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A PARAMETER-BASED OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS WHOSE DERIVATIVES BELONGS TO $L_p([a,b])$ INVOLVING MULTIPLE POINTS

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Abstract. A new generalization of Ostrowski's inequality for functions whose derivatives belong to $L_p([a,b])$ $(1 \leq p < \infty)$ for k points via a parameter is provided. Some particular integral inequalities are derived as by products. Our results generalize some results in the literature.

1. Introduction

In 1938, Ostrowski [17] obtained the following inequality which is known in the literature as Ostrowski's inequality.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b). If $M := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)M,$$

for all $x \in [a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Due to the numerous applications of the Ostrowski's inequality, many authors have studied and generalized the inequality in several different ways. For more information

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about the Ostrowski's inequality and its associates, we refer the interested reader to the papers [1–16, 18].

In [5], Dragomir and Wang provided the following extension of Theorem 1.1 for functions whose derivatives belong to L_1 as follows.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \|f'\|_1$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

The same authors in [7], obtained an Ostrowski type inequality for differentiable mappings whose derivatives belong to L_p-spaces as follows.

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_p(a, b)$ $(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right] \|f'\|_p,$$

for all $x \in [a, b]$, where $\| \cdot \|_p$ is the $L_p([a, b])$-norm.

In [2], Dragomir obtained the following generalization of Theorem 1.2 as follows.

Theorem 1.4. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, 1, \ldots, k+1)$ be $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then we have the inequality*

$$\begin{aligned}
&\left| \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)dt \right| \\
\leq &\left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| : i = 0, \ldots, k-1 \right\} \right] \|f'\|_1 \\
\leq &\nu(h)\|f'\|_1,
\end{aligned}$$

where $\nu(h) := \max\{h_i | i = 0, 1, \ldots, k-1\}$, $h_i = x_{i+1} - x_i$ $(i = 0, \ldots, k-1)$.

In [3], Dragomir obtained the following generalization of Theorem 1.3 as follows.

Theorem 1.5. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, 1, \ldots, k+1)$ be $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then we have the inequality*

$$\left| \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)dt \right|$$

$$\leq \frac{1}{(q+1)^{1/q}} \|f'\|_p \left\{ \sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right\}^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}} \|f'\|_p \left\{ \sum_{i=0}^{k-1} h_i^{q+1} \right\}^{1/q}$$
$$\leq \frac{\nu(h)(b-a)^{1/q}}{(q+1)^{1/q}} \|f'\|_p,$$

where $\nu(h) := \max\{h_i | i = 0, 1, \ldots, k-1\}$, $h_i = x_{i+1} - x_i$ $(i = 0, \ldots, k-1)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\| \cdot \|_p$ *is the ususal* $L_p([a, b])$*-norm.*

Motivated by the numerous research on the Ostrowski's inequality in the past years, our main goal in this paper is to provide a generalization of Theorem 1.1 involving multiple points by introducing a parameter $\lambda \in [0, 1]$ for functions whose derivative belongs to L_p for $1 \leq p < \infty$ such that when $\lambda = 0$, we recapture Theorem 1.4 and Theorem 1.5.

2. Main Results

To prove our main results, we need the following lemma which is the case when the time scale $\mathbb{T} = \mathbb{R}$ in [18, Lemma 1] but the proof is provided here for completion.

Lemma 2.1 (Montgomery Identity)**.** *Let*

(a) $a, b \in \mathbb{R}$, $\lambda \in [0, 1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a, b]$*;*
(b) $\alpha_i \in \mathbb{R}$ $(i = 0, 1, \ldots, k+1)$ *is* $k + 2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = b$*;*
(c) $f : [a, b] \to \mathbb{R}$ *is a differentiable function;*
(d) *define the kernel function* $K(\cdot, I_k) : [a, b] \to \mathbb{R}$ *as follows*

$$K(t, I_k) = \begin{cases} t - \left(\alpha_1 - \lambda \frac{\alpha_1 - a}{2} \right), & t \in [a, \alpha_1), \\ t - \left(\alpha_1 + \lambda \frac{\alpha_2 - \alpha_1}{2} \right), & t \in [\alpha_1, x_1), \\ t - \left(\alpha_2 - \lambda \frac{\alpha_2 - \alpha_1}{2} \right), & t \in [x_1, \alpha_2), \\ \vdots \\ t - \left(\alpha_{k-1} + \lambda \frac{\alpha_k - \alpha_{k-1}}{2} \right), & t \in [\alpha_{k-1}, x_{k-1}), \\ t - \left(\alpha_k - \lambda \frac{\alpha_k - \alpha_{k-1}}{2} \right), & t \in [x_{k-1}, \alpha_k), \\ t - \left(\alpha_k + \lambda \frac{\alpha_{k+1} - \alpha_k}{2} \right), & t \in [\alpha_k, b], \end{cases}$$

for all $t \in [a, b]$.

Then we have the identity

$$\int_a^b K(t,I_k)f'(t)dt = (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) \\ + \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right) - \int_a^b f(t)dt. \tag{2.1}$$

Proof. We observe that

$$\int_a^b K(t,I_k)f'(t)dt = \sum_{i=0}^{k-1}\Bigg[\int_{x_i}^{\alpha_{i+1}}\left[t-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f'(t)dt \\ + \int_{\alpha_{i+1}}^{x_{i+1}}\left[t-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f'(t)dt\Bigg].$$

By integrating by parts, we have

$$\begin{aligned}\int_a^b K(t,I_k)f'(t)dt = &\sum_{i=0}^{k-1}\Bigg[\left[\alpha_{i+1}-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f(\alpha_{i+1}) \\ &- \left[x_i-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f(x_i) - \int_{x_i}^{\alpha_{i+1}} f(t)dt \\ &+ \left[x_{i+1}-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f(x_{i+1}) \\ &- \left[\alpha_{i+1}-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f(\alpha_{i+1}) - \int_{\alpha_{i+1}}^{x_{i+1}} f(t)dt\Bigg] \\ = &\sum_{i=0}^{k-1}\Bigg[\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(\alpha_{i+1}) - (x_i-\alpha_{i+1})f(x_i) - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i) \\ &+ (x_{i+1}-\alpha_{i+1})f(x_{i+1}) - \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1}) \\ &+ \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(\alpha_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)dt\Bigg].\end{aligned}$$

It follows that,

$$\begin{aligned}\int_a^b K(t,I_k)f'(t)dt = &\sum_{i=0}^{k-1}\left[\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)dt\right] \\ &+ \sum_{i=0}^{k-1}\left[-(x_i-\alpha_{i+1})f(x_i) + (x_{i+1}-\alpha_{i+1})f(x_{i+1})\right] \\ &+ \sum_{i=0}^{k-1}\left[-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i) - \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})\right] \\ = &\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1}) - \int_a^b f(t)dt\end{aligned}$$

$$- x_0 f(x_0) + x_k f(x_k) + \sum_{i=0}^{k-1} \alpha_{i+1}(f(x_i) - f(x_{i+1}))$$

$$+ \sum_{i=0}^{k-1} -\frac{\lambda}{2}\left[(\alpha_{i+1} - \alpha_i) f(x_i) + (\alpha_{i+2} - \alpha_{i+1}) f(x_{i+1})\right].$$

That is,

$$\begin{aligned}
\int_a^b K(t, I_k) f'(t)dt =& \sum_{i=0}^{k-1} \lambda \frac{\alpha_{i+2} - \alpha_i}{2} f(\alpha_{i+1}) - \int_a^b f(t)dt \\
&+ (\alpha_1 - a) f(a) + (b - \alpha_k) f(b) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \\
&- \frac{\lambda}{2}\left[(\alpha_1 - a) f(a) + (b - \alpha_k) f(b) + 2\sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i)\right] \\
=& \sum_{i=0}^{k-1} \lambda \frac{\alpha_{i+2} - \alpha_i}{2} f(\alpha_{i+1}) - \int_a^b f(t)dt + (1-\lambda) \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \\
&+ \left(1 - \frac{\lambda}{2}\right)\left[(\alpha_1 - a) f(a) + (b - \alpha_k) f(b)\right]. \qquad (2.2)
\end{aligned}$$

Now, consider the following

$$\begin{aligned}
&\sum_{i=0}^{k-1} (\alpha_{i+2} - \alpha_i) f(\alpha_{i+1}) \\
&= \sum_{i=0}^{k-1} (\alpha_{i+2} - \alpha_{i+1}) f(\alpha_{i+1}) + \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) \qquad (2.3) \\
&= \sum_{i=1}^{k} (\alpha_{i+1} - \alpha_i) f(\alpha_i) + \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) \\
&= \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(\alpha_i) - (\alpha_1 - \alpha_0) f(\alpha_0) \\
&\quad + \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) - (\alpha_{k+1} - \alpha_k) f(\alpha_{k+1}) \\
&= \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i)(f(\alpha_i) + f(\alpha_{i+1})) - \left[(\alpha_1 - \alpha_0) f(\alpha_0) + (\alpha_{k+1} - \alpha_k) f(\alpha_{k+1})\right].
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{i=0}^{k-1} \frac{\lambda}{2} (\alpha_{i+2} - \alpha_i) f(\alpha_{i+1}) =& \frac{\lambda}{2} \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i)\left(f(\alpha_i) + f(\alpha_{i+1})\right) \\
&- \frac{\lambda}{2}\left[(\alpha_1 - a) f(a) + (b - \alpha_k) f(b)\right]. \qquad (2.4)
\end{aligned}$$

Substituting (2.4) in (2.2) gives the identity

$$\int_a^b K(t,I_k)f'(t)dt = (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) + \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right) - \int_a^b f(t)dt. \qquad \square$$

Lemma 2.2. *Under the conditions of Lemma* 2.1*, we have the following inequality:*

$$\left|\int_a^b K(t,I_k)f'(t)dt\right| \le \sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\,|f'(t)|\,dt\right] + \frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\dots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\int_a^b |f'(t)|dt. \tag{2.5}$$

Proof. First, we observe that for any real numbers δ and γ, the following holds:

$$\max\{\gamma,\delta\} = \frac{\gamma+\delta}{2} + \frac{|\gamma-\delta|}{2}. \tag{2.6}$$

Now, by using the property of the absolute value and the definition of $K(\cdot, I_k)$, we have that

$$\begin{aligned}\left|\int_a^b K(t,I_k)f'(t)dt\right| &\le \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}} |K(t,I_k)|\,|f'(t)|\,dt + \int_{\alpha_{i+1}}^{x_{i+1}} |K(t,I_k)|\,|f'(t)|\,dt\right] \\ &\le \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}} \left|t-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right|\,|f'(t)|\,dt \right.\\ &\quad\left. + \int_{\alpha_{i+1}}^{x_{i+1}} \left|t-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right|\,|f'(t)|\,dt\right] \\ &\le \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}} |t-\alpha_{i+1}|\,|f'(t)|\,dt + \frac{\lambda}{2}(\alpha_{i+1}-\alpha_i)\int_{x_i}^{\alpha_{i+1}} |f'(t)|\,dt \right.\\ &\quad\left. + \int_{\alpha_{i+1}}^{x_{i+1}} |t-\alpha_{i+1}|\,|f'(t)|\,dt + \frac{\lambda}{2}(\alpha_{i+2}-\alpha_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}} |f'(t)|\,dt\right] \\ &\le \sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}||f'(t)|dt \right.\\ &\quad\left. + \frac{\lambda}{2}\max\left\{\alpha_{i+1}-\alpha_i, \alpha_{i+2}-\alpha_{i+1}\right\}\int_{x_i}^{x_{i+1}} |f'(t)|dt\right].\end{aligned}$$

Thus,

$$\left|\int_a^b K(t,I_k)f'(t)dt\right| \le \sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}||f'(t)|dt\right] + \frac{\lambda}{2}\sum_{i=0}^{k-1}\left[\max\left\{\alpha_{i+1}-\alpha_i, \alpha_{i+2}-\alpha_{i+1}\right\}\int_{x_i}^{x_{i+1}} |f'(t)|dt\right]. \tag{2.7}$$

By using (2.10), we deduce that

$$\begin{aligned}
&\sum_{i=0}^{k-1}\max\{\alpha_{i+1}-\alpha_i,\alpha_{i+2}-\alpha_{i+1}\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&=\sum_{i=0}^{k-1}\left(\frac{1}{2}(\alpha_{i+2}-\alpha_i)+\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right)\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&\le \max_{i=0,1,\ldots,k-1}\left\{\frac{1}{2}\tau_i+\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&\le\left(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\int_a^b|f'(t)|dt. \qquad (2.8)
\end{aligned}$$

Using (2.7) and (2.8) yields the desired result. Hence, the proof is complete. □

We now state and prove our first theorem which is for the case $p=1$.

Theorem 2.1. *Under the conditions of Lemma* 2.1, *suppose that* $f'\in L_1[a,b]$, *then the following inequalities hold:*

$$\begin{aligned}
&\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt\right|\\
&\le\left[\frac{1}{2}\nu(h)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right\}\right]\|f'\|_1\\
&\quad+\left[\frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\right]\|f'\|_1\\
&\le\left(\nu(h)+\frac{\lambda}{2}\nu(\tau)\right)\|f'\|_1, \qquad (2.9)
\end{aligned}$$

where $h_i=x_{i+1}-x_i$, $\tau_i=\alpha_{i+2}-\alpha_i$ $(i=0,1,\ldots,k-1)$, $\nu(h)=\max\{h_i:i=0,1,\ldots,k-1\}$ *and* $\nu(\tau)=\max\{\tau_i:i=0,1,\ldots,k-1\}$.

Proof. First, we observe that for any real numbers β,δ and γ, the following holds:

$$\sup_{t\in[\gamma,\delta]}|t-\beta|=\max\{|\gamma-\beta|,|\delta-\beta|\}. \qquad (2.10)$$

By using (2.10), we have

$$\begin{aligned}
&\sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt\right]\\
&\le\sum_{i=0}^{k-1}\left[\sup_{t\in[x_i,x_{i+1}]}|t-\alpha_{i+1}|\int_{x_i}^{x_{i+1}}|f'(t)|dt\right]\\
&=\sum_{i=0}^{k-1}\left[\max\{|x_i-\alpha_{i+1}|,|x_{i+1}-\alpha_{i+1}|\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\right]
\end{aligned}$$

$$= \sum_{i=0}^{k-1} \left[\max\left\{\alpha_{i+1} - x_i, x_{i+1} - \alpha_{i+1}\right\} \int_{x_i}^{x_{i+1}} |f'(t)|dt \right].$$

That is,

$$\sum_{i=0}^{k-1} \left[\int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}||f'(t)|dt \right]$$
$$\leq \sum_{i=0}^{k-1} \left[\max\{\alpha_{i+1} - x_i, x_{i+1} - \alpha_{i+1}\} \int_{x_i}^{x_{i+1}} |f'(t)|dt \right]. \tag{2.11}$$

Now, by using (2.6) in (2.11), we have that

$$\sum_{i=0}^{k-1} \left[\int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}||f'(t)|dt \right]$$
$$\leq \sum_{i=0}^{k-1} \left[\left(\frac{1}{2}(x_{i+1} - x_i) + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right) \int_{x_i}^{x_{i+1}} |f'(t)|dt \right]$$
$$= \sum_{i=0}^{k-1} \left[\left(\frac{1}{2}h_i + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right) \int_{x_i}^{x_{i+1}} |f'(t)|dt \right]$$
$$\leq \max_{i=0,1,\ldots,k-1} \left\{ \frac{1}{2}h_i + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |f'(t)|dt$$
$$\leq \left[\frac{1}{2}\nu(h) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \right] \|f'\|_1. \tag{2.12}$$

Using (2.5) and (2.12), we have

$$\left| \int_a^b K(t, I_k) f'(t)dt \right| \leq \left[\frac{1}{2}\nu(h) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \right.$$
$$\left. + \frac{\lambda}{2} \left(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2} \right| \right\} \right) \right] \|f'\|_1. \tag{2.13}$$

By using (2.1) and (2.13), we obtained the first inequality in (2.9). To obtain the second inequality, we observe that

$$\left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2}(x_{i+1} - x_i) = \frac{1}{2}h_i$$

and

$$\left| \alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2} \right| \leq \frac{1}{2}(\alpha_{i+2} - \alpha_i) = \frac{1}{2}\tau_i.$$

So, it follows that

$$\max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \leq \frac{1}{2}\nu(h)$$

and

$$\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\le\frac{1}{2}\nu(\tau).$$

This completes the proof of the theorem. □

Remark 2.1. If we take $\lambda = 0$ in Theorem 2.1, then we recover Theorem 1.4.

Lemma 2.3. *Under the conditions of Lemma* 2.1 *with $f' \in L_p([a,b])$, we have the following inequalities*

$$\begin{aligned}\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt &\le \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}}\\ &\le \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{\frac{1}{q}} \qquad (2.14)\\ &\le \frac{\nu(h)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.2. The inequalities in Lemma 2.3 were established in the proof of [3, Theorem 3], and hence the proof is omited.

Theorem 2.2. *Under the conditions of Lemma* 2.1*, suppose that $f' \in L_p([a,b])$, for $1 < p < \infty$, then the following inequalities hold:*

$$\begin{aligned}&\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt\right|\\ \le&\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}}+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p\\ \le&\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{\frac{1}{q}}+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p\\ \le&\frac{\nu(h)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p,\end{aligned}$$

where $h_i = x_{i+1}-x_i$, $\tau_i=\alpha_{i+2}-\alpha_i$ $(i=0,1,\ldots,k-1)$, $\nu(h)=\max_{i=0,1,\ldots,k-1}h_i$, $\nu(\tau)=\max_{i=0,1,\ldots,k-1}\tau_i$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 2.2 and the fact that

$$\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\le\frac{1}{2}\nu(\tau),$$

we deduce that

$$\left|\int_a^b K(t,I_k)f'(t)dt\right| \le \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\,|f'(t)|\,dt + \frac{\lambda}{2}\nu(\tau)\int_a^b |f'(t)|dt. \tag{2.15}$$

Using the Hölder's inequality, we obtain

$$\int_a^b |f'(t)|dt \le (b-a)^{\frac{1}{q}}\|f'\|_p. \tag{2.16}$$

We obtain the desired results by using (2.1), (2.14), (2.15) and (2.16). □

Remark 2.3. If we choose $\lambda = 0$ in Theorem 2.2, then we recover Theorem 1.5.

3. Some Particular Cases

In this section, we consider some particular cases of our main results.

Corollary 3.1. *Under the conditions of Theorem* 2.1*, if we choose* $\alpha_0 = a$*,* $\alpha_{i+1} = \frac{x_i+x_{i+1}}{2}$ $(i = 0,\dots,k-1)$ *and* $\alpha_{k+1} = b$*, then we have the inequalities*

$$\begin{aligned}
&\Bigg|\frac{1-\lambda}{2}\left[(x_1-a)f(a) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i) + (b-x_{k-1})f(b)\right] \\
&\quad + \frac{\lambda}{4}\Bigg[(x_1-a)\left(f(a)+f\left(\frac{x_1+a}{2}\right)\right) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})\Bigg(f\left(\frac{x_i+x_{i-1}}{2}\right) \\
&\quad + f\left(\frac{x_{i+1}+x_i}{2}\right)\Bigg) + (b-x_{k-1})\left(f(b)+f\left(\frac{b+x_{k-1}}{2}\right)\right)\Bigg] - \int_a^b f(t)dt\Bigg| \\
&\le \left[\frac{1}{2}\nu(h) + \frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\dots,k-1}\left\{\left|\alpha_{i+1} - \frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\right]\|f'\|_1 \\
&\le \frac{\nu(h)+\lambda\nu(\tau)}{2}\|f'\|_1,
\end{aligned}$$

where $h_i = x_{i+1}-x_i$*,* $\tau_i = \alpha_{i+2}-\alpha_i$ $(i = 0,1,\dots,k-1)$*,* $\nu(h) = \max\{h_i : i = 0,1,\dots, k-1\}$ *and* $\nu(\tau) = \max\{\tau_i : i = 0,1,\dots,k-1\}$ *.*

Proof. In this case, we have $\alpha_1-\alpha_0 = \frac{x_1-a}{2}$, $\alpha_{i+1}-\alpha_i = \frac{x_{i+1}-x_{i-1}}{2}$ $(i = 1,\dots,k-1)$, $\alpha_{k+1}-\alpha_k = \frac{b-x_{k-1}}{2}$ and $\alpha_{i+1} - \frac{x_i+x_{i+1}}{2} = 0$ $(i = 0,\dots,k-1)$. □

Now, if we choose I_k to be the equidistant partition of $[a,b]$, then we have the following corollary.

Corollary 3.2. *Let* $I_k : x_i = a+(b-a)\frac{i}{k}$ $(i = 0,1,\dots,k)$ *be the equidistant partitioning of* $[a,b]$ *and the* α_i'*s be as in Corollary* 3.1*. Then the following inequality holds*

$$\begin{aligned}
&\Bigg|\frac{1-\lambda}{2}\left[\frac{b-a}{k}\left(f(a)+f(b)\right) + \frac{2(b-a)}{k}\sum_{i=1}^{k-1} f\left(\frac{(k-i)a+bi}{k}\right)\right] \\
&\quad + \frac{\lambda}{4}\Bigg[\frac{b-a}{k}\left(f(a)+f(b)+f\left(\frac{(2k-1)a+b}{2k}\right)+f\left(\frac{a+(2k-1)b}{2k}\right)\right)
\end{aligned}$$

$$+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}\left(f\left(\frac{(2k-2i+1)a+(2i-1)b}{2k}\right)\right.$$
$$\left.\left.+f\left(\frac{(2k-2i-1)a+(2i+1)b}{2k}\right)\right)\right]-\int_a^b f(t)dt\Bigg|$$
$$\leq\frac{b-a}{2k}(1+2\lambda)\|f'\|_1.$$

Proof. This follows from the second inequality in Corollary 3.1 and the following computations:

$$x_1-a=\frac{b-a}{k},\quad x_1+a=\frac{(2k-1)a+b}{k},$$
$$x_{i+1}-x_{i-1}=\frac{2(b-a)}{k}\quad(i=1,\ldots,k-1),\quad \frac{x_i+x_{i-1}}{2}=\frac{(2k-2i+1)a+(2i-1)b}{k},$$
$$\frac{x_i+x_{i+1}}{2}=\frac{(2k-2i-1)a+(2i+1)b}{k}\quad(i=1,\ldots,k-1),$$
$$b-x_{k-1}=\frac{b-a}{k},\quad b+x_{k-1}=\frac{a+(2k-1)b}{k},\quad h_i=\frac{b-a}{k},$$
$$(i=0,\cdots,k-1),\quad \tau_0=\frac{3(b-a)}{2k},\quad \tau_i=\frac{2(b-a)}{k}\quad(i=1,\ldots,k-2)$$
$$\text{and}\quad \tau_{k-1}=\frac{3(b-a)}{2k}.$$

Thus, we deduce that $\nu(h)=\frac{b-a}{k}$ and $\nu(\tau)=\dfrac{2(b-a)}{k}$. □

Corollary 3.3. *Under the conditions of Theorem* 2.2*, if we choose I_k to be the equidistant partitioning of* $[a,b]$ *and the α_i's be as in Corollary* 3.1*, then the following inequality holds;*

$$\left|\frac{1-\lambda}{2}\left[\frac{b-a}{k}(f(a)+f(b))+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}f\left(\frac{(k-i)a+bi}{k}\right)\right]\right.$$
$$+\frac{\lambda}{4}\left[\frac{b-a}{k}\left(f(a)+f(b)+f\left(\frac{(2k-1)a+b}{2k}\right)+f\left(\frac{a+(2k-1)b}{2k}\right)\right)\right.$$
$$+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}\left(f\left(\frac{(2k-2i+1)a+(2i-1)b}{2k}\right)\right.$$
$$\left.\left.+f\left(\frac{(2k-2i-1)a+(2i+1)b}{2k}\right)\right)\right]-\int_a^b f(t)dt\Bigg|$$
$$\leq\left[\frac{1}{(q+1)^{\frac{1}{q}}}+\lambda\right]\frac{(b-a)^{\frac{1}{q}+1}}{k}\|f'\|_p.$$

Proof. The result follows from Theorem 2.2 and using the computations in the proves of Corollaries 3.1 and 3.2. □

In what follows, we consider some special cases of Theorem 2.1. Similar results could also be derived from Theorem 2.2 as well.

Corollary 3.4. *Let $a, b \in \mathbb{R}$, $a < b$, $\lambda \in [0,1]$, $a \le \alpha_1 \le x \le \alpha_2 \le b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1([a,b])$. Then the following inequalities hold*

$$\begin{aligned}
&\Big|(1-\lambda)\left[(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f(x) + (b-\alpha_2)f(b)\right] \\
&+ \frac{\lambda}{2}[(\alpha_1 - a)(f(a)+f(\alpha_1)) + (\alpha_2-\alpha_1)(f(\alpha_1)+f(\alpha_2)) \\
&+ (b-\alpha_2)(f(\alpha_2)+f(b))] - \int_a^b f(t)dt\Big| \\
\le&\Bigg[\frac{1}{2}\max\{x-a, b-x\} + \max\left\{\left|\alpha_1 - \frac{a+x}{2}\right|, \left|\alpha_2 - \frac{x+b}{2}\right|\right\} \\
&+ \frac{\lambda}{2}\left(\frac{1}{2}\max\{\alpha_2 - a, b-\alpha_1\} + \max\left\{\left|\alpha_1 - \frac{a+\alpha_2}{2}\right|, \left|\alpha_2 - \frac{\alpha_1+b}{2}\right|\right\}\right)\Bigg]\|f'\|_1 \\
\le&\left(\max\{x-a, b-x\} + \frac{\lambda}{2}\max\{\alpha_2 - a, b-\alpha_1\}\right)\|f'\|_1.
\end{aligned}$$

Proof. The proof follows directly from Theorem 2.1 by choosing $k = 2$. □

Corollary 3.5. (a) *If we choose $\alpha_1 = a$ and $\alpha_2 = b$ in Corollary* 3.4*, then we have the inequality*

$$\begin{aligned}
&\left|(b-a)\left[(1-\lambda)f(x) + \frac{\lambda}{2}(f(a)+f(b))\right] - \int_a^b f(t)dt\right| \\
\le&\left(\max\{x-a, b-x\} + \frac{\lambda}{2}(b-a)\right)\|f'\|_1 \\
=&\left(\frac{(1+\lambda)(b-a)}{2} + \left|x - \frac{a+b}{2}\right|\right)\|f'\|_1
\end{aligned}$$

for all $x \in [a,b]$.

(b) *If we choose $x = \frac{a+b}{2}$ in part* (a)*, then we have the following perturbed "midpoint inequality":*

$$\begin{aligned}
&\left|(b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \frac{\lambda}{2}(f(a)+f(b))\right] - \int_a^b f(t)dt\right| \\
\le&\frac{(1+\lambda)(b-a)}{2}\|f'\|_1.
\end{aligned}$$

(c) *If we choose $\alpha_1 = \frac{5a+b}{6}$, $\alpha_2 = \frac{a+5b}{6}$ and $x_1 = x$ in Corollary* 3.4*, then we have*

$$\left|(1-\lambda)\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x)\right]\right.$$

$$+\frac{\lambda(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+\frac{5}{2}f\left(\frac{5a+b}{6}\right)+\frac{5}{2}f\left(\frac{a+5b}{6}\right)\right]-\int_a^b f(t)dt\Bigg|$$
$$\leq\left[\frac{1}{2}\max\{x-a,b-x\}+\frac{1}{2}\max\left\{\left|x-\frac{2a+b}{3}\right|,\left|x-\frac{a+2b}{3}\right|\right\}+\frac{\lambda}{3}(b-a)\right]\|f'\|_1$$
$$\leq\left(\max\{x-a,b-x\}+\frac{5\lambda(b-a)}{12}\right)\|f'\|_1$$
$$=\left[\frac{(b-a)(6+5\lambda)}{12}+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_1.$$

(d) *In particular, if we choose* $x=\frac{a+b}{2}$ *in the first inequality in part* (c)*, then we have the following perturbed "Simpson's inequality":*

$$\Bigg|(1-\lambda)\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]+\frac{\lambda(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+\frac{5}{2}f\left(\frac{5a+b}{6}\right)\right.$$
$$\left.+\frac{5}{2}f\left(\frac{a+5b}{6}\right)\right]-\int_a^b f(t)dt\Bigg|\leq\frac{(b-a)(1+\lambda)}{3}\|f'\|_1.$$

4. Conclusion

Some new integral inequalities of Ostrowski type involving a parameter $\lambda\in[0,1]$ for functions whose derivatives belong to L_p involving multiple points have been established. Some particular cases have be considered as examples. By considering different partitions, different points and/or different values of the parameter we will obtain several interesting inequalities. For $\lambda=0$, our results reduces to some results in the literature and for $\lambda\in(0,1]$, we obtain new results. It is worth noting that the Ostrowski inequality plays a very important role in numerical integration such as applications to the numerical quadrature rule. So, we believe that the inequalities obtained in this paper could be applied in numerical integration and other areas of Mathematical Analysis.

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