Kragujevac Journal of Mathematics
Volume 40(1) (2016), Pages 113–124.

INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL WITH RESTRICTED ZEROS

AHMAD ZIREH[1] AND MAHMOOD BIDKHAM[2]

Abstract. For a polynomial $p(z)$ of degree n, we consider an operator D_α which map a polynomial $p(z)$ into $D_\alpha p(z) := (\alpha - z)p'(z) + np(z)$ with respect to α. It was proved by Liman et al. [A. Liman, R. N. Mohapatra and W. M. Shah, Inequalities for the Polar Derivative of a Polynomial, Complex Analysis and Operator Theory, 2010] that if $p(z)$ has no zeros in $|z| < 1$ then for all α, $\beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)\right| \leq \frac{n}{2}\left\{ \left[\left|\alpha + \beta\frac{|\alpha|-1}{2}\right| + \left|z + \beta\frac{|\alpha|-1}{2}\right|\right]\max_{|z|=1}|p(z)| - \left[\left|\alpha + \beta\frac{|\alpha|-1}{2}\right| - \left|z + \beta\frac{|\alpha|-1}{2}\right|\right]\min_{|z|=1}|p(z)|\right\}.$$

In this paper we extend above inequality for the polynomials having no zeros in $|z| < 1$, except s-fold zeros at the origin. Our result generalize certain well-known polynomial inequalities.

1. Introduction and Statement of Results

According to a well known result as Bernstein's inequality on the derivative of a polynomial $p(z)$ of degree n, we have

$$\max_{|z|=1}|p'(z)| \leq n\max_{|z|=1}|p(z)|. \tag{1.1}$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin (see [13] and [4]). The inequality (1.1) can be sharpened, by considering

Key words and phrases. Polynomial, inequality, maximum modulus, polar derivative, restricted zeros.

2010 *Mathematics Subject Classification.* Primary: 30A10. Secondary: 30C10, 30D15.

Received: February 20, 2015.

Accepted: December 15, 2015.

the class of polynomials having no zeros in $|z| < 1$. In fact, P. Erdős conjectured and later Lax [10] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

As a refinement of (1.2), Aziz and Dawood [1] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}. \tag{1.3}$$

As an improvement of inequality (1.3) Dewan and Hans [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for any complex number β with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq & \frac{n}{2} \left\{ \left(\left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right) \max_{|z|=1} |p(z)| \right. \\ & \left. - \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}. \end{aligned} \tag{1.4}$$

Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[\frac{D_\alpha p(z)}{\alpha} \right] = p'(z).$$

For the polar derivative $D_\alpha p(z)$, Aziz and Shah [2] proved that if $p(z)$ having all its zeros in $|z| \leq 1$, then

$$|D_\alpha p(z)| \geq n|\alpha||z|^{n-1} \min_{|z|=1} |p(z)|, \quad |z| \geq 1, \tag{1.5}$$

and as an extension to inequality (1.3) they proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (|\alpha| + 1) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=1} |p(z)| \right\}. \tag{1.6}$$

Recently Dewan et al. [9] generalized the inequality (1.6) to the polynomial of the form $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leq t \leq n$, and proved if $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$ then for $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1 + s_0} \left\{ (|\alpha| + s_0) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=k} |p(z)| \right\}, \tag{1.7}$$

where

$$s_0 = k^{t+1}\left\{\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\},$$

and $m = \min_{|z|=k}|p(z)|$.

As a generalization of the inequality (1.7), Bidkham et al. [5] proved, if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$ then for $0 < r \le R \le k$ and $|\alpha| \ge R$,

$$\max_{|z|=R}|D_\alpha p(z)| \le \frac{n}{1+s_0'}\left\{\left(\frac{|\alpha|}{R}+s_0'\right)\exp\left\{n\int_r^R A_t dt\right\}\max_{|z|=r}|p(z)|\right.$$
$$\left. +(s_0'+1-\left(\frac{|\alpha|}{R}+s_0'\right)\exp\left\{n\int_r^R A_t dt\right\}\min_{|z|=k}|p(z)|\right\},$$

where

$$A_t = \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\left(\frac{\mu}{n}\right)\left(\frac{|a_\mu|}{|a_0|-m}\right)(k^{\mu+1}t^\mu+k^{2\mu}t)},$$

$$s_0' = \left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}Rk^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{R(|a_0|-m)}k^{\mu+1}+1}\right\},$$

and $m = \min_{|z|=k}|p(z)|$.

As an improvement and generalization to the inequalities (1.6) and (1.4), Liman et al. [11] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for all α, β with $|\alpha| \ge 1$, $|\beta| \le 1$ and $|z| = 1$,

(1.8)
$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)\right| \le \frac{n}{2}\left\{\left(\left|\alpha+\beta\frac{|\alpha|-1}{2}\right| + \left|z+\beta\frac{|\alpha|-1}{2}\right|\right)\max_{|z|=1}|p(z)|\right.$$
$$\left. -\left(\left|\alpha+\beta\frac{|\alpha|-1}{2}\right| - \left|z+\beta\frac{|\alpha|-1}{2}\right|\right)\min_{|z|=1}|p(z)|\right\}.$$

In this paper, we first obtain the following generalization of polynomial inequality (1.5), as follows:

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \le 1$, with s-fold zeros at the origin, then*

$$\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \ge \left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right||z|^n \min_{|z|=1}|p(z)|, \tag{1.9}$$

for every real or complex numbers β, α with $|\beta| \le 1$, $|\alpha| \ge 1$ and $|z| \ge 1$. The result is best possible and equality holds for the polynomials $p(z) = az^n$.

If we take $s = 0$ in Theorem 1.1, we have

Corollary 1.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then for $|\beta| \leq 1$, $|\alpha| \geq 1$ and $|z| \geq 1$, we have*

$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)\right| \geq n\left|\alpha + \beta\frac{|\alpha|-1}{2}\right| |z|^n \min_{|z|=1}|p(z)|. \tag{1.10}$$

For $\beta = 0$ the inequality (1.10) reduces to inequality (1.5).

Next by using Theorem 1.1, we generalize the inequality (1.8).

Theorem 1.2. *Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < 1$, except s-fold zeros at the origin, then for all α, $\beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\begin{aligned}\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \\ \leq \frac{1}{2}\Bigg[\left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right| + \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\}\max_{|z|=1}|p(z)| \\ - \left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right| - \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\}\min_{|z|=1}|p(z)|\Bigg].\end{aligned} \tag{1.11}$$

If we take $s = 0$ in Theorem 1.2, then the inequality (1.11) reduces to the inequality (1.8).

Theorem 1.2 simplifies to the following result by taking $\beta = 0$.

Corollary 1.2. *Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < 1$, except s-fold zeros at the origin, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $|z| = 1$, we have*

$$|D_\alpha p(z)| \leq \frac{1}{2}\left\{n|\alpha| + |(n-s)z + s\alpha|\right) \max_{|z|=1}|p(z)| - (n|\alpha| - |(n-s)z + s\alpha|)\min_{|z|=1}|p(z)|\right\}.$$

Dividing two sides of inequality (1.11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following generalization of the inequality (1.4).

Corollary 1.3. *Let $p(z)$ be a polynomial of degree n, having no zeros in $|z| < 1$, except s-fold zeros at the origin, then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, and $|z| = 1$ we have*

$$\begin{aligned}\left|zp'(z) + \frac{\beta(n+s)}{2}p(z)\right| \leq \frac{1}{2}\Bigg\{\left(\left|n + \beta\frac{n+s}{2}\right| + \left|s + \beta\frac{n+s}{2}\right|\right)\max_{|z|=1}|p(z)| \\ - \left(\left|n + \beta\frac{n+s}{2}\right| - \left|s + \beta\frac{n+s}{2}\right|\right)\min_{|z|=1}|p(z)|\Bigg\}.\end{aligned}$$

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Laguerre [12].

Lemma 2.1. *If all the zeros of an n^{th} degree polynomial $p(z)$ lie in a circular region C and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

Lemma 2.2. *Let $p(z)$ is a polynomial of degree n, has no zero in $|z| < 1$, then on $|z| = 1$,*

$$|p'(z)| \leq |q'(z)|,$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

The above lemma is due to Chan and Malik [6].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq 1$, then on $|z| = 1$,*

$$|q'(z)| \leq |p'(z)|,$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

Proof. Since $p(z)$ has all its zeros in $|z| \leq 1$, therefore $q(z)$ has no zero in $|z| < 1$. Now applying Lemma 2.2 to the polynomial $q(z)$ and the result follows. □

The following lemma is due to Aziz and Shah [3].

Lemma 2.4. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq 1$, with s-fold zeros at the origin, then*

$$|p'(z)| \geq \frac{n+s}{2}|p(z)|, \quad |z| = 1.$$

Lemma 2.5. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq 1$, with s-fold zeros at the origin, then for all real or complex number α with $|\alpha| \geq 1$ and $|z| = 1$, we have*

$$|D_\alpha p(z)| \geq \frac{(n+s)(|\alpha|-1)}{2}|p(z)|.$$

The above lemma is due to K. K. Dewan and A. Mir [8].

Lemma 2.6. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, then for all α, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $|\alpha| \geq 1$ and $|z| = 1$, we have*

$$\left| zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \leq \left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| \max_{|z|=1}|p(z)|. \tag{2.1}$$

Proof. Let $M = \max_{|z|=1}|p(z)|$, if $|\lambda| < 1$, then $|\lambda p(z)| < |Mz^n|$ for $|z| = 1$. Therefore it follows by Rouche's Theorem that the polynomial $G(z) = Mz^n - \lambda p(z)$ has all its zeros in $|z| < 1$ with s-fold zeros at the origin. By using Lemma 2.5, to the polynomial $G(z)$, we have for every real or complex number α with $|\alpha| \geq 1$ and for $|z| = 1$,

$$|zD_\alpha G(z)| \geq \frac{(n+s)(|\alpha|-1)}{2}|G(z)|,$$

or

$$|n\alpha Mz^n - \lambda zD_\alpha p(z)| \geq \frac{(n+s)(|\alpha|-1)}{2}|Mz^n - \lambda p(z)|.$$

On the other hand by Lemma 2.1 all the zeros of $D_\alpha G(z) = n\alpha M z^{n-1} - \lambda D_\alpha p(z)$ lie in $|z| < 1$, where $|\alpha| \geq 1$. Therefore for any β with $|\beta| \leq 1$, Rouche's Theorem implies that all the zeros of

$$n\alpha M z^n - \lambda z D_\alpha p(z) + \beta \frac{(n+s)(|\alpha|-1)}{2}(Mz^n - \lambda p(z)),$$

lie in $|z| < 1$. This conclude that the polynomial

(2.2)
$$T(z) = \left(n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right) Mz^n - \lambda\left(zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right),$$

will have no zeros in $|z| \geq 1$. This implies that for every β with $|\beta| < 1$ and $|z| = 1$,

$$\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \leq \left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| M. \tag{2.3}$$

If the inequality (2.3) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| M < \left|z_0 D_\alpha p(z_0) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z_0)\right|.$$

Take

$$\lambda = \frac{\left(n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right) M}{z_0 D_\alpha p(z_0) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z_0)},$$

then $|\lambda| < 1$ and with this choice of λ, we have $T(z_0) = 0$ for $|z_0| \geq 1$, from (2.2). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For β with $|\beta| = 1$, inequality (2.3) follows by continuity. This completes the proof of Lemma 2.6. □

Lemma 2.7. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, then for all α, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $|\alpha| \geq 1$ and $|z| = 1$, we have*

$$\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| + \left|zD_\alpha Q(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|$$
$$\leq \left\{\left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| + \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\} \max_{|z|=1}|p(z)|,$$

where $Q(z) = z^{n+s}\overline{p(1/\overline{z})}$.

Proof. Let $M = \max_{|z|=1}|p(z)|$. For λ with $|\lambda| > 1$, it follows by Rouche's Theorem that the polynomial $G(z) = p(z) - \lambda M z^s$ has no zeros in $|z| < 1$, except s-fold zeros at the origin. Consequently the polynomial

$$H(z) = z^{n+s}\overline{G(1/\overline{z})},$$

has all its zeros in $|z| \leq 1$ with s-fold zeros at the origin, also $|G(z)| = |H(z)|$ for $|z| = 1$. Since all the zeros of $H(z)$ lie in $|z| \leq 1$, therefore, for δ with $|\delta| > 1$, by

Rouche's Theorem all the zeros of $G(z)+\delta H(z)$ lie in $|z|\le 1$. Hence by Lemma 2.5 for every α with $|\alpha|\ge 1$, and $|z|=1$, we have

$$\frac{(n+s)(|\alpha|-1)}{2}|G(z)+\delta H(z)|\le |zD_\alpha(G(z)+\delta H(z))|.$$

Now using a similar argument as used in the proof of Lemma 2.6, we get for every real or complex number β with $|\beta|\le 1$ and $|z|\ge 1$,

$$\left|zD_\alpha G(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}G(z)\right|\le\left|zD_\alpha H(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}H(z)\right|. \tag{2.4}$$

Therefore by the equalities

$$H(z)=z^{n+s}\overline{G\left(1/\overline{z}\right)}=z^{n+s}\overline{p\left(1/\overline{z}\right)}-\overline{\lambda}Mz^n=Q(z)-\overline{\lambda}Mz^n,$$

or

$$H(z)=Q(z)-\overline{\lambda}Mz^n,$$

and substitute for $G(z)$ and $H(z)$ in (2.4) we get

$$\left|\left(zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right)-\lambda\left((n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right)Mz^s\right|$$
$$\le\left|\left(zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right)-\overline{\lambda}\left(n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right)Mz^n\right|.$$

This implies

$$\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|-\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right||\lambda Mz^s|$$
$$\le\left|\left(zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right)-\overline{\lambda}\left(n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right)Mz^n\right|. \tag{2.5}$$

As $|p(z)|=|Q(z)|$ for $|z|=1$, i.e., $\max_{|z|=1}|p(z)|=\max_{|z|=1}|Q(z)|=M$, by using Lemma 2.6 for $Q(z)$, we obtain for $|z|=1$,

$$\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|<|\lambda|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|M.$$

Thus taking suitable choice of argument of λ, result is

$$\left|\left(zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right)-\overline{\lambda}\left(n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right)Mz^n\right|$$
$$=|\lambda|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|M-\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|. \tag{2.6}$$

By combining right hand side of (2.5) and (2.6) we get for $|z|=1$ and $|\beta|\le 1$,

$$\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|-|\lambda|\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|M$$
$$\le|\lambda|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|M-\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|,$$

i.e.,

$$\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|+\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|$$
$$\leq|\lambda|\left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|+\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\}M.$$

Taking $|\lambda|\to 1$, we have

$$\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|+\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|$$
$$\leq\left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|+\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\}M.$$

This gives the result. □

The following lemma is due to Zireh [14].

Lemma 2.8. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n, having all its zeros in $|z|<k$, ($k>0$), then $m<k^n|a_n|$, where $m=\min\limits_{|z|=k}|p(z)|$.*

3. Proof of the Theorems

Proof of Theorem 1.1. If $p(z)$ has a zero on $|z|=1$, then the inequality (1.9) is trivial. Therefore we assume that $p(z)$ has all its zeros in $|z|<1$. Let $m=\min_{|z|=1}|p(z)|$, then $m>0$ and $|p(z)|\geq m$ where $|z|=1$. Therefore, for $|\lambda|<1$, it follows by Rouche's Theorem and Lemma 2.8 that the polynomial $G(z)=p(z)-\lambda m z^n$ is of degree n and has all its zeros in $|z|<1$ with s-fold zeros at the origin. By using Lemma 2.1, $D_\alpha G(z)=D_\alpha p(z)-\alpha\lambda mnz^{n-1}$, has all its zeros in $|z|<1$, where $|\alpha|\geq 1$. Applying Lemma 2.5 to the polynomial $G(z)$, yields

$$|zD_\alpha G(z)|\geq\frac{(n+s)(|\alpha|-1)}{2}|G(z)|,\quad |z|=1. \tag{3.1}$$

Since $zD_\alpha G(z)$ has all its zeros in $|z|<1$, by using Rouche's Theorem, it can be easily verifies from (3.1), that the polynomial

$$zD_\alpha G(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}G(z),$$

has all its zeros in $|z|<1$, where $|\beta|<1$.

Substituting for $G(z)$, we conclude that the polynomial

$$T(z)=\left(zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right)-\lambda m z^n\left(n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right), \tag{3.2}$$

will have no zeros in $|z|\geq 1$. This implies for every β with $|\beta|<1$ and $|z|\geq 1$,

$$\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|\geq m|z^n|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|. \tag{3.3}$$

If the inequality (3.3) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left|z_0 D_\alpha p(z_0) + \beta \frac{(n+s)(|\alpha|-1)}{2} p(z_0)\right| < m|z_0^n| \left|n\alpha + \beta \frac{(n+s)(|\alpha|-1)}{2}\right|.$$

Take

$$\lambda = \frac{z_0 D_\alpha p(z_0) + \beta \frac{(n+s)(|\alpha|-1)}{2} p(z_0)}{m z_0^n \left(n\alpha + \beta \frac{(n+s)(|\alpha|-1)}{2}\right)},$$

then $|\lambda| < 1$ and with this choice of λ, we have $T(z_0) = 0$ for $|z_0| \geq 1$, from (3.2). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For β with $|\beta| = 1$, inequality (3.3) follows by continuity. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Under the assumption of Theorem 1.2, we can write $p(z) = z^s h(z)$, where the polynomial $h(z) \neq 0$ in $|z| < 1$, and thus if $m = \min_{|z|=1} |h(z)| = \min_{|z|=1} |p(z)|$, then $m \leq |h(z)|$ for $|z| \leq 1$. Now for λ with $|\lambda| < 1$, we have

$$|\lambda m| < m \leq |h(z)|,$$

where $|z| = 1$.

It follows by Rouche's Theorem that the polynomial $h(z) - \lambda m$ has no zero in $|z| < 1$. Hence the polynomial $G(z) = z^s(h(z) - \lambda m) = p(z) - \lambda m z^s$, has no zero in $|z| < 1$ except s-fold zeros at the origin. Therefore the polynomial

$$H(z) = z^{n+s}\overline{G(1/\overline{z})} = Q(z) - \overline{\lambda} m z^n,$$

will have all its zeros in $|z| \leq 1$ with s-fold zeros at the origin, where $Q(z) = z^{n+s}\overline{p(1/\overline{z})}$. Also $|G(z)| = |H(z)|$ for $|z| = 1$.

Now, using a similar argument as used in the proof of Lemma 2.7 (inequality (2.4)), for the polynomials $H(z)$ and $G(z)$, we have

$$\left|z D_\alpha G(z) + \beta \frac{(n+s)(|\alpha|-1)}{2} G(z)\right| \leq \left|z D_\alpha H(z) + \beta \frac{(n+s)(|\alpha|-1)}{2} H(z)\right|,$$

where $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$. Substituting for $G(z)$ and $H(z)$ in the above inequality, we conclude that for every α, β, with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}&\left|z D_\alpha p(z) - \lambda((n-s)z + s\alpha) m z^s + \beta \frac{(n+s)(|\alpha|-1)}{2}(p(z) - \lambda m z^s)\right| \\ &\leq \left|z D_\alpha Q(z) - \overline{\lambda}\alpha n m z^n + \beta \frac{(n+s)(|\alpha|-1)}{2}(Q(z) - \overline{\lambda} m z^n)\right|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\left|z D_\alpha p(z) + \beta \frac{(n+s)(|\alpha|-1)}{2} p(z) - \lambda\left((n-s)z + s\alpha + \beta \frac{(n+s)(|\alpha|-1)}{2}\right) m z^s\right| \\ &\leq \left|z D_\alpha Q(z) + \beta \frac{(n+s)(|\alpha|-1)}{2} Q(z) - \overline{\lambda}\left(n\alpha + \beta \frac{(n+s)(|\alpha|-1)}{2}\right) m z^n\right|. \qquad (3.4)\end{aligned}$$

Since all the zeros of $Q(z)$ lie in $|z|\le 1$ with s-fold zeros at the origin, and $|p(z)| = |Q(z)|$ for $|z|=1$, therefore by applying Theorem 1.1 to $Q(z)$, we have

$$\begin{aligned}\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right| &\ge \left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\min_{|z|=1}|Q(z)|\\ &=\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|m.\end{aligned}$$

Then for an appropriate choice of the argument of λ, we have

$$\begin{aligned}&\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)-\overline{\lambda}\left(n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right)mz^n\right|\\ &=\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|-|\lambda|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|m, \qquad (3.5)\end{aligned}$$

where $|z|=1$.

Then combining the right hand sides of (3.4) and (3.5), we can rewrite (3.4) as

$$\begin{aligned}&\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|-|\lambda|\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|m\\ &\le\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|-|\lambda|\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|m, \qquad (3.6)\end{aligned}$$

where $|z|=1$.

Equivalently

$$\begin{aligned}&\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|\\ &\le\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|-|\lambda|\left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right.\\ &\quad\left.-\left|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\}m.\end{aligned}$$

As $|\lambda|\to 1$ we have

$$\begin{aligned}&\left|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right|\\ &\le\left|zD_\alpha Q(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right|-\left\{\left|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right.\\ &\quad\left.-\left|(n-s)z+s\alpha+\beta\frac{|\alpha|-1}{2}\right|\right\}m.\end{aligned}$$

It implies for every real or complex number β with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} 2\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \\ \leq \left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| + \left|zD_\alpha Q(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}Q(z)\right| \\ - \left\{\left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| - \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\} m. \end{aligned}$$

This in conjunction with Lemma 2.7 gives for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} 2\left|zD_\alpha p(z) + \beta\frac{(n+s)(|\alpha|-1)}{2}p(z)\right| \\ \leq \left\{\left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| + \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\} \max_{|z|=1}|p(z)| \\ - \left\{\left|n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right| - \left|(n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2}\right|\right\} \min_{|z|=1}|p(z)|. \end{aligned}$$

The proof is complete. □

Acknowledgements. The authors wish to thank the referee, for the careful reading of the paper and for the helpful suggestions and comments.

References

[1] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory **54** (1998), 306–313.

[2] A. Aziz and W. M. Shah, *Some inequalities for the polar derivative of a polynomial*, Proc. Indian Acad. Sci. **107** (1997), 263–270.

[3] A. Aziz and W. M. Shah, *Inequalities for a polynomial and its derivative*, Math. Ineq. Appl. **7** (2004), 379–391.

[4] S. Bernstein, *Sur la limitation des derivees des polnomes*, C. R. Acad. Sci. Paris. **190** (1930), 338–341.

[5] M. Bidkham, M. Shakeri and M. E. Gordji, *Inequalities for the polar derivative of a polynomial*, J. Ineq. Appl. **1155** (2009), 1–9.

[6] T. N. Chan and M. A. Malik, *On Erdös-Lax theorem*, Proc. Indian. Acad. Sci. **92** (1983), 191–193.

[7] K. K. Dewan and S. Hans, *Generalization of certain well-known polynomial inequalities*, J. Math. Anal. Appl. **363** (2010), 38–41.

[8] K. K. Dewan and A. Mir, *Inequalities for the polar derivative of a polynomial*, J. Interdisciplinary Math. **10** (2007), 525–531.

[9] K. K. Dewan, N. Singh and A. Mir, *Extension of some polynomial inequalities to the polar derivative*, J. Math. Anal. Appl. **352** (2009), 807–815.

[10] P. D. Lax, *Proof of a conjecture of P. Erdos on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[11] A. Liman, R. N. Mohapatra and W. M. Shah, *Inequalities for the polar derivative of a polynomial*, Complex Anal. Oper. Theory. **6** (2012), 1199–1209.
[12] M. Marden, *Geometry of Polynomials: Mathematical Surveys*, American Mathematical Society, Providence, USA, 1966.
[13] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Oxford University Press, New York, 2002.
[14] A. Zireh, *On the maximum modulus of a polynomial and its polar derivative*, J. Inequal. Appl. **111** (2011), 1–9.

[1]Department of Mathematics,
Shahrood University of Technology, Shahrood, Iran.
E-mail address: `azireh@shahroodut.ac.ir`

[2]Department of Mathematics,
University of Semnan, Semnan, Iran
E-mail address: `mdbidkham@gmail.com`