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SERIES EXPANSION OF A COTANGENT SUM RELATED TO THE ESTERMANN ZETA FUNCTION

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Abstract. In this paper, we study the cotangent sum $c_0\left(\frac{q}{p}\right)$ related to the Estermann zeta function for the special case when the numerator is equal to 1 and get two useful series expansions of $c_0\left(\frac{1}{p}\right)$.

1. Introduction

For a positive integer p and $q = 1, 2, \ldots, p-1$, such that $(p, q) = 1$, let the cotangent sum (see [10])

$$c_0\left(\frac{q}{p}\right) = -\sum_{k=1}^{p-1} \frac{k}{p} \cot \frac{\pi k q}{p}.$$

$c_0\left(\frac{q}{p}\right)$ is the value at $s = 0$,

$$E_0\left(0, \frac{q}{p}\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{q}{p}\right)$$

of the Estermann zeta function

$$E_0\left(s, \frac{q}{p}\right) = \sum_{k\geq 1} \frac{d(k)}{k^s} \exp\left(\frac{2\pi i k q}{p}\right).$$

It is well-known that the sum $c_0\left(\frac{q}{p}\right)$ satisfies the reciprocity formula (see [2])

$$c_0\left(\frac{q}{p}\right) + \frac{p}{q} c_0\left(\frac{p}{q}\right) - \frac{1}{\pi q} = \frac{i}{2}\psi_0\left(\frac{q}{p}\right).$$

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The Vasyunin cotangent sum (see [11])

$$V\left(\frac{q}{p}\right)=\sum_{r=1}^{p-1}\left\{\frac{rq}{p}\right\}\cot\left(\frac{\pi r}{p}\right)=-c_0\left(\frac{\overline{q}}{p}\right)$$

arises in the study of the Riemann zeta function by virtue of the formula (see [2, 9])

$$\begin{aligned}&\frac{1}{2\pi\sqrt{pq}}\int_{-\infty}^{+\infty}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left(\frac{q}{p}\right)^{it}\frac{dt}{\frac{1}{4}+t^2}\\ =&\frac{\log 2\pi-\gamma}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\frac{p-q}{2pq}\log\frac{q}{p}-\frac{\pi}{2pq}\left(V\left(\frac{p}{q}\right)+V\left(\frac{q}{p}\right)\right).\end{aligned}$$

This formula is connected to the approach of Nyman, Beurling and Báez-Duarte to the Riemann hypothesis (see [8]), which states that the Riemann hypothesis is true if and only if $\lim\limits_{n\to\infty}d_N=0$, where

$$d_N^2=\inf_{A_N}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|1-\zeta A\left(\frac{1}{2}+it\right)\right|^2\frac{dt}{\frac{1}{4}+t^2},$$

and the infimum is taken over all Dirichlet polynomials

$$A_N(s)=\sum_{n=1}^{N}\frac{a_n}{n^s}.$$

In a recent work with A. Bayad [7], we have proved that the sum $V\left(\frac{q}{p}\right)$ satisfies the reciprocity formula

$$V\left(\frac{q}{p}\right)+V\left(\frac{p}{q}\right)=\frac{1}{\pi}\left(G(p,p)+G(q,q)+G(p,q)+(q-p)\log\frac{q}{p}\right),\tag{1.1}$$

where

$$G(p,q)=\sum_{k\geq 1}\frac{pq}{k(k+1)}\left\{\frac{k}{p}\right\}\left\{\frac{k}{q}\right\}.$$

Thereafter the restriction of the relationship (1.1) to $q=1$ gives

$$c_0\left(\frac{1}{p}\right)=-\frac{1}{\pi}G(p,p)-(p-1)\log p.$$

Exactly our interest in this work is the case $q=1$ in order to get two series expansions of $c_0\left(\frac{1}{p}\right)$. First we recall the different asymptotical writings of $c_0\left(\frac{1}{p}\right)$ in the literature. In [10, Theorem 1.2, Theorrem 1.3] M. Th. Rassias proved that

$$c_0\left(\frac{1}{p}\right)=\frac{1}{\pi}p\log p-\frac{p}{\pi}(\log 2\pi-\gamma)+\left\{\mathcal{O}(\log p)\ \text{ or }\ \mathcal{O}(1)\right\}.$$

In [9, Theorem 1.7] H. Maier and M. Th. Rassias provide the following improvement. Let $b,n\in\mathbb{N}$, $b\geq 6N$, with $N=\left\lfloor\frac{n}{2}\right\rfloor+1$. There exist absolute real constants $A_1,A_2\geq 1$

and absolute real constants E_l, l, with $|E_l| \leq (A_1 l)^{2l}$, such that for each $n \in \mathbb{N}$ we have

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi}(\log 2\pi - \gamma) - \frac{1}{\pi} + \sum_{l=1}^{n} E_l p^{-l} + R_n^{\star}(p),$$

where $|R_n^{\star}(p)| \leq (A_2 n)^{4n} p^{-(n+1)}$.

Only in [9, Theorem 1.9] H. Maier and M. Th. Rassias provide another improvement,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi}(\log 2\pi - \gamma) + C_1 p + \mathcal{O}(1).$$

We draw attention that S. Bettin finds other reformulations of $c_0\left(\frac{1}{p}\right)$ inspired from continued fraction theory (see [3]).

Finally from another point of view we show in [5] with A. Bayad and M. O. Hernane that

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\left(\log\frac{2\pi}{p} - \gamma\right)p + \frac{1}{\pi} + \frac{\pi}{36p}$$
$$- \frac{1}{2}\sum_{k=2}^{\left\lfloor \frac{N}{2} \right\rfloor} (-1)^k \frac{4^k \pi^{2k-1} B_{2k}^2}{k(2k)!}\left(\frac{1}{p}\right)^{2k-1} + \mathcal{O}\left(\frac{1}{p^N}\right).$$

There is a misprint in the formula (1.22) Corollary 1.2 in [5] the correct one is in the formula (1.21) Corollary 1.2.

Otherwise in the same paper [5], an integral representation of $c_0(\frac{1}{p})$ is given by

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}\int_0^1 \frac{(p-2)x^p - px^{p-1} + px - p + 2}{(x-1)^2(x^p-1)}dx. \tag{1.2}$$

In this work we prove that

$$(p-2)x^p - px^{p-1} + px - p + 2 = (x-1)^3 \sum_{r=1}^{p-1}(p-r-1)rx^{r-1}$$

and we get another formulation that is

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}\int_0^1 \frac{\sum_{r=1}^{p-1}(p-r-1)rx^{r-1}}{1 + x + \cdots + x^{p-1}}dx.$$

Applying some techniques from the generating function theory [4] to previous integrals; we find two series expansions of $c_0\left(\frac{1}{p}\right)$, as they are well explained in the next section.

2. Series Expansion of $c_0\left(\frac{1}{p}\right)$

Let b_k be the integer sequence defined by $b_0 = 1, b_1 = 2$ and the recursive formulae:

$$b_k - 2b_{k-1} + b_{k-2} = 0, \quad 2 \leq k \leq p-1,\ k = p+1,$$
$$b_p - 2b_{p-1} + b_{p-2} = 1$$

and

$$b_k - 2b_{k-1} + b_{k-2} - b_{k-p} + 2b_{k-p-1} - b_{k-p-2} = 0, \quad k \geq p+2.$$

According to the terms b_k we get the first series expansion in the following theorem.

Theorem 2.1.

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}p\left(p-1\right)\left(p-2\right)\sum_{k\geq 0}\frac{b_k}{\left(k+1\right)\left(k+p+1\right)\left(k+2\right)\left(k+p\right)}. \tag{2.1}$$

For $p \geq 1$ we define the arithmetic function a_p in the form

$$a_p(k) = \begin{cases} 1, & \text{if } p \mid k, \\ -1, & \text{if } k \equiv 1 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases}$$

This function is not multiplicative. In general the arithmetical functions are defined from the set of natural integers $\mathbb{N}$ into $\mathbb{C}$. We can extend this definition to $\mathcal{F}\left(\mathbb{C},\mathbb{C}\right)$; set of functions from $\mathbb{C}$ to $\mathbb{C}$. In that case the corresponding function is $A : \mathbb{N} \to \mathcal{F}\left(\mathbb{C},\mathbb{C}\right)$ with $A(p) = a_p$. Furthermore, $A\left(pq\right) = \pm A(p)A(q)$ and $|A|$ is multiplicative.

Let the function $M\left(p,k\right)$ defined by

$$M\left(p,0\right) = \frac{1}{2}p^2 - \frac{3}{2}p + 1$$

and

$$M\left(p,k\right) = \left(p-1\right)\left(\frac{1}{2}p + k - 1\right) - k\left(p+k-1\right)\left(H_{p+k-1} - H_k\right), \quad k \geq 1,$$

where H_k is the Harmonic number

$$H_k = \sum_{j=1}^{k}\frac{1}{j}.$$

Following this function a second series expansion of $c_0\left(\frac{1}{p}\right)$ is given in the following theorem.

Theorem 2.2.

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}\sum_{k\geq 0}a_p(k)M\left(p,k\right). \tag{2.2}$$

2.1. **Proof of Theorem** 2.1. We take inspiration from the theory of generating functions [4,6], and prove that the sequence (b_k) is generated by the rational function:

$$f\left(x\right) = \frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}}.$$

More precisely we get the following lemma.

Lemma 2.1.

$$\frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k\geq 0}b_kx^k, \quad |x| < 1. \tag{2.3}$$

Proof. It is well known that

$$\frac{1}{1-x}=\sum_{k\geq 0}x^k,\quad |x|<1. \tag{2.4}$$

Since for $0\leq x<1$

$$0<(x-1)^2(1-x^p)<1$$

and

$$(x-1)^2(1-x^p)=1-\left(2x-x^2+x^p-2x^{p+1}+x^{p+2}\right),$$

then we have

$$0<2x-x^2+x^p-2x^{p+1}+x^{p+2}<1.$$

Furthermore, $f(x)$ is developable on entire series to get the result we have to take the quantity $2x-x^2+x^p-2x^{p+1}+x^{p+2}$ instead of x in the last formula (2.4). Now, writing

$$\frac{1}{1-2x+x^2-x^p+2x^{p+1}-x^{p+2}}=\sum_{k\geq 0}d_kx^k$$

and then

$$\left(1-2x+x^2-x^p+2x^{p+1}-x^{p+2}\right)\left(\sum_{k\geq 0}d_kx^k\right)=1.$$

To compute this we use the well known Cauchy product of two entire series

$$\left(\sum_{k\geq 0}a_kx^k\right)\left(\sum_{j\geq 0}d_jx^j\right)=\sum_{k\geq 0}\left(\sum_{j=0}^{k}a_jd_{k-j}\right)x^k,$$

which generates the product of a polynomial of degree n with an entire series that also gives an entire series as follows

$$\left(\sum_{k=0}^{n}a_kx^k\right)\left(\sum_{j\geq 0}d_jx^j\right)=\sum_{k\geq 0}\left(\sum_{j=0}^{\min\{n,k\}}a_jd_{k-j}\right)x^k.$$

We return to $f(x)$ in writing

$$1-2x+x^2-x^p+2x^{p+1}-x^{p+2}=\sum_{k=0}^{p+2}a_kx^k,$$

with $a_0=1,\ a_1=-2,\ a_2=1,\ a_p=-1,\ a_{p+1}=2,\ a_{p+2}=-1$, and the others are zero. We conclude that $d_0=1,\ d_1=2$. The formula

$$\sum_{j=0}^{\min\{p+2,k\}}a_jd_{k-j}=0$$

states that

$$d_k-2d_{k-1}+d_{k-2}=0,\quad 2\leq k\leq p-1,\ k=p+1,$$
$$d_p-2d_{p-1}+d_{p-2}=1$$

and

$$d_k-2d_{k-1}+d_{k-2}-d_{k-p}+2d_{k-p-1}-d_{k-p-2}=0,\quad k\geq p+2.$$

Finally, we see that d_k and b_k are identical for every integer $k \geq 0$. For more information on this approach we refer to [6]. $\square$

To get the result (2.1) of Theorem 2.1 we must substitute the expression (2.3) in the identity (1.2) and one obtains

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\sum_{k\geq 0} b_k \int_0^1 \left((p-2)\,x^{k+p} - px^{k+p-1} + px^{k+1} + (2-p)\,x^k\right) dx.$$

Furthermore,

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\sum_{k\geq 0} b_k \left(\frac{p-2}{k+p+1} - \frac{p}{k+p} + \frac{p}{k+2} - \frac{p-2}{k+1}\right).$$

Finally,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}p\,(p-1)\,(p-2)\sum_{k\geq 0}\frac{b_k}{(k+1)\,(k+p+1)\,(k+2)\,(k+p)}$$

and $c_0\,(1) = c_0\left(\frac{1}{2}\right) = 0$ is compatible with the definition of c_0.

Regarding the identity (2.3) Lemma 2.1 we remark that

$$\frac{1}{(1-x)^2\,(1-x^p)} = \sum_{k\geq 0} b_k x^k, \quad |x| < 1.$$

Furthermore, for $x = \frac{1}{2}$ we deduce that the coefficients b_k satisfy the following statements

$$\sum_{k\geq 0}\frac{b_k}{2^k} = \frac{2^{p+2}}{2^p - 1} \quad \text{and} \quad \lim_{k\to\infty}\frac{b_k}{2^k} = 0.$$

2.2. **Proof of Theorem** 2.2. First we began by proving another integral representation of $c_0\left(\frac{1}{p}\right)$.

Lemma 2.2.

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}\int_0^1 \frac{\sum_{r=1}^{p-1}(p-r-1)\,rx^{r-1}}{1+x+\cdots+x^{p-1}}dx. \tag{2.5}$$

Proof.

$$\begin{aligned}(x-1)^3\sum_{r=1}^{q-1}(q-r-1)\,rx^{r-1} &= \sum_{r=3}^{q}(q-r+1)\,(r-2)\,x^r - 3\sum_{r=2}^{q-1}(q-r)\,(r-1)\,x^r \\ &\quad + 3\sum_{r=1}^{q-2}(q-r-1)\,rx^r - \sum_{r=0}^{q-3}(q-r-2)\,(r+1)\,x^r.\end{aligned}$$

It's obvious to remark that

$$(q-r+1)\,(r-2) - 3\,(q-r)\,(r-1) + 3\,(q-r-1)\,r - (q-r-2)\,(r+1) = 0$$

and the quantity

$$(t-1)^3 \sum_{r=1}^{q-1} (q-r-1)\, r x^{r-1}$$

is reduced to

$$(q-2)\, x^q + 2\,(q-3)\, x^{q-1} + 3(q-4)x^{q-2} - 3(q-2)x^{q-1} - 6(q-3)x^{q-2} - 3(q-2)x^2$$
$$+\, 3(q-2)x^{q-2} + 3(q-2)x + 6(q-3)x^2 - q + 2 - 2(q-3)x - 3(q-4)x^2.$$

After simplification we obtain

$$(t-1)^3 \sum_{r=1}^{q-1} (q-r-1)\, r x^{r-1} = (q-2)\, x^q - q x^{q-1} + qx - q + 2. \qquad \square$$

The Theorem 2.2 is immediate from the Lemma 2.2 in the following way. Since

$$\frac{1}{1 + x + \cdots + x^{p-1}} = \frac{1-x}{1-x^p}$$

and $|x| < 1$, then

$$\frac{1}{1 + x + \cdots + x^{p-1}} = \frac{1-x}{1-x^p} = \sum_{k \geq 0} (1-x)\, x^{pk}.$$

Furthermore,

$$\frac{1}{1 + x + \cdots + x^{p-1}} = \sum_{k \geq 0} a_p\,(k)\, x^k$$

and we have

$$\frac{\sum_{r=1}^{p-1} (p-r-1)\, r x^{r-1}}{1 + x + \cdots + x^{p-1}} = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p\,(k)\,(p-r-1)\, r x^{k+r-1}.$$

The passage to the integral inducts

$$c_0 \left(\frac{1}{p}\right) = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p\,(k)\, \frac{(p-r-1)\, r}{k+r}.$$

But

$$\sum_{r=1}^{p-1} \frac{(p-r-1)\, r}{k+r} = (p-1) \left(\frac{1}{2} p + k - 1\right) - k\,(p+k-1) \sum_{r=k+1}^{p+k-1} \frac{1}{r}$$

and the result (2.2) is deduced.

3. Connection to Digamma Function

We finish this work by revisiting the proof of the expression of $c_0\left(\frac{1}{p}\right)$ according to the function digamma and Bernoulli polynomials in the work [1] of L. Báez Duarte et al.

$$c_0\left(\frac{1}{p}\right)=\frac{2}{\pi}\sum_{r=1}^{p-1}B_1\left(\frac{r}{p}\right)\psi\left(\frac{r}{p}\right),$$

where B_1 is the reduced Bernoulli polynomial

$$B_1(x)=\begin{cases}0, & \text{if } x\in\mathbb{Z},\\ \{x\}-\frac{1}{2}, & \text{otherwise,}\end{cases}$$

and ψ the digamma function defined by

$$\psi\left(z\right)=-\gamma-\frac{1}{z}+\sum_{k\geq 1}\left(\frac{1}{k}-\frac{1}{k+z}\right).$$

Starting with the demonstration of a property of ψ that will be used later.

Proposition 3.1.

$$\psi\left(\frac{r+1}{p}\right)-\psi\left(\frac{r}{p}\right)=p\int_0^1\frac{x^{r-1}}{1+x+\cdots+x^{p-1}}dx. \tag{3.1}$$

Proof. We quote from [5] the formula

$$\psi\left(\frac{r+1}{p}\right)-\psi\left(\frac{r}{p}\right)=p\sum_{k\geq 0}\frac{1}{\left(pk+r+1\right)\left(pk+r\right)}.$$

The general term $\frac{1}{(pk+r+1)(pk+r)}$ can be written as following

$$\frac{1}{\left(pk+r+1\right)\left(pk+r\right)}=\frac{1}{pk+r}-\frac{1}{pk+r+1}=\int_0^1\left(x^{pk+r-1}-x^{pk+r}\right)dx$$

and the passage to the sum states that

$$\sum_{k\geq 0}\frac{1}{\left(pk+r+1\right)\left(pk+r\right)}=\int_0^1\frac{x^{r-1}-x^r}{1-x^p}dx.$$

Finally,

$$\sum_{k\geq 0}\frac{1}{\left(pk+r+1\right)\left(pk+r\right)}=\int_0^1\frac{x^{r-1}}{1+x+\cdots+x^{p-1}}dx$$

and we have (3.1). Proposition 3.1 follows. □

In [5], it is shown that

$$\log p=\frac{1}{p}\sum_{r=1}^{p-1}r\left(\psi\left(\frac{r+1}{p}\right)-\psi\left(\frac{r}{p}\right)\right).$$

This identity conducts to the following interesting lemma.

Lemma 3.1.

$$\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) = -\gamma p - p\log p. \tag{3.2}$$

Proof. Since

$$\sum_{r=1}^{p-1} r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right) = p\log p,$$

then

$$-\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) + \psi(1)\, p = p\log p.$$

Furthermore,

$$\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) = -\gamma p - p\log p. \qquad \square$$

According to the identity (3.1) Proposition 3.1 and the integral representation (2.5) we conclude that

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p}\sum_{r=1}^{p-1} (p-r-1)\, r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right).$$

Furthermore combining this result with the identity (3.2) Lemma 3.1 we get

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\log p + \frac{1}{\pi p}\sum_{r=1}^{p-1} (p-r)\, r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right)$$

and

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\log p - \gamma\frac{p-1}{\pi p} + \frac{1}{\pi p}\sum_{r=1}^{p-1} (2r-p-1)\, \psi\left(\frac{r}{p}\right),$$

then

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p}\sum_{r=1}^{p-1} (2r-p)\, \psi\left(\frac{r}{p}\right).$$

But

$$2r - p = 2p\left(\frac{r}{p} - \frac{1}{2}\right) = 2pB_1\left(\frac{r}{p}\right),$$

which means that

$$c_0\left(\frac{1}{p}\right) = \frac{2}{\pi}\sum_{r=1}^{p} B_1\left(\frac{r}{p}\right)\psi\left(\frac{r}{p}\right).$$

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