A GENERALIZATION OF HERMITE-HADAMARD’S INEQUALITY

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ABSTRACT. In literature the Hermite-Hadamard inequality was eligible for many reasons, one of the most surprising and interesting that the Hermite-Hadamard inequality combine the midpoint and trapezoid formulae in an inequality. In this work, a Hermite-Hadamard like inequality that combines the composite trapezoid and composite midpoint formulae is proved. So that, the classical Hermite-Hadamard inequality becomes a special case of the presented result. Some Ostrowski’s type inequalities for convex functions are proved as well.

1. Introduction

Let \( f : [a, b] \to \mathbb{R} \), be a twice differentiable mapping such that \( f''(x) \) exists on \((a, b)\) and \( \|f''\|_{\infty} = \sup_{x \in (a, b)} |f''(x)| < \infty \). Then the midpoint inequality is known as:

\[
\left| \int_a^b f(x) \, dx - (b - a) f \left( \frac{a + b}{2} \right) \right| \leq \frac{(b - a)^3}{24} \|f''\|_{\infty},
\]

and, the trapezoid inequality

\[
\left| \int_a^b f(x) \, dx - (b - a) \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)^3}{12} \|f''\|_{\infty},
\]

also hold. Therefore, the integral \( \int_a^b f(x) \, dx \) can be approximated in terms of the midpoint and the trapezoidal rules, respectively such as:

\[
\int_a^b f(x) \, dx \approx (b - a) f \left( \frac{a + b}{2} \right),
\]

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and
\[ \int_a^b f(x) \, dx \equiv (b - a) \frac{f(a) + f(b)}{2}, \]
which are combined in a useful and famous relationship, known as the Hermite-Hadamard’s inequality. That is,
\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}, \]
which hold for all convex functions \( f \) defined on a real interval \([a, b]\).

The real beginning was (almost) in the last twenty five years, where, in 1992 Dragomir [9] published his article about (1.1). The main result in [9] was

**Theorem 1.1.** Let \( f : [a, b] \) is convex function one can define the following mapping on \([0, 1]\) such as:
\[ H(t) = \frac{1}{b - a} \int_a^b f \left( tx + (1 - t) \frac{a + b}{2} \right) \, dx, \]
then:

(a) \( H \) is convex and monotonic non-decreasing on \([0, 1]\);

(b) one has the bounds for \( H \)
\[ \sup_{t \in [0, 1]} H(t) = \frac{1}{b - a} \int_a^b f(x) \, dx = H(1), \]
and
\[ \inf_{t \in [0, 1]} H(t) = f \left( \frac{a + b}{2} \right) = H(0). \]

Few years after 1992, many authors have took (a real) attention to the Hermite-Hadamard inequality and sequence of several works under various assumptions for the function involved such as bounded variation, convex, differentiable functions whose \( n \)-derivative(s) belong to \( L_p[a, b]; \) \( 1 \leq p \leq \infty \), Lipschitz, monotonic, etc., have been published. For a comprehensive list of results and excellent bibliography we recommend the interested to refer to [3,4,13].

In 1997, Yang and Hong [15], continued on Dragomir result (Theorem 1.1) and they proved the following theorem.

**Theorem 1.2.** Suppose that \( f : [a, b] \rightarrow \mathbb{R} \), is convex and the mapping \( F : [0, 1] \rightarrow \mathbb{R} \) is defined by
\[ F(t) = \frac{1}{b - a} \int_a^b \left[ f \left( \frac{1 + t}{2} a + \frac{1 - t}{2} u \right) + f \left( \frac{1 + t}{2} b + \frac{1 - t}{2} u \right) \right] \, du, \]
then:
(a) the mapping \( F \) is convex and monotonic nondecreasing on \([0, 1]\);
(b) we have the bounds

\[
\inf_{t \in [0, 1]} F(t, s) = \frac{1}{(b - a)} \int_a^b f(x) \, dx = F(0),
\]

\[
\sup_{t \in [0, 1]} F(t) = \frac{f(a) + f(b)}{2} = F(1).
\]

For other closely related results see [1, 5–8, 10–12].

In terms of composite numerical integration, we recall the composite midpoint rule
[2, p. 202]

\[
\int_a^b f(x) \, dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{(b - a)}{6} h^2 f''(\mu),
\]

for some \( \mu \in (a,b) \), where \( f \in C^2[a,b] \), \( n \) is even, \( h = \frac{b - a}{n} \) and \( x_j = a + (j + 1)h \), for each \( j = -1, 0, \ldots, n + 1 \); and, the composite trapezoid rule
[2, p. 203]

\[
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{(b - a)}{12} h^2 f''(\mu),
\]

for some \( \mu \in (a,b) \), where \( f \in C^2[a,b] \), \( h = \frac{b - a}{n} \) and \( x_j = a + jh \), for each \( j = 0, 1, \ldots, n \).

The main purpose of this work, is to combine the composite trapezoid and composite midpoint formulae in an inequality that is similar to the classical Hermite-Hadamard inequality (1.1) for convex functions defined on a real interval \([a,b]\). In this way, we establish a conventional generalization of (1.1) which is in turn most useful and has a very constructional form.

2. A Generalization of Hermite-Hadamard’s Inequality

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\), then the double inequality

\[
h \sum_{k=1}^{n} \frac{f(x_{k-1} + x_k)}{2} \leq \int_a^b f(t) \, dt \leq \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right],
\]

holds, where \( x_k = a + k \frac{b - a}{n}, k = 0, 1, 2, \ldots, n \); with \( h = \frac{b - a}{n}, n \in \mathbb{N} \). The constant ‘1’ in the left-hand side and ‘\( \frac{1}{2} \)’ in the right-hand side are the best possible for all \( n \in \mathbb{N} \). If \( f \) is concave then the inequality is reversed.

*Nota bene:* after revision of this paper, the anonymous referee informed us that the inequality (2.1) was proved in more general case in [14] (see also [13, p. 22–23]). We appreciate this remark from the reviewer.
Proof. Since $f$ is convex on $[a, b]$, then $f$ so is on each subinterval $[x_{j-1}, x_j]$, $j = 1, \ldots, n$, then for all $t \in [0, 1]$, we have

$$f \left( t x_{j-1} + (1 - t) x_j \right) \leq t f \left( x_{j-1} \right) + (1 - t) f \left( x_j \right).$$

Integrating (2.2) with respect to $t$ on $[0, 1]$ we get

$$\int_0^1 f \left( t x_{j-1} + (1 - t) x_j \right) dt \leq \frac{f \left( x_{j-1} \right) + f \left( x_j \right)}{2}.$$ 

Substituting $u = t x_{j-1} + (1 - t) x_j$, in the left hand side of (2.3), we get

$$\int_{x_{j-1}}^{x_j} f (u) du \leq \frac{x_j - x_{j-1}}{2} \left( f \left( x_{j-1} \right) + f \left( x_j \right) \right).$$

Taking the sum over $j$ from 1 to $n$, we get

$$\sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f (u) du \leq \frac{1}{2} \max_j \{x_j - x_{j-1}\} \cdot \sum_{j=1}^{n} \left( f \left( x_{j-1} \right) + f \left( x_j \right) \right)$$

$$= \frac{h}{2} \left[ f \left( x_0 \right) + f \left( x_1 \right) + \sum_{j=2}^{n-1} \left\{ f \left( x_{j-1} \right) + f \left( x_j \right) \right\} + f \left( x_{n-1} \right) + f \left( x_n \right) \right]$$

$$= \frac{h}{2} \left[ f \left( a \right) + 2 \sum_{j=1}^{n-1} f \left( x_j \right) + f \left( b \right) \right].$$

On the other hand, again since $f$ is convex on $I_x$, then for $t \in [0, 1]$, we have

$$f \left( \frac{x_{j-1} + x_j}{2} \right) = f \left( t x_j + (1 - t) x_{j-1} \right) \leq \frac{1}{2} \left[ f \left( t x_j + (1 - t) x_{j-1} \right) + f \left( (1 - t) x_j + t x_{j-1} \right) \right].$$
Integrating inequality (2.5) with respect to $t$ on $[0, 1]$ we get

$$f \left( \frac{x_{j-1} + x_j}{2} \right) \leq \frac{1}{2} \int_0^1 \left[ f(tx_j + (1 - t)x_{j-1}) + f((1 - t)x_j + tx_{j-1}) \right] dt$$

$$= \frac{1}{2} \int_0^1 f(tx_j + (1 - t)x_{j-1}) dt + \frac{1}{2} \int_0^1 f((1 - t)x_j + tx_{j-1}) dt.$$  

By putting $1 - t = s$ in the second integral on the right-hand side of (2.6), we have

$$f \left( \frac{x_{j-1} + x_j}{2} \right) \leq \frac{1}{2} \int_0^1 f(tx_j + (1 - t)x_{j-1}) dt + \frac{1}{2} \int_0^1 f((1 - t)x_j + tx_{j-1}) dt$$

$$= \int_0^1 f(tx_j + (1 - t)x_{j-1}) dt.$$  

Substituting $u = tx_j + (1 - t)x_{j-1}$, in the left hand side of (2.7), and then taking the sum over $j$ from 1 to $n$, we get

$$h \sum_{k=1}^n f \left( \frac{x_{k-1} + x_k}{2} \right) \leq \int_a^b f(t) dt.$$  

From (2.4) and (2.8), we get the desired inequality (2.1).

To prove the sharpness let (2.1) hold with another constants $C_1, C_2 > 0$, which gives

$$C_1 \cdot h \sum_{k=1}^n f \left( \frac{x_{k-1} + x_k}{2} \right) \leq \int_a^b f(t) dt$$

$$\leq C_2 \cdot h \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right].$$
Let \( f : [a, b] \to \mathbb{R} \) be the identity map \( f(x) = x \), then the right-hand side of (2.9) reduces to

\[
\frac{b^2 - a^2}{2} \leq C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} x_k + b \right] 
= C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} \left( a + k \frac{b-a}{n} \right) + b \right] 
= C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} a + 2 \frac{b-a}{n} \sum_{k=1}^{n-1} k + b \right] 
= C_2 \cdot \frac{b-a}{n} \left[ a + 2 (n-1) a + 2 \frac{b-a}{n} \cdot \frac{n(n-1)}{2} + b \right] 
= C_2 \cdot (b-a) (a+b). 
\]

It follows that \( \frac{1}{2} \leq C_2 \), i.e., \( \frac{1}{2} \) is the best possible constant in the right-hand side of (2.1).

For the left-hand side, we have

\[
\frac{b^2 - a^2}{2} \geq C_1 \cdot h \sum_{k=1}^{n} \frac{x_{k-1} + x_k}{2} 
= C_1 \cdot h \sum_{k=1}^{n} \left\{ a + (2k-1) \frac{b-a}{2n} \right\} 
= C_1 \cdot \frac{b-a}{n} \cdot \left[ na + \frac{b-a}{2n} \left( 2 \cdot \frac{n(n+1)}{2} - n \right) \right] 
= C_1 \cdot \frac{b^2 - a^2}{2}, 
\]

which means that \( 1 \geq C_1 \), and thus 1 is the best possible constant in the left-hand side of (2.1). Thus the proof of Theorem 2.1 is completely finished. \( \square \)

**Remark 2.1.** In Theorem 2.1, if we take \( n = 1 \), then we refer to the original Hermite-Hadamard inequality (1.1).

In viewing of (2.1), next we give direct sharp refinements of Hermite-Hadamard’s type inequalities for convex functions defined on a real interval \([a, b]\), according to the number of division ‘n’ (in our case \( n = 1, 2, 3, 4 \)) in Theorem 2.1.

**Corollary 2.1.** In Theorem 2.1, we have

(a) if \( n = 1 \), then

\[
(b - a) f \left( \frac{a+b}{2} \right) \leq \int_{a}^{b} f(t) \, dt \leq (b - a) \frac{f(a) + f(b)}{2};
\]
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(b) if \( n = 2 \), then
\[
\frac{(b - a)}{2} \left[ f \left( \frac{3a + b}{4} \right) + f \left( \frac{a + 3b}{4} \right) \right] \\
\leq \int_a^b f(t) \, dt \\
\leq \frac{(b - a)}{4} \left[ f(a) + 2f \left( \frac{a + b}{2} \right) + f(b) \right];
\]

(c) if \( n = 3 \), then
\[
\frac{(b - a)}{3} \left[ f \left( \frac{5a + b}{6} \right) + f \left( \frac{a + b}{2} \right) + f \left( \frac{a + 5b}{6} \right) \right] \\
\leq \int_a^b f(t) \, dt \\
\leq \frac{(b - a)}{6} \left[ f(a) + 2f \left( \frac{2a + b}{3} \right) + 2f \left( \frac{a + 2b}{3} \right) + f(b) \right];
\]

(d) if \( n = 4 \), then
\[
\frac{(b - a)}{4} \left[ f \left( \frac{7a + b}{8} \right) + f \left( \frac{5a + 3b}{8} \right) + f \left( \frac{3a + 5b}{8} \right) + f \left( \frac{a + 7b}{8} \right) \right] \\
\leq \int_a^b f(t) \, dt \\
\leq \frac{(b - a)}{12} \left[ f(a) + 2f \left( \frac{3a + b}{4} \right) + 2f \left( \frac{a + b}{2} \right) + 2f \left( \frac{a + 3b}{4} \right) + f(b) \right].
\]

Now, let \( f : [a,b] \to \mathbb{R} \) be a convex function on \([a,b]\. Define the mappings \( H_j, F_j : [0,1] \to \mathbb{R} \), given by
\[
(2.10) \quad H_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} f \left( tu + (1 - t) \frac{x_{j-1} + x_j}{2} \right) \, du, \quad u \in [x_{j-1}, x_j],
\]
and
\[
(2.11) \quad F_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} \left[ f \left( \frac{1 + t}{2} x_{j-1} + \frac{1 - t}{2} u \right) + f \left( \frac{1 + t}{2} x_j + \frac{1 - t}{2} u \right) \right] \, du,
\]
where \( u \in [x_{j-1}, x_j]\. Applying Theorems 1.1 and 1.2, for \( f : [x_{j-1}, x_j] \to \mathbb{R}, j = 1, 2, \ldots, n \). Then the following statements hold:

(a) \( H_j(t) \) and \( F_j(t) \) are convex for all \( t \in [0,1] \) and \( u \in [x_{j-1}, x_j] \);
(b) \( H_j(t) \) and \( F_j(t) \) are monotonic nondecreasing for all \( t \in [0,1] \) and \( u \in [x_{j-1}, x_j] \);
(c) we have the following bounds for \( H_j(t) \)
\[
(2.12) \quad \frac{1}{h} \int_{x_{j-1}}^{x_j} f(u) \, du = H_j(1),
\]
and
\begin{equation}
(2.13) \quad f \left( \frac{x_{k-1} + x_k}{2} \right) = H_j (0).
\end{equation}
and the following bounds for $F_j(t)$
\begin{equation}
(2.14) \quad \frac{f(x_{j-1}) + f(x_j)}{2} = F_j (1),
\end{equation}
and
\begin{equation}
(2.15) \quad \frac{1}{h} \int_{x_{j-1}}^{x_j} f(u) \, du = F_j (0).
\end{equation}
Hence, we may establish two related mappings for the inequality (2.1).

**Proposition 2.1.** Let $f$ be as in Theorem 2.1, define the mappings $H, F : [0, 1] \to \mathbb{R}$, given by

\[ H(t) = \sum_{j=1}^{n} H_j (t) \quad \text{and} \quad F(t) = \sum_{j=1}^{n} F_j (t), \]

where $H_j(t)$ and $F_j(t)$ are defined in (2.10) and (2.11), respectively; then the following statements hold:

(a) $H(t)$ and $F(t)$ are convex for all $t \in [0, 1]$ and $u \in [a, b]$;
(b) $H(t)$ and $F(t)$ are monotonic nondecreasing for all $t \in [0, 1]$ and $u \in [a, b]$;
(c) We have the following bounds for $H(t)$
\begin{equation}
\sup_{t \in [0,1]} H(t) = \frac{1}{h} \int_{a}^{b} f(u) \, du = H(1),
\end{equation}
and
\begin{equation}
\inf_{t \in [0,1]} H(t) = \sum_{k=1}^{n} f \left( \frac{x_{k-1} + x_k}{2} \right) = H(0),
\end{equation}
and the following bounds for $F(t)$
\begin{equation}
\sup_{t \in [0,1]} F(t) = \frac{1}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right] = F(1),
\end{equation}
and
\begin{equation}
\inf_{t \in [0,1]} F(t) = \frac{1}{h} \int_{a}^{b} f(u) \, du = F(0).
\end{equation}

**Proof.** Taking the sum over $j$ from 1 to $n$, in (2.12)–(2.15) we get the required results, and we shall omit the details. \qed

**Remark 2.2.** The inequality (2.1) may written in a convenient way as follows:
\[ \sum_{k=1}^{n} f \left( \frac{x_{k-1} + x_k}{2} \right) - \sum_{k=1}^{n-1} f(x_k) \leq \frac{1}{h} \int_{a}^{b} f(t) \, dt - \sum_{k=1}^{n-1} f(x_k) \leq \frac{f(a) + f(b)}{2}, \]
which is of Ostrowski's type.

Some sharps Ostrowski's type inequalities for convex functions defined on a real interval \([a, b]\), are proposed in the next theorems.

**Theorem 2.2.** Let \(f : [a, b] \to \mathbb{R}_+\) be a convex function on \([a, b]\), then the inequality

\[
\int_a^b f(x) \, dx - (b - a) f(y) \leq \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right],
\]

holds for all \(y \in [a, b]\), where, \(x_k = a + k \frac{b-a}{n}, \ k = 0, 1, 2, \ldots, n; \) with \(h = \frac{b-a}{n}, \ n \in \mathbb{N}\). The constant \(\frac{1}{2}\) in the right-hand side is the best possible, in the sense that it cannot be replaced by a smaller one for all \(n \in \mathbb{N}\). If \(f\) is concave then the inequality is reversed.

**Proof.** Fix \(y \in [x_{j-1}, x_j]\), \(j = 1, \ldots, n\). Since \(f\) is convex on \([a, b]\), then \(f\) so is on each subinterval \([x_{j-1}, x_j]\), in particular on \([x_{j-1}, y]\), then for all \(t \in [0, 1]\), we have

\[
f(tx_{j-1} + (1-t)y) \leq tf(x_{j-1}) + (1-t)f(y), \quad j = 1, \ldots, n.
\]

Integrating (2.17) with respect to \(t\) on \([0, 1]\) we get

\[
\int_0^1 f(tx_{j-1} + (1-t)y) \, dt \leq \frac{f(x_{j-1}) + f(y)}{2}.
\]

Substituting \(u = tx_{j-1} + (1-t)y\), in the left hand side of (2.18), we get

\[
\int_{x_{j-1}}^y f(u) \, du \leq \frac{y-x_{j-1}}{2} (f(x_{j-1}) + f(y)).
\]

Now, we do similarly for the interval \([y, x_j]\), we therefore have

\[
f(ty + (1-t)x_j) \leq tf(y) + (1-t)f(x_j), \quad j = 1, \ldots, n.
\]

Integrating (2.20) with respect to \(t\) on \([0, 1]\) we get

\[
\int_0^1 f(ty + (1-t)x_j) \, dt \leq \frac{f(y) + f(x_j)}{2}.
\]

Substituting \(u = ty + (1-t)x_j\), in the left hand side of (2.21), we get

\[
\int_y^{x_j} f(u) \, du \leq \frac{x_j-y}{2} (f(y) + f(x_j)).
\]
Adding the inequalities (2.19) and (2.22), we get

\begin{equation}
\int_{x_{j-1}}^{x_j} f(u) \, du + \int_{x_j}^{x_{j+1}} f(u) \, du = \int_{x_{j-1}}^{x_j} f(u) \, du \\
\leq \frac{y - x_{j-1}}{2} (f(x_{j-1}) + f(y)) + \frac{x_j - y}{2} (f(y) + f(x_j)) \\
\leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_j) + (x_j - x_{j-1}) f(y) \\
\leq \frac{x_j - x_{j-1}}{2} (f(x_{j-1}) + f(x_j)) + hf(y).
\end{equation}

Taking the sum over \( j \) from 1 to \( n \), we get

\[ \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(u) \, du = \int_{a}^{b} f(u) \, du \\
= \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{2} \{ f(x_{j-1}) + f(x_j) \} + \sum_{j=1}^{n} hf(y) \\
\leq \frac{1}{2} \max_j \{ x_j - x_{j-1} \} \cdot \sum_{j=1}^{n} (f(x_{j-1}) + f(x_j)) + (b - a) f(y) \\
= \frac{h}{2} \left[ f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{ f(x_{j-1}) + f(x_j) \} + f(x_{n-1}) + f(x_n) \right] + (b - a) f(y) \\
= \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + (b - a) f(y), \]

which gives that

\[ \int_{a}^{b} f(u) \, du - (b - a) f(y) \leq \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right], \]

for all \( y \in [x_{j-1}, x_j] \subseteq [a, b] \) for all \( j = 1, 2, \ldots, n \), which gives the desired result (2.16).

To prove the sharpness let (2.16) hold with another constants \( C > 0 \), which gives

\begin{equation}
\int_{a}^{b} f(x) \, dx - (b - a) f(y) \leq C \cdot h \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right].
\end{equation}
Let \( f : [0, 1] \to \mathbb{R} \) be the identity map \( f(x) = x \), then the right-hand side of (2.24) reduces to

\[
\frac{1}{2} - y \leq C \cdot \frac{1}{n} \left[ 2 \sum_{k=1}^{n-1} x_k + 1 \right] = C \cdot \frac{1}{n} \left[ 2 \cdot \frac{n(n-1)}{2} + 1 \right] = C.
\]

Choose \( y = 0 \), it follows that \( \frac{1}{2} \leq C \), i.e., \( \frac{1}{2} \) is the best possible constant in the right-hand side of (2.16).

Theorem 2.3. Under the assumptions of Theorem 2.2, we have

\[
\int_a^b f(x) \, dx - (b - a) f(y) \leq \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\]

for all \( y \in [a,b] \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible for all \( n \in \mathbb{N} \). If \( f \) is concave then the inequality is reversed.

In particular, if \( n = 1 \) then

\[
\int_a^b f(x) \, dx - (b - a) f(y) \leq \left[ \frac{b - a}{2} + \left| y - \frac{a + b}{2} \right| \right] \cdot \left[ f(a) + f(b) \right],
\]

for all \( y \in [a,b] \).

Proof. Repeating the steps of the proof of Theorem 2.2, therefore by (2.23)

\[
\int_{x_{j-1}}^{x_j} f(u) \, du \leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + h f(y)
\]

\[
\leq \max \left\{ \frac{y - x_{j-1}}{2}, \frac{x_j - y}{2} \right\} \cdot (f(x_{j-1}) + f(x_{j-1})) + h f(y)
\]

\[
\leq \left[ \frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot (f(x_{j-1}) + f(x_{j-1})) + h f(y)
\]
Taking the sum over \( j \) from 1 to \( n \), we get
\[
\int_a^b f(u) \, du \quad \leq \quad \sum_{j=1}^{n} \left[ \frac{x_j - x_{j-1}}{2} + \left| \frac{y - x_{j-1} + x_j}{2} \right| \right] \cdot \{ f(x_{j-1}) + f(x_j) \} + \sum_{j=1}^{n} hf(y) \quad \leq \quad \max_{1 \leq j \leq n} \left[ \frac{x_j - x_{j-1}}{2} + \left| \frac{y - x_{j-1} + x_j}{2} \right| \right] \cdot \sum_{j=1}^{n} (f(x_{j-1}) + f(x_j)) + (b - a) f(y) \quad \leq \quad \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \times \left[ f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{ f(x_{j-1}) + f(x_j) \} + f(x_{n-1}) + f(x_n) \right] + (b - a) f(y) \quad = \quad \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + (b - a) f(y),
\]
which gives that
\[
\int_a^b f(u) \, du - (b - a) f(y) \quad \leq \quad \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] m
\]
for all \( y \in [x_{j-1}, x_j] \subseteq [a, b] \) for all \( j = 1, 2, \ldots, n \), which gives the desired result (2.25). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. \( \square \)

**Corollary 2.2.** Let \( \alpha_i \geq 0 \), for all \( i = 0, 1, 2, \ldots, n \), be positive real numbers such that \( \sum_{i=0}^{n} \alpha_i = 1 \), then under the assumptions of Theorem 2.3, we have
\[
\int_a^b f(x) \, dx - (b - a) f \left( \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i \right) \quad \leq \quad \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\]
for all \( y \in [a, b] \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible. If \( f \) is concave then the inequality is reversed.

**Theorem 2.4.** Under the assumptions of Theorem 2.3, we have
\[
(2.27) \quad \frac{1}{b-a} \int_a^b f(u) \, du - \frac{1}{n} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) - \frac{1}{n} \sum_{j=1}^{n-1} f(x_j) \leq \frac{f(a) + f(b)}{2n},
\]
for all \( j = 1, 2, \ldots, n \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible. If \( f \) is concave then the inequality is reversed.

**Proof.** Repeating the steps of the proof of Theorem 2.3, (2.26) if we choose \( y = \frac{x_{j-1}+x_j}{2} \), then we get
\[
\int_{x_{j-1}}^{x_j} f(u) du \leq \frac{1}{2} \left( \frac{x_j-x_{j-1}}{2} \cdot (f(x_{j-1}) + f(x_j)) + h f \left( \frac{x_{j-1}+x_j}{2} \right) \right).
\]
Taking the sum over \( j \) from 1 to \( n \), we get
\[
\int_{a}^{b} f(u) du \leq \sum_{j=1}^{n} \frac{x_j-x_{j-1}}{2} \cdot \left( f(x_{j-1}) + f(x_j) \right) + \sum_{j=1}^{n} h f \left( \frac{x_{j-1}+x_j}{2} \right)
\leq \frac{h}{2} \sum_{j=1}^{n} \left( f(x_{j-1}) + f(x_j) \right) + \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1}+x_j}{2} \right),
\]
which gives that
\[
\int_{a}^{b} f(u) du - h \sum_{j=1}^{n} f \left( \frac{x_{j-1}+x_j}{2} \right) \leq \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\]
for all \( j = 1, 2, \ldots, n \), which gives the desired result (2.27). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. \( \square \)

**Theorem 2.5.** Let \( I \subset \mathbb{R} \) be an open interval and \( a, b \in I, a < b \). Let \( f : I \to \mathbb{R} \) be an increasing convex function on \([a, b]\), then the inequality
\[
\int_{a}^{b} f(t) dt - \frac{(b-a)}{2} f(y) \geq \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1}+x_j}{2} \right) \geq 0,
\]
is valid for all \( y \in [a, b] \subset I \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible, in the sense that it cannot be replaced by a greater one. If \( f \) is concave then the inequality is reversed. In particular, if \( n = 1 \) then
\[
\int_{a}^{b} f(t) dt - \frac{(b-a)}{2} f(y) \geq \frac{b-a}{2} f \left( \frac{a+b}{2} \right) \geq 0,
\]

**Proof.** Let \( y \in [x_{j-1}, x_j] \) be an arbitrary point such that \( x_{j-1} < y-p \leq y \leq y+p < x_j \) for all \( j = 1, 2, \ldots, n \) with \( p > 0 \).

It is well known that \( f \) is convex on \( I \) iff
\[
f(y) \leq \frac{1}{2p} \int_{y-p}^{y+p} f(t) dt,
\]
for every subinterval \([y-p, y+p] \subset [a, b] \subset I \) for some \( p > 0 \). But since \( f \) increases on \([a, b]\), we also have
\[
f(y) \leq \frac{1}{2p} \int_{x_{j-1}}^{x_j} f(t) dt \leq \frac{1}{2p} \int_{x_{j-1}}^{x_j} f(t) dt.
\]
Choosing \( p \geq \frac{h}{2} \), (this choice is available since it is true for every subinterval in \( I \)), therefore from the last inequality we get

\[
f(y) \leq \frac{1}{h} \int_{y-p}^{y+p} f(t) \, dt \leq \frac{1}{h} \int_{x_{j-1}}^{x_j} f(t) \, dt.
\]

Again by convexity we have

\[
hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq \int_{x_{j-1}}^{x_j} f(t) \, dt.
\]

Adding the last two inequalities, we get

\[
hf(y) + hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) \, dt,
\]

or we write

\[
hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) \, dt - hf(y).
\]

Taking the sum over \( j \) from 1 to \( n \), we get

\[
h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(t) \, dt - \sum_{j=1}^{n} hf(y)
\]

hence,

\[
\int_{a}^{b} f(t) \, dt - \frac{(b-a)}{2} f(y) \geq h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0,
\]

holds by positivity of \( f \) and this proves our assertion.

To prove the sharpness let (2.28) holds with another constant \( C > 0 \), which gives

\[
(2.29) \quad \int_{a}^{b} f(t) \, dt - \frac{(b-a)}{2} f(y) \geq C \cdot h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0.
\]

Let \( f : [0, 1] \rightarrow \mathbb{R}_+ \) be the identity map \( f(x) = x \), then the right-hand side of (2.29) reduces to

\[
\frac{1}{2} - \frac{1}{2} y \geq C \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{2j - 1}{2n}
\]

\[
= C \cdot \frac{1}{n} \left[ \sum_{j=1}^{n} \frac{j}{n} - \sum_{j=1}^{n} \frac{1}{2n} \right]
\]

\[
= C \cdot \frac{1}{n} \left[ \frac{1}{n} \cdot \frac{n(n+1)}{2} - \frac{1}{2n} \cdot n \right]
\]

\[
= \frac{1}{2} C.
\]

Choose \( y = \frac{1}{2} \), it follows that \( \frac{1}{3} \geq \frac{1}{2} C \) which means that \( \frac{1}{2} \geq C \), i.e., \( \frac{1}{2} \) is the best possible constant in the right-hand side of (2.28). □
Corollary 2.3. Let \( \alpha_i \geq 0 \), for all \( i = 0, 1, 2, \ldots, n \), be positive real numbers such that \( \sum_{i=0}^{n} \alpha_i = 1 \), then under the assumptions of Theorem 2.5, we have

\[
\int_{a}^{b} f(t) \, dt - \frac{(b-a)}{2} f \left( \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i \right) \geq \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_{j}}{2} \right) \geq 0.
\]

The constant \( \frac{1}{2} \) in the right-hand side is the best possible. If \( f \) is concave then the inequality is reversed. In particular case if \( n = 1 \), then

\[
\int_{a}^{b} f(t) \, dt - \frac{(b-a)}{2} f \left( \alpha a + (1-\alpha) b \right) \geq \frac{b-a}{2} f \left( \frac{a+b}{2} \right) \geq 0,
\]

for all \( \alpha \in [0, 1] \).

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References


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