\gamma_1'}, \ldots, |z_r|^{\gamma_r'} \right), \quad \max\{|z_1|, \ldots, |z_r|\} \to 0,$$

$$I(z_1, \ldots, z_r) = 0 \left(|z_1|^{\beta_1'}, \ldots, |z_r|^{\beta_s'} \right), \quad \min\{|z_1|, \ldots, |z_r|\} \to +\infty,$$

where $k = 1, \ldots, z,\ \alpha_k' = \min\left[\mathrm{Re}\left(b_j^{(k)}/\beta_j^{(k)}\right)\right],\ j = 1, \ldots, m_k$, and

$$\beta_k' = \max\left[\mathrm{Re}\left(\left(a_j^{(k)} - 1\right)/\alpha_j^{(k)}\right)\right], \quad j = 1, \ldots, n_k.$$

In this paper, we also use the following notations:

$$A = (a_{2k}, \alpha_{2k}', \alpha_{2k}'')_{1,p_2}; \ldots; \left(a_{(r-1)k}, \alpha_{(r-1)k}', \alpha_{(r-1)k}'', \ldots, \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}},$$
$$B = (b_{2k}, \beta_{2k}', \beta_{2k}'')_{1;q_2}; \ldots; \left(b_{(r-1)k}, \beta_{(r-1)k}', \beta_{(r-1)k}'', \ldots, \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}},$$
$$\mathbb{A} = (a_{sk}; \alpha_{rk}', \alpha_{rk}'', \ldots, \alpha_{rk}^{r})_{1,p_r}; \quad \mathbb{B} = (b_{rk}; \beta_{rk}', \beta_{rk}'', \ldots, \beta_{rk}^{r})_{1,q_r},$$
$$A' = (a_k', \alpha_k')_{1,p'}; \ldots; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}; \quad B' = (b_k', \beta_k')_{1,q'}; \ldots; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}.$$

The Mellin transform of $f(x)$ will be denoted by $M[f(x)]$ or $F(s)$. If p and y are real, we write $s = p^{-1} + iy$. If $p \geq 1$, $f(x) \in \mathcal{L}_p(0, +\infty)$, then, for $p = 1$, we have

$$M[f(x)] = F(s) = \int_0^{+\infty} x^{s-1} f(x) \mathrm{d}x \quad \text{and} \quad f(x) = \frac{1}{2i\pi} \int_{\mathcal{L}} F(s) x^{-s} \mathrm{d}s.$$

For $p > 1$,

$$M[f(x)] = F(s) = l.i.m. \int_{1/x}^{x} x^{s-1} f(x) \mathrm{d}x,$$

where $l.i.m.$ denotes the usual limit in the mean for $\mathcal{L}_p$-spaces.

2. Definitions

In the context of fractional calculus, extended fractional integral operators can enhance our understanding and application of fractional integrals, especially when dealing with functions that exhibit complex behavior.

The pair of new extended fractional integral operators are defined by the following equations:

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta} I\left(\begin{array}{c}\gamma_1\upsilon_1\\ \vdots\\ \gamma_n\upsilon_n\end{array}\middle|\begin{array}{c}A;\mathbb{A}:A'\\ \\ B;\mathbb{B}:B'\end{array}\right)$$
$$\times\prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j};r_j,s_j,q_j,A_j,B_j,m_j,k_j,l_j\right]$$
$$\times\prod_{j=1}^{r} S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c}z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}}\\ \vdots\\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}}\end{array}\right)\Psi\left(\frac{y^t}{x^t}\right)f(y)\,\mathrm{d}y \tag{2.1}$$

and

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = tx^{\rho}\int_x^{+\infty} y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta} I\left(\begin{array}{c}\gamma_1\mu_1\\ \vdots\\ \gamma_n\mu_n\end{array}\middle|\begin{array}{c}A;\mathbb{A}:A'\\ \\ B;\mathbb{B}:B'\end{array}\right)$$
$$\times\prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j};r_j,s_j,q_j,A_j,B_j,m_j,k_j,l_j\right]$$
$$\times\prod_{j=1}^{r} S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c}z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \vdots\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right)\Psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y, \tag{2.2}$$

where $\upsilon_i=\left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu_i=\left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$, t, u_i and v_i, $g_i^{(j)}, h_i^{(j)}$, a_j and b_j are positive numbers.

The kernels $\Psi\left(\frac{y^t}{x^t}\right)$ and $\Psi\left(\frac{x^t}{y^t}\right)$ appearing in (2.1) and (2.2), respectively, are assumed to be continuous functions such that integrals make sense for a wide class of functions f.

The existence conditions of these operators are given as follows:

(a) $f\in\mathcal{L}_p(0,+\infty)$;

(b) $1\le p,q<+\infty,\ p^{-1}+q^{-1}=1$;

(c) $\mathrm{Re}\left[\alpha+ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^n u_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1}$;

(d) $\mathrm{Re}\left[\beta+tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^n v_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1}$;

(e) $\mathrm{Re}\left[\rho+ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^n v_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-p^{-1}$,

where $i=1,\dots,k$. Condition (a) ensures that both operators defined in (2.1) and (2.2) exist and belong to $\mathcal{L}_p(0,+\infty)$. These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [7], Love [14], Saigo et al. [19], Saxena and Kiryakova [20], Saxena and Kumbhat [21, 22], etc.

3. Main Results

In this section, we investigate a unified and extended fractional integral operator involving the multivariable I-function defined by Prasad, Raizada's generalized polynomial set and a general class of multivariable polynomials. Here, we derive results pertaining to the Mellin transforms of these operators.

Theorem 3.1. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \le p \le 2$, or $f \in \mathcal{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied: $p^{-1}+q^{-1}=1$,*

$$\operatorname{Re}\left[\alpha + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} u_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

and the integrals involved are absolutely convergent, then

$$M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = M\left\{f(x)\right\}\, R_{\gamma_n}^{\alpha-s+1,\beta}\left[1\right], \tag{3.1}$$

where $M_p(0,+\infty)$ stands for the class of all functions f from $\mathcal{L}_p(0,+\infty)$ with $p>2$, which are the inverse Mellin-transforms of functions from $\mathcal{L}_p(-\infty,+\infty)$.

Proof. By making use of the Mellin transform of (2.1), we get

$$\begin{aligned} M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = \int_0^{+\infty} x^{s-1} \Bigg\{ & tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta} I\left(\begin{array}{c} \gamma_1 v_1 \\ \vdots \\ \gamma_n v_n \end{array}\middle| \begin{array}{c} A;\mathbb{A}:A' \\ \\ B;\mathbb{B}:B' \end{array}\right) \\ & \times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}; r_j,s_j,q_j,A_j,B_j,m_j,k_j,l_j\right] \\ & \times \prod_{j=1}^{r} S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}} \end{array}\right)\Psi\left(\frac{y^t}{x^t}\right)f(y)\mathrm{d}y\Bigg\}\mathrm{d}x. \end{aligned}$$

By interchanging the order of integration, which is permissible under the conditions, the result (3.1) follows in view of (2.2). □

Theorem 3.2. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \le p \le 2$, or $f \in \mathbb{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied : $p^{-1}+q^{-1}=1$,*

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\operatorname{Re}\left[\rho+ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}v_i\min_{1\leq j\leq m^{(i)}}\operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-p^{-1},$$

and the integrals involved are absolutely convergent, then

$$M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\}=M\left\{f(x)\right\}\,Q_{\gamma_n}^{\rho+s-1,\beta}\left[1\right]. \tag{3.2}$$

Proof. By making use of Mellin transform of (2.2), we get

$$\begin{aligned}M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\}=&\int_0^{+\infty}x^{s-1}\left\{tx^{\rho}\int_x^{+\infty}y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta}I\left(\begin{array}{c}\gamma_1\mu_1\\ \vdots\\ \gamma_n\mu_n\end{array}\middle|\begin{array}{c}A;\mathbb{A}:A'\\ \\ B;\mathbb{B}:B'\end{array}\right)\right.\\&\times\prod_{j=1}^{k}S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j};r_j,s_j,q_j,A_j,B_j,m_j,k_j,l_j\right]\\&\left.\times\prod_{j=1}^{r}S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c}z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \vdots\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right)\Psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y\right\}\mathrm{d}x.\end{aligned}$$

By interchanging the order of integration, which is permissible under the conditions, the result (3.2) follows in view of (2.1). □

Theorem 3.3. *If* $f\in\mathcal{L}_p(0,+\infty)$, $v\in\mathcal{L}_p(0,+\infty)$, *and the following conditions are satisfied:*

$$\operatorname{Re}\left[\alpha+ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}u_i\min_{1\leq j\leq m^{(i)}}\operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$$\operatorname{Re}\left[\beta+tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}v_i\min_{1\leq j\leq m^{(i)}}\operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$p^{-1}+q^{-1}=1$, *and the integrals involved are absolutely convergent, then*

$$\int_0^{+\infty}v(x)\,Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\mathrm{d}x=\int_0^{+\infty}f(x)\,R_{\gamma_n}^{\alpha,\beta}\left[v(x)\right]\mathrm{d}x. \tag{3.3}$$

Proof. The result of (3.3) can be obtained in view of (2.1) and (2.2). □

4. Inversion Formulae

In this section, we we provide inversion formulas for the main results.

Theorem 4.1. *If* $f\in\mathcal{L}_p(0,+\infty)$, $1\leq p\leq 2$, *or* $f(x)\in\mathcal{L}_p(0,+\infty)$, $p>2$, *and the following conditions are satisfied:* $p^{-1}+q^{-1}=1$,

$$\operatorname{Re}\left[\alpha+ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}u_i\min_{1\leq j\leq m^{(i)}}\operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

and the integrals involved are absolutely convergent and

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = v(x), \tag{4.1}$$

then

$$f(x) = \int_0^{+\infty} y^{-1}\left[v(y)\right]\left[h\left(\frac{x}{y}\right)\right] \mathrm{d}y, \tag{4.2}$$

where

$$\begin{aligned} h(x) =& \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y^{-1}\frac{x^{-s}}{R(s)}\,\mathrm{d}s, \\ R(s) =& R_{\gamma_n}^{\alpha-s+1,\beta}\left[1\right]. \end{aligned} \tag{4.3}$$

Proof. By taking the Mellin transform of (4.1) and then applying Theorem 3.1, we get

$$M\left\{f(x)\right\} = \frac{M\left\{v(x)\right\}}{R(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M\left\{v(x)\right\}}{R(s)}\,\mathrm{d}s = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{R(s)}\left\{\int_0^{+\infty}\left[v(y)\right]\mathrm{d}y\right\}\mathrm{d}s.$$

Upon interchanging the order of integration, we have

$$f(x) = \int_0^{+\infty}\frac{v(y)}{y}\left\{\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^s\frac{1}{R(s)}\,\mathrm{d}s\right\}\mathrm{d}y.$$

Now in view of (4.3), we obtain the desired result (4.2). □

Theorem 4.2. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \le p \le 2$, or $f \in \mathcal{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied: $p^{-1}+q^{-1} = 1$,*

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\operatorname{Re}\left[\rho + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} u_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -p^{-1},$$

and the integrals involved are absolutely convergent and

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = w(x), \tag{4.4}$$

then

$$f(x) = \int_0^{+\infty} y^{-1}\left[w(y)\right]\left[g\left(\frac{x}{y}\right)\right] \mathrm{d}y, \tag{4.5}$$

$$g(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y^{-1}\frac{x^{-s}}{T(s)}\,\mathrm{d}s \tag{4.6}$$

and

$$T(s) = Q_{\gamma_n}^{\rho+s-1,\beta}[1].$$

Proof. By taking the Mellin transform into account of (4.4) and then applying Theorem 3.2, we get

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M\{w(x)\}}{T(s)}\,\mathrm{d}s = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{T(s)}\left\{\int_0^{+\infty}[w(y)]\,\mathrm{d}y\right\}\mathrm{d}s,$$

on interchanging the order of integration, then we have

$$f(x) = \int_0^{+\infty}\frac{w(y)}{y}\left\{\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^s\frac{1}{T(s)}\,\mathrm{d}s\right\}\mathrm{d}y.$$

Now in view of (4.6), we obtain the desired result (4.5). □

5. General Properties

The properties given below are consequences of Definitions 2.1 and 2.2

$$\begin{aligned}
x^{-1}Q_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] &= R_{\gamma_n}^{\alpha,\beta}[f(x)],\\
x^{-1}R_{\gamma_n}^{\rho,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] &= Q_{\gamma_n}^{\rho,\beta}[f(x)],\\
x^{\mu}Q_{\gamma_n}^{\alpha,\beta}[f(x)] &= Q_{\gamma_n}^{\alpha-\mu,\beta}[x^{\mu}f(x)],\\
x^{\mu}R_{\gamma_n}^{\rho,\beta}[f(x)] &= R_{\gamma_n}^{\rho+\mu,\beta}[x^{\mu}f(x)].
\end{aligned}$$

The properties given below express the homogeneity of the operators Q and R, respectively. If $Q_{\gamma_n}^{\alpha,\beta}[f(x)] = v(x)$, then $Q_{\gamma_n}^{\alpha,\beta}[f(cx)] = v(cx)$, and $R_{\gamma_n}^{\rho,\beta}[f(x)] = w(x)$, then $R_{\gamma_n}^{\rho,\beta}[f(cx)] = w(cx)$.

6. Multivariable H-Function

In this section, we give special case concerning multivariable H-function.

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad reduces to multivariable H-function [4, 6, 13], see also [8, 17]. We obtain the two following operators.

$$\begin{aligned}
Q_{\gamma_n}^{\alpha,\beta}[f(x)] =& tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}(x^t-y^t)^{\beta}\, H\left(\begin{array}{c}\gamma_1 v_1\\ \vdots\\ \gamma_n v_n\end{array}\middle|\begin{array}{c}\mathbb{A}:A'\\ \\ \mathbb{B}:B'\end{array}\right)\\
&\times\prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j\right]
\end{aligned}$$

$$\times \prod_{j=1}^{r} S_{N_1^{(j)},\ldots,N_s^{(j)}}^{M_1^{(j)},\ldots,M_s^{(j)}} \begin{pmatrix} z_1^{(j)} \left(\frac{y^t}{x^t}\right)^{g_1^{(j)}} \left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{y^t}{x^t}\right)^{g_s^{(j)}} \left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}} \end{pmatrix} \Psi\left(\frac{y^t}{x^t}\right) f(y)\,\mathrm{d}y,$$

under the same notations and conditions that (2.1) with $U = V = A = B = 0$.

$$R_{\gamma_n}^{\rho,\beta}[f(x)] = tx^{\rho} \int_x^{+\infty} y^{-\rho-t\beta-1} \left(y^t - x^t\right)^{\beta} H \left(\begin{array}{c|c} \gamma_1\mu_1 & \mathbb{A} : A' \\ \vdots & \\ \gamma_n\mu_n & \mathbb{B} : B' \end{array} \right)$$

$$\times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j} \left[z_j \left(\frac{x^t}{y^t}\right)^{a_j} \left(1-\frac{x^t}{y^t}\right)^{b_j}; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j \right]$$

$$\times \prod_{j=1}^{r} S_{N_1^{(j)},\ldots,N_s^{(j)}}^{M_1^{(j)},\ldots,M_s^{(j)}} \begin{pmatrix} z_1^{(j)} \left(\frac{x^t}{y^t}\right)^{g_1^{(j)}} \left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{x^t}{y^t}\right)^{g_s^{(j)}} \left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}} \end{pmatrix} \Psi\left(\frac{x^t}{y^t}\right) f(y)\,\mathrm{d}y,$$

under the same notations and conditions as given in (2.2) with $U = V = A = B = 0$. We can obtain the similar theorems and properties for multivariable H-function concerning multivariable I-function.

7. Concluding Remarks

Fractional calculus involving multivariable I-functions is a rich area of study that combines advanced mathematical concepts with practical applications. Understanding the definitions and methods of fractional differentiation and integration is essential for effectively leveraging these tools in real-world problems. The functions involved in the results established in the present paper are of a unified and general nature, hence a large number of known results lying in the literature follow as particular cases. Further, on suitable specifications of the parameters involved, many new results involving simpler functions can also be derived.

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