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RIGHT AND LEFT MAPPINGS IN EQUALITY ALGEBRAS

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Abstract. The notion of (right) left mapping on equality algebras is introduced, and related properties are investigated. In order for the kernel of (right) left mapping to be filter, we investigate what conditions are required. Relations between left mapping and $\rightarrow$-endomorphism are investigated. Using left mapping and $\rightarrow$-endomorphism, a characterization of positive implicative equality algebra is established. By using the notion of left mapping, we define $\rightarrow$-endomorphism and prove that the set of all $\rightarrow$-endomorphisms on equality algebra is a commutative semigroup with zero element. Also, we show that the set of all right mappings on positive implicative equality algebra makes a dual BCK-algebra.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. A crucial question for every many-valued logic is, what should be structure of its truth values. It is generally accepted that in fuzzy logic, it should be a residuated lattice, possibly fulfilling some additional properties. On the basis of that, we may now distinguish various kinds of formal fuzzy logics. Most important among them seem to be BL-logics, MTL-logics and IMTL-logics. The answer to the above question is positive and the fuzzy type theory (FTT) has indeed been introduced in [12]. However, the basic connective in FTT is a fuzzy equality since it is developed as a generalization of the elegant classical formal system originated by Henkin (see [5]). So Novák in [13] introduced a special

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algebra so called EQ-algebra and that reflects directly the syntax of FTT. Viewing the axioms of EQ-algebras with a purely algebraic eye it appears that unlike in the case of residuated lattices where the adjointness condition ties product with implication, the product in EQ-algebras is quite loosely related to the other connectives. For instance, a moment's reflection shows that one can replace the product of an EQ-algebra by any other binary operation which is smaller or equal than the original product (viewed as a two-place function) and still obtains an EQ-algebra. However, the huge freedom in choosing the product might prohibit to find deep related algebraic results, hence our aim was to find something similar to EQ-algebras but without a product: an axiomatic treatment of equality/equivalence. Because of that Jenei in [9] introduced a new structure, called equality algebras. It has two connectives, a meet operation and an equivalence, and a constant 1.

Left and right mappings are very important concepts and mathematicians have used them in various mathematical fields. For example, Kondo [11] introduced the notion of left mapping on BCK-algebras and investigated some properties of it. He showed that in a positive implicative BCK-algebra, if a left map is surjective, then it is also an injective one. Borzooei and Aaly [2], introduced left and right stabilizers by using a fixed point sets of right and left mappings. They investigated that under which conditions these sets can be equal. Also, by using the (right) left stabilizers, produced a basis for a topology on hoops and showed that the generated topology by this basis is Baire, connected, locally connected and separable. Moreover, Hail, Abu baker and Mohd [4], by using the notion of (right) left mapping defined different kinds of derivation on BCK/BCI-algebras. The notion of derivation and extended of that are introduced on different kinds of logical algebras such as UP-algebras, MV-algebras and etc. In UP-algebras, Iampan in [6] proved that the fixed point set and the kernel of left derivation are UP-subalgebras and investigated under which condition they can be an ideal or filter. Kamali in [10], extended the notion of derivation on MV-algebras by using left and right mappings and investigate some properties of them.

Now, in this paper, we introduce the concept of (right) left mapping on equality algebras and investigate several properties. Then by using of left and right mapping on equality algebras, we construct a commutative monoid, a commutative semigroup with zero element and a dual BCK-algebra.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1 ([8])**.** By an *equality algebra*, we mean an algebra $(X, \wedge, \sim, 1)$ satisfying the following conditions:

(E1) $(X, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., meet semilattice with the top element 1);

(E2) The operation "$\sim$" is commutative;

(E3) $(\forall a \in X)(a \sim a = 1)$;
(E4) $(\forall a \in X)(a \sim 1 = a)$;
(E5) $(\forall a, b, c \in X)(a \leq b \leq c \Rightarrow a \sim c \leq b \sim c, a \sim c \leq a \sim b)$;
(E6) $(\forall a, b, c \in X)(a \sim b \leq (a \wedge c) \sim (b \wedge c))$;
(E7) $(\forall a, b, c \in X)(a \sim b \leq (a \sim c) \sim (b \sim c))$,

where $a \leq b$ if and only if $a \wedge b = a$. The equality algebra $(X, \wedge, \sim, 1)$ is simply denoted by X only.

In an equality algebra $(X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$a \rightarrow b := a \sim (a \wedge b),$$
$$a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a).$$

Proposition 2.1 ([8])**.** *Let $(X, \wedge, \sim, 1)$ be an equality algebra. Then for all $a, b, c \in X$, the following assertions are valid:*

$$a \rightarrow b = 1 \Leftrightarrow a \leq b,$$
$$a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c), \tag{2.1}$$
$$1 \rightarrow a = a, \quad a \rightarrow 1 = 1, \quad a \rightarrow a = 1, \tag{2.2}$$
$$a \leq b \rightarrow c \Leftrightarrow b \leq a \rightarrow c,$$
$$a \leq b \rightarrow a, \tag{2.3}$$
$$a \leq (a \rightarrow b) \rightarrow b, \tag{2.4}$$
$$a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c), \tag{2.5}$$
$$b \leq a \Rightarrow a \leftrightarrow b = a \rightarrow b = a \sim b,$$
$$a \sim b \leq a \leftrightarrow b \leq a \rightarrow b,$$
$$a \leq b \ \Rightarrow \begin{cases} b \rightarrow c \leq a \rightarrow c, \\ c \rightarrow a \leq c \rightarrow b, \end{cases} \tag{2.6}$$
$$((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b. \tag{2.7}$$

An equality algebra X is said to be *bounded* if there exists an element $0 \in X$ such that $0 \leq a$ for all $a \in X$. In a bounded equality algebra X, we define the negation "$\neg$" on X by $\neg a = a \rightarrow 0 = a \sim 0$ for all $a \in X$.

A subset A of X is called a *deductive system* (or *filter*) of X (see [9]) if it satisfies

$$\begin{aligned} &1 \in A, \\ &(\forall a, b \in X)(a \in A, a \leq b \ \Rightarrow \ b \in A), \\ &(\forall a, b \in X)(a \in A, a \sim b \in A \ \Rightarrow \ b \in A). \end{aligned} \tag{2.8}$$

Denote by $\mathcal{DS}(X)$ the set of all deductive systems of X.

Lemma 2.1 ([7]). *Let X be an equality algebra. A subset A of X is a deductive system of X if and only if it satisfies* (2.8) *and*

$$(\forall a, b \in X)(a \in A, a \to b \in A \Rightarrow b \in A).$$

Definition 2.2 ([14]). An equality algebra X is said to be *commutative* if it satisfies:

$$(\forall x, y \in X)((x \to y) \to y = (y \to x) \to x).$$

Definition 2.3 ([1]). Given an equality algebra $(X, \wedge, \sim, 1)$ and $a, b \in X$, we define

$$X(a, b) := \{x \in X \mid a \leq b \to x\}.$$

It is clear that 1, a and b are contained in $X(a, b)$.

Definition 2.4 ([1]). An equality algebra $(X, \wedge, \sim, 1)$ is called an *&-equality algebra* if for all $a, b \in X$, the set $X(a, b)$ has the least element which is denoted by $a \odot b$.

Proposition 2.2 ([1]). *If $X = (X, \wedge, \sim, 1)$ is an &-equality algebra, then*

$$(\forall a, b \in X)(a \odot b = b \odot a),$$
$$(\forall a, b, c \in X)((a \odot b) \odot c = a \odot (b \odot c)),$$
$$(\forall a, b, c \in X)(a \leq b \Rightarrow a \odot c \leq b \odot c).$$

Lemma 2.2 ([1]). *Let $X = (X, \wedge, \sim, 1)$ be an equality algebra in which there exists a binary operation "$\odot$" such that*

$$(\forall a, b, c \in X)(a \to (b \to c) = (a \odot b) \to c).$$

Then $\mathcal{X} = (X, \wedge, \sim, 1)$ is an &-equality algebra.

3. Left Mappings

In this section, we define the notion of left mapping on equality algebra and investigate some properties of it. Moreover, we define the notions of $\to$-homomorphism, positive implicative and &-equality algebras and study the relation among them.

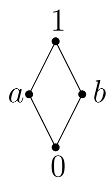
Definition 3.1. Given a fixed element a in an equality algebra X, we define a self-mapping f_a of X by

$$f_a : X \to X, \quad x \mapsto a \to x,$$

and we say that f_a is a *left mapping* on X.

Let $\mathcal{L}(X)$ denote the set of all left mappings on an equality algebra X.

Example 3.1. Let $X = \{0, a, b, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 1. Then $(X, \wedge, \sim, 1)$ is an equality algebra, and the

TABLE 1. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

implication "$\rightarrow$" is given by Table 2.

TABLE 2. Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Let f_a and f_b be self mappings of X defined by

$$f_a(0) = f_a(b) = b, \quad f_a(a) = f_a(1) = 1$$

and

$$f_b(0) = f_b(a) = a, \quad f_b(b) = f_b(1) = 1,$$

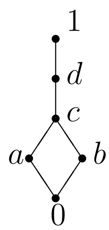
respectively. It is routine to verify that f_a and f_b are left mappings on X.

Remark 3.1. It is clear that $f_0(x) = 1$ and $f_1(x) = x$ for all x in a bounded equality algebra X.

Question 1. If f_a is a left mapping on X, then is f_a^2 a left mapping on X?

The following example shows that the answer to the above question is false.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 3. Then $\mathcal{E} = (X, \wedge, \sim, 1)$ is an equality algebra, and

TABLE 3. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

TABLE 4. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

the implication "$\to$" is given by Table 4. Define a mapping $f_a : X \to X$ by $f_a(0) = f_a(b) = d$ and $f_a(a) = f_a(c) = f_a(d) = f_a(1) = 1$. Then f_a is a left mapping on X, but f_a^2 is not a left mapping on X since

$$d = a \to 0 = f_a(0) \neq f_a^2(0) = a \to (a \to 0) = a \to d = 1.$$

Definition 3.2. An equality algebra X is said to be *positive implicative* if it satisfies

$$(\forall x, y, z \in X)(x \to (y \to z) = (x \to y) \to (x \to z)). \tag{3.1}$$

Theorem 3.1. *In a positive implicative equality algebra X, if f_a is a left mapping on X, then so is f_a^2.*

Proof. For any $x \in X$, we have

$$f_a^2(x) = f_a(f_a(x)) = a \to (a \to x) = (a \to a) \to (a \to x) = 1 \to (a \to x) = a \to x.$$

Therefore, f_a^2 is a left mapping on X. □

Corollary 3.1. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then so is f_a^n for every $n \in \mathbb{N}$.*

Proof. It is by mathematical induction. □

Proposition 3.1. *Let X be an equality algebra and f_a be a left mapping on X. Then the following statements hold:*

(1) $f_a(x) \to f_a(y) \le f_a(x \to y)$ *and the equality is true when* X *is positive implicative;*

(2) *the left mapping* f_a *on* X *is isotone, that is, if* $x \le y$*, then* $f_a(x) \le f_a(y)$*;*

(3) $x \le f_a(x) \le f_a^2(x) \le \cdots$ *and the equality is true when* $a = 1$*;*

(4) $x \to y \le f_a^n(x) \to f_a^n(y)$ *for any* $n \in \mathbb{N}$ *and the equality is true when* $a = 1$*;*

(5) $\mathrm{Im}(f_a^n) \subseteq \cdots \subseteq \mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a)$*;*

(6) $\mathrm{Fix}(f_a) \subseteq \mathrm{Fix}(f_a^2) \subseteq \cdots$, *where* $\mathrm{Fix}(f_a) := \{x \in X \mid f_a(x) = x\}$*;*

(7) $\ker(f_a) \subseteq \ker(f_a^2) \subseteq \cdots$ *and the equality is true when* $a = 1$, *where* $\ker(f_a) := \{x \in X \mid f_a(x) = 1\}$*;*

(8) $\mathrm{Fix}(f_a^n) \subseteq \mathrm{Im}(f_a^n)$ *for any* $n \in \mathbb{N}$ *and the equality is true when* X *is positive implicative;*

(9) $\mathrm{Fix}(f_a^n) \cap \ker(f_a^n) = \{1\}$ *for any* $n \in \mathbb{N}$, *for all* $a, x, y \in X$*;*

(10) *if* X *is an* &*-equality algebra, then* $f_a^2 = f_a$ *for any* $a \in X$, *with* $a \odot a = a$.

Proof. Let $a, x, y \in X$. Using (2.1) and (2.6), we have

$$f_a(x) \to f_a(y) = (a \to x) \to (a \to y) \le a \to (x \to y) = f_a(x \to y),$$

which proves (1).

(2) and (3) are straightforward by (2.6) and (2.3), respectively.

(4) Using (2.1) and (2.5), we have $x \to y \le (a \to x) \to (a \to y) = f_a(x) \to f_a(y)$. Suppose that $x \to y \le f_a^k(x) \to f_a^k(y)$ for $k \in \mathbb{N}$. Then

$$x \to y \le f_a^k(x) \to f_a^k(y) \le f_a(f_a^k(x)) \to f_a(f_a^k(y)) = f_a^{k+1}(x) \to f_a^{k+1}(y),$$

and so $x \to y \le f_a^n(x) \to f_a^n(y)$ by mathematical induction.

(5) If $y \in \mathrm{Im}(f_a^2)$, then $y = f_a^2(x) = f_a(f_a(x))$ for some $x \in X$ and so $y \in \mathrm{Im}(f_a)$, which shows that $\mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a)$. Repeating this process induces

$$\mathrm{Im}(f_a^n) \subseteq \cdots \subseteq \mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a).$$

By the similar way to the proof of (5), we have (6), (7) and (8).

(9) Let $x \in \mathrm{Fix}(f_a^n) \cap \ker(f_a^n)$. Then $x = f_a^n(x) = 1$. Hence, $\mathrm{Fix}(f_a^n) \cap \ker(f_a^n) = \{1\}$ for any $n \in \mathbb{N}$.

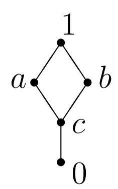
(10) Let $a \in X$ with $a \odot a = a$. Then

$$f_a^2(x) = a \to (a \to x) = (a \odot a) \to x = a \to x = f_a(x),$$

for all $x \in X$ and so $f_a^2 = f_a$. $\square$

We pose a question as follows. Given a left mapping f_a on X, is the subset $\mathrm{Fix}(f_a)$ of X a filter of X? But the following example shows that the answer is negative.

Example 3.3. Let $Y = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram.



Then $(Y, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on Y by Table 5. Then $(Y, \wedge, \sim, 1)$ is an equality algebra which is not

TABLE 5. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

commutative, and the implication $(\rightarrow)$ is given by Table 6. We know that the map

TABLE 6. Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

$f_b : Y \to Y$ given by $f_b(0) = 0$, $f_b(a) = f_b(c) = a$ and $f_b(b) = f_b(1) = 1$ is a left mapping on Y. Then $\mathrm{Fix}(f_b) = \{0, a, 1\}$, which is not a filter of Y.

Proposition 3.2. *Given a left mapping f_a on X, the following statements are equivalent.*

(1) $(\forall x, y \in X)(\forall n \in \mathbb{N})$ $(y^n \to x = y^{n+1} \to x)$, *where*

$$y^n \to x = \underbrace{y \to (y \to \cdots (y}_{n \text{ times}} \to x)).$$

(2) $(\forall n \in \mathbb{N})(\mathrm{Im}(f_a^n) = \mathrm{Fix}(f_a^n))$.

(3) $(\forall n \in \mathbb{N})(f_a^n = f_a^{n+1})$.

Proof. (1) $\Rightarrow$ (2) By Proposition 3.1 (8), we have $\mathrm{Fix}(f_a^n) \subseteq \mathrm{Im}(f_a^n)$. If $y \in \mathrm{Im}(f_a^n)$, then there exists $x \in X$ such that $f_a^n(x) = y$. By (1), we get

$$y = f_a^n(x) = f_a^{n+1}(x) = \cdots = f_a^{2n}(x) = f_a^n(f_a^n(x)) = f_a^n(y).$$

Hence, $y \in \mathrm{Fix}(f_a^n)$, and so $\mathrm{Im}(f_a^n) \subseteq \mathrm{Fix}(f_a^n)$. Therefore, $\mathrm{Im}(f_a^n) = \mathrm{Fix}(f_a^n)$.

(2) $\Rightarrow$ (1) Let $x, y \in X$. It is clear that $y^n \to x \leq y^{n+1} \to x$. On the other hand,

$$(y^{n+1} \to x) \to (y^n \to x) = f_y^{n+1}(x) \to f_y^n(x) = f_y^n(f_y(x)) \to f_y^n(x).$$

Then $f_y(x) \in \mathrm{Im}(f_y^n) = \mathrm{Fix}(f_y^n)$ and so $f_y^n(f_y(x)) = f_y(x)$. Since $f_y(x) \leq f_y^n(x)$, we have

$$(y^{n+1} \to x) \to (y^n \to x) = f_y(x) \to (f_y^n(x)) = 1,$$

and so $(y^{n+1} \to x) \leq (y^n \to x)$. Therefore, $y^n \to x = y^{n+1} \to x$.

(1) $\Leftrightarrow$ (3) The proof is clear. $\square$

Corollary 3.2. *In a positive implicative equality algebra X, the conditions* (2) *and* (3) *of Proposition* 3.2 *are always valid.*

Definition 3.3. Let X and Y be equality algebras. A mapping $f : X \to Y$ is called a *$\to$-homomorphism* if $f(a \to x) = f(a) \to f(x)$ for all $a, x \in X$.

By a $\to$-endomorphism on X we mean a $\to$-homomorphism from X to X. It is clear that the left mapping f_1 on X is a $\to$-homomorphism.

Example 3.4. Let X be the equality algebra as in Example 3.1 and Y be the equality algebra as in Example 3.3. We define a mapping $f : X \to Y$ by $f(0) = f(b) = 0$ and $f(a) = f(1) = 1$. Then f is a $\to$-homomorphism.

Theorem 3.2. *Let $(X, \wedge_X, \sim_X, 1_X)$ and $(Y, \wedge_Y, \sim_Y, 1_Y)$ be equality algebras. Then every homomorphism from X to Y is a $\to$-homomorphism.*

Proof. Let $f : X \to Y$ be a homomorphism. Then

$$\begin{aligned} f(x \to_X y) &= f((x \wedge_X y) \sim_X x) \\ &= f(x \wedge_X y) \sim_Y f(x) \\ &= (f(x) \wedge_Y f(y)) \sim_Y f(x) \\ &= f(x) \to_Y f(y), \end{aligned}$$

for all $x, y \in X$. Hence, f is a $\to$-homomorphism. $\square$

The following example shows that a left mapping is not a $\to$-endomorphism.

Example 3.5. The left mapping f_a in Example 3.2 is not a $\to$-endomorphism since

$$1 = f_a(c) = f_a(d \to 0) \neq f_a(d) \to f_a(0) = (a \to d) \to (a \to 0) = 1 \to d = d.$$

We provide a condition for a left mapping to be a $\to$-endomorphism, and consider a characterization of a positive implicative equality algebra by using the notion of left mapping.

Theorem 3.3. *An equality algebra X is a positive implicative if and only if every left mapping on X is a $\to$-endomorphism of X.*

Proof. Let X be a positive implicative equality algebra and $f_a : X \to X$ be a left mapping on X where $a \in X$. Then

$$f_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = f_a(x) \to f_a(y),$$

for all $x, y \in X$, and so f_a is a $\to$-endomorphism of X.

Conversely, assume that every left mapping on X is a $\to$-endomorphism of X. Let f_a be a left mapping on X for each $a \in X$. Then f_a is a $\to$-endomorphism of X and so

$$a \to (x \to y) = f_a(x \to y) = f_a(x) \to f_a(y) = (a \to x) \to (a \to y).$$

Therefore, X is a positive implicative equality algebra. $\square$

Corollary 3.3. *Let f_a be a left mapping on X. If $f_a^2 = f_a$, then f_a is a $\to$-endomorphism.*

Corollary 3.4. *If f_a is a $\to$-endomorphism on X, then $f_a^n = f_a^{n+1}$ for any $n \in \mathbb{N}$.*

Theorem 3.4. *Let X be an &-equality algebra. Then $\mathcal{L}(X)$ is a commutative monoid under the composition of mappings with the zero element f_1.*

Proof. For any $f_a, f_b, f_c \in \mathcal{L}(X)$, where $a, b, c \in X$, we have

$$(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(b \to x) = a \to (b \to x) = (a \odot b) \to x = f_{a\odot b}(x),$$

for all $x \in X$. Hence, $\mathcal{L}(X)$ is closed under the operation $\circ$. Also, we have

$$\begin{aligned}(f_a \circ (f_b \circ f_c))(x) &= f_a(f_{(b\odot c)}(x)) = f_{(a\odot(b\odot c))}(x) = f_{((a\odot b)\odot c)}(x)\\ &= f_{(a\odot b)} \circ f_c(x) = ((f_a \circ f_b) \circ f_c)(x),\\ (f_a \circ f_b)(x) &= f_a(b \to x) = a \to (b \to x) = b \to (a \to x) = f_b(a \to x)\\ &= (f_b \circ f_a)(x),\end{aligned}$$

and $(f_a \circ f_1)(x) = f_{a\odot 1}(x) = f_a(x)$ for all $x \in X$. Therefore, $\mathcal{L}(X)$ is a commutative monoid. $\square$

Theorem 3.5. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then* $\mathrm{Im}(f_a)$, $\mathrm{Fix}(f_a)$ *and* $\ker(f_a)$ *are closed under the operation $\to$.*

Proof. If $x, y \in \mathrm{Im}(f_a)$, then there exist $u, v \in X$ such that $f_a(u) = x$ and $f_a(v) = y$. It follows that

$$x \to y = f_a(u) \to f_a(v) = (a \to u) \to (a \to v) = a \to (u \to v) = f_a(u \to v) \in \mathrm{Im}(f_a).$$

Thus, $\mathrm{Im}(f_a)$ is closed under $\to$. Let $x, y \in \ker(f_a)$. Then $f_a(x) = 1 = f_a(y)$ and thus

$$f_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = f_a(x) \to f_a(y) = 1.$$

Hence, $x \to y \in \ker(f_a)$ and so $\ker(f_a)$ is closed under $\to$. Let $x, y \in \mathrm{Fix}(f_a)$. Then $f_a(x) = x$ and $f_a(y) = y$. Thus,

$$x \to y = f_a(x) \to f_a(y) = (a \to x) \to (a \to y) = a \to (x \to y) = f_a(x \to y),$$

and so $x \to y \in \mathrm{Fix}(f_a)$. Hence, $\mathrm{Fix}(f_a)$ is closed under $\to$. $\square$

Using mathematical induction, we have the following corollary.

Corollary 3.5. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then* $\mathrm{Im}(f_a^n)$, $\mathrm{Fix}(f_a^n)$ *and* $\ker(f_a^n)$ *are closed under the operation $\to$ for all $n \in \mathbb{N}$.*

We define an order "$\le$" and equality "$=$" on $\mathcal{L}(X)$ as follows.

$$f_a \le f_b \Leftrightarrow f_a(x) \le f_b(x) \text{ for all } x \in X,$$
$$f_a = f_b \Leftrightarrow f_a \le f_b \;\&\; f_b \le f_a,$$

for all $f_a, f_b \in \mathcal{L}(X)$.

Proposition 3.3. *If X is a positive implicative equality algebra, then the following assertions are true in $\mathcal{L}(X)$:*

(1) $f_a \circ f_b = f_b \circ f_a$;
(2) $f_a \circ f_a = f_a$;
(3) $f_1 \circ f_a = f_a = f_a \circ f_1$;
(4) $a \le b \Rightarrow f_b \le f_a,\ f_a \circ f_b = f_a$.

Proof. (1) Let $a, b, x \in X$. Then by (2.1), it is clear that

$$f_a \circ f_b(x) = f_a(f_b(x)) = a \to (b \to x) = b \to (a \to x) = f_b(f_a(x)) = f_b \circ f_a(x).$$

(2) Let $a, x \in X$. Since X is a positive equality algebra, we get that

$$f_a \circ f_a(x) = f_a(f_a(x)) = a \to (a \to x) = a \to x = f_a(x).$$

(3) The proof is clear.

(4) Let $a, b \in X$ such that $a \le b$. Then for any $x \in X$, by (2.6), we get $f_b(x) = b \to x \le a \to x = f_a(x)$. Moreover, since X is positive implicative, we have

$$f_a \circ f_b(x) = a \to (b \to x) = (a \to b) \to (a \to x) = 1 \to (a \to x) = a \to x = f_a.$$

This completes the proof. □

Let $\mathrm{End}_{\to}(X)$ denote the set of all left mappings on X which is a $\to$-homomorphism, that is,

$$\mathrm{End}_{\to}(X) = \{f_a \in \mathcal{L}(X) \mid f_a \text{ is a } \to\text{-homomorphism}\}.$$

Theorem 3.6. *If X is a positive implicative &-equality algebra, then $(\mathrm{End}_{\to}(X), \circ)$ is a commutative semigroup with the zero element f_1.*

Proof. Let $x \in X$. Since X is an &-equality algebra, we get

$$(f_a \circ f_b)(x) = a \to (b \to x) = (a \odot b) \to x = f_{a\odot b}(x),$$

and so, $f_{a\odot b}(x) \in \mathcal{L}(X)$. Since X is a positive implicative equality algebra, we have

$$\begin{aligned}(f_a \circ f_b)(x \to y) &= f_{a\odot b}(x \to y) = (a \odot b) \to (x \to y)\\ &= ((a \odot b) \to x) \to ((a \odot b) \to y)\\ &= (f_a \circ f_b)(x) \to (f_a \circ f_b)(y).\end{aligned}$$

Let $f_a, f_b, f_c \in \mathrm{End}_{\to}(X)$. Since X is an &-equality algebra, we have

$$(f_a \circ (f_b \circ f_c))(x) = f_{a\odot(b\odot c)}(x) = f_{(a\odot b)\odot c}(x) = ((f_a \circ f_b) \circ f_c)(x).$$

Also $f_a \circ f_b = f_b \circ f_a$ and $f_a \circ f_1 = f_1$ by Proposition 3.3. Therefore, $(\mathrm{End}_{\to}(X), \circ)$ is a commutative semigroup with the zero element f_1. □

4. Right Mappings

In this section, we introduce the notion of right mapping and investigate some properties of it. Also, we prove that kernel of g_a^2 is a filter of X. Finally we show that the set of all right mappings on positive implicative equality algebra is a dual BCK-algebra.

Definition 4.1. Given a fixed element a in an equality algebra X, we define a self-mapping g_a of X by

$$g_a : X \to X, \quad x \mapsto x \to a, \tag{4.1}$$

and we say that g_a is a *right mapping* on X.

Let $\mathcal{R}(X)$ denote the set of all right mappings on an equality algebra X.

Example 4.1. Let X be the equality algebra as in Example 3.1. Then define a self mapping $g_a : X \to X$ by $g_a(0) = g_a(a) = 1$ and $g_a(b) = g_a(1) = a$. It is routine to verify that g_a is a right mapping on X.

Proposition 4.1. *Every right mapping g_β on X, where β is any element of X, satisfies the following conditions:*

1. $(\forall a \in X)(g_a(a) = 1, g_a(1) = a)$;
2. $(\forall a, b \in X)(g_a(1) \leq g_a(b))$;
3. *If X is bounded, then $g_a(0) = 1$ and $g_0(a) = \neg a$ for all $a \in X$;*
4. $(\forall x \in X)(g_1(x) = 1)$;
5. $(\forall a, x, y \in X)(x \leq y \Rightarrow g_a(y) \leq g_a(x))$.

Proof. Straightforward. □

Proposition 4.2. *For any right mapping g_β on X where β is any element of X, we have the following assertions.*

1. *If X is a commutative equality algebra, then $g_a^2(x) = g_x^2(a)$ for all $x, a \in X$.*
2. *For any natural number $n \in \mathbb{N}$ and $a \in X$, we have*
$$g_a^n = \begin{cases} g_a & n \text{ is odd,} \\ g_a^2 & n \text{ is even.} \end{cases}$$
3. $g_a^2(x) \to g_a(y) = g_a^2(y) \to g_a(x)$ *for any $a, x, y \in X$.*
4. $y \to g_a^2(x) = g_a(x) \to g_a(y)$ *and* $g_a^2(x) \to g_a^2(y) = x \to g_a^2(y)$ *for any $a, x, y \in X$.*
5. $g_a^2(x) = 1$ *if and only if $f_x(a) = a$, where f_x is a left mapping on X.*
6. *The mapping g_a^2 is isotone.*

Proof. (1) Since X is a commutative equality algebra, we have

$$g_a^2(x) = (x \to a) \to a = (a \to x) \to x = g_x^2(a),$$

for all $a, x \in X$.

(2) Let $x, a \in X$ and $n \in \mathbb{N}$. Suppose $n = 4$. Then

$$g_a^4(x) = (((x \to a) \to a) \to a) \to a = (x \to a) \to a = g_a^2(x),$$

by (2.7). By the similar way, we can prove that $g_a^n(x) = g_a^2(x)$ for any even number $n \in \mathbb{N}$. Now, if $n = 3$, then

$$g_a^3(x) = ((x \to a) \to a) \to a = x \to a = g_a(x),$$

by (2.7). By the similar way, we can prove that $g_a^n(x) = g_a(x)$ for any odd number $n \in \mathbb{N}$.

(3) Let $a, x, y \in X$. Then

$$\begin{aligned} g_a^2(x) \to g_a(y) &= ((x \to a) \to a) \to (y \to a) \\ &= y \to (((x \to a) \to a) \to a) \\ &= y \to (x \to a) \\ &= y \to g_a(x), \end{aligned}$$

by (2.7). By the similar way, we can prove that $g_a^2(y) \to g_a(x) = y \to g_a(x)$. Hence, $g_a^2(x) \to g_a(y) = g_a^2(y) \to g_a(x)$.

(4) Let $x, y, a \in X$. Then

$$y \to g_a^2(x) = y \to ((x \to a) \to a) = (x \to a) \to (y \to a) = g_a(x) \to g_a(y),$$

by (2.7). Also, we have

$$\begin{aligned} g_a^2(x) \to g_a^2(y) &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= g_a(y) \to g_a(x) = x \to g_a^2(y). \end{aligned}$$

(5) and (6) are straightforward. □

Theorem 4.1. *For any right mapping g_a on X, the following are equivalent.*

(1) *g_a^2 is a $\to$-endomorphism.*
(2) *$g_a^2(x \to y) = x \to g_a^2(y)$ for all $x, y \in X$.*
(3) *$g_a^2(x \to y) = g_a(y) \to g_a(x)$ for all $x, y \in X$.*

Proof. (1) ⇒ (2). Let g_a^2 be a $\to$-endomorphism and $x, y \in X$. Then

$$\begin{aligned} g_a^2(x \to y) &= g_a^2(x) \to g_a^2(y) \\ &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= x \to ((y \to a) \to a) \\ &= x \to g_a^2(y), \end{aligned}$$

by (2.7).

(2) $\Rightarrow$ (3). For any $x, y \in X$ we have

$$\begin{aligned} g_a^2(x \to y) =& x \to g_a^2(y) = x \to ((y \to a) \to a) = (y \to a) \to (x \to a) \\ =& g_a(y) \to g_a(x), \end{aligned}$$

by (2).

(3) $\Rightarrow$ (1). For any $a, x, y \in X$ we have

$$\begin{aligned} g_a^2(x) \to g_a^2(y) &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= g_a(y) \to g_a(x) \\ &= g_a^2(x \to y), \end{aligned}$$

by (2.7) and (3). Therefore, g_a^2 is a $\to$-endomorphism on X. ☐

Theorem 4.2. *For any right mapping g_a on X, the following are equivalent.*

(1) *g_a^2 is an identity map.*
(2) *g_a is an injective map.*
(3) *g_a is a surjective map.*

Proof. (1) $\Rightarrow$ (2). Let g_a^2 be an identity map. Let $x, y \in X$ be such that $g_a(x) = g_a(y)$. Then $x \to a = y \to a$ and so

$$x = g_a^2(x) = (x \to a) \to a = (y \to a) \to a = g_a^2(y) = y.$$

Hence, g_a is an injective map on X.

(2) $\Rightarrow$ (3). For any $x, y \in X$, we have $g_a((x \to a) \to a) = g_a(x)$ by (2.7). Since g_a is an injective map on X, it follows that $(x \to a) \to a = x$. Moreover, we know that $\mathrm{Im}(g_a) \subseteq X$. Let $y \in X$. Then $g_a(y \to a) = (y \to a) \to a = y$ and so $y \in \mathrm{Im}(g_a)$. Hence, $X = \mathrm{Im}(g_a)$. Therefore, g_a is a surjective map on X.

(3) $\Rightarrow$ (1). Using (2.4), we have $x \leq (x \to a) \to a = g_a^2(x)$ for any $x \in X$. Since g_a is a surjective map, for any $y \in X$, there exists $x \in X$ such that $g_a(x) = y$, i.e., $x \to a = y$. It follows from (2.1) and (2.7) that

$$g_a^2(y) \to y = ((y \to a) \to a) \to (x \to a) = x \to (y \to a) = y \to y = 1,$$

that is, $g_a^2(y) \leq y$ for all $y \in X$. Hence, $g_a^2(y) = y$ for all $y \in X$ and therefore g_a^2 is an identity map. ☐

Corollary 4.1. *For any right mapping g_β on X where β is any element of X, the following are equivalent.*

(1) *g_a^2 is an injective map for all $a \in X$.*
(2) *g_a^2 is an identity map for all $a \in X$.*
(3) *g_a^2 is a surjective map for all $a \in X$.*

Proof. By Theorem 4.2 and Proposition 4.2 (2), the proof is clear. ☐

Theorem 4.3. *For any right map g_a on X, the set* $\ker(g_a^2) = \{x \in X \mid g_a^2(x) = 1\}$ *is a filter of X.*

Proof. Let $a \in X$. Since $g_a^2(1) = (1 \to a) \to a = a \to a = 1$, we get that $1 \in \ker(g_a^2)$. Let $x, y \in X$ be such that $x, x \to y \in \ker(g_a^2)$. Then $g_a^2(x) = g_a^2(x \to y) = 1$. It follows from (2.1), (2.5) and (2.7) that

$$\begin{aligned}
g_a^2(y) &= (y \to a) \to a \\
&= 1 \to ((y \to a) \to a) \\
&= (((x \to y) \to a) \to a) \to ((y \to a) \to a) \\
&= (y \to a) \to ((((x \to y) \to a) \to a) \to a) \\
&= (y \to a) \to ((x \to y) \to a) \\
&= (x \to y) \to ((y \to a) \to a) \\
&= (x \to y) \to (1 \to ((y \to a) \to a)) \\
&= (x \to y) \to (((x \to a) \to a) \to ((y \to a) \to a)) \\
&= (x \to y) \to ((y \to a) \to (((x \to a) \to a) \to a)) \\
&= (x \to y) \to ((y \to a) \to (x \to a)) \\
&= (x \to y) \to (x \to ((y \to a) \to a)) \\
&\geq y \to ((y \to a) \to a) \\
&= (y \to a) \to (y \to a) \\
&= 1
\end{aligned}$$

Hence, $g_a^2(y) = 1$, and so $y \in \ker(g_a^2)$. Therefore, $\ker(g_a^2)$ is a filter of X. □

Corollary 4.2. *For any right map g_a^{2k} on X, the set* $\ker(g_a^{2k})$ *is a filter of X, where k is any natural number.*

Proposition 4.3. *Let g_β be a right mapping on X where β is any element of X. If F and G are filters of X such that $F \cap G = \{1\}$, then $g_x^2(y) = g_y^2(x) = 1$ for all $x \in F$ and $y \in G$.*

Proof. Let F and G be filters of X such that $F \cap G = \{1\}$. Suppose $x \in F$ and $y \in G$. Since $x \leq (x \to y) \to y$ and $y \leq (x \to y) \to y$ by (2.1), (2.2) and (2.4), we have $(x \to y) \to y \in F \cap G = \{1\}$ and so $g_y^2(x) = (x \to y) \to y = 1$. By the similar way we can prove that $g_x^2(y) = 1$. □

Let X be a positive implicative equality algebra. We define the implication "$\hookrightarrow$" on $\mathcal{R}(X)$ as follows:

$$\hookrightarrow : \mathcal{R}(X) \times \mathcal{R}(X) \to \mathcal{R}(X), \quad (g_a, g_b) \mapsto g_a(x) \to g_b(x).$$

Using the positive implicativity of X, we have

$$(g_a \hookrightarrow g_b)(x) = g_a(x) \to g_b(x) = (x \to a) \to (x \to b) = x \to (a \to b) = g_{a \to b}(x),$$

and so $g_a \to g_b \in \mathcal{R}(X)$.

Theorem 4.4. *If X is a positive implicative equality algebra, then $(\mathcal{R}(X), \hookrightarrow, g_1)$ is a dual BCK-algebra (see [3] for the notion of dual BCK-algebra).*

Proof. Let $g_a, g_b, g_c \in \mathcal{R}(X)$. Then

$$\begin{aligned}
&((g_b \hookrightarrow g_c) \hookrightarrow ((g_c \hookrightarrow g_a) \hookrightarrow (g_b \hookrightarrow g_a)))(x) \\
=&(g_b(x) \to g_c(x)) \to ((g_c(x) \to g_a(x)) \to (g_b(x) \to g_a(x))) \\
=&((x \to b) \to (x \to c)) \to (((x \to c) \to (x \to a)) \to ((x \to b) \to (x \to a))) \\
=&(x \to (b \to c)) \to ((x \to (c \to a)) \to (x \to (b \to a))) \\
=&(x \to (b \to c)) \to (x \to ((c \to a) \to (b \to a))) \\
=&x \to ((b \to c) \to ((c \to a) \to (b \to a))) \\
=&x \to 1 = g_1(x)
\end{aligned}$$

and

$$\begin{aligned}
&(g_b \hookrightarrow ((g_b \hookrightarrow g_a) \hookrightarrow g_a))(x) = g_b(x) \to ((g_b(x) \to g_a(x)) \to g_a(x)) \\
=&(x \to b) \to (((x \to b) \to (x \to a)) \to (x \to a)) \\
=&(x \to b) \to ((x \to (b \to a)) \to (x \to a)) \\
=&(x \to b) \to (x \to ((b \to a) \to a)) \\
=&x \to (b \to ((b \to a) \to a)) \\
=&x \to ((b \to a) \to (b \to a)) \\
=&x \to 1 = g_1(x),
\end{aligned}$$

for all $x \in X$ by (2.1), (2.2), (2.5) and (3.1). Thus,

$$(g_b \hookrightarrow g_c) \hookrightarrow ((g_c \hookrightarrow g_a) \hookrightarrow (g_b \hookrightarrow g_a)) = g_1,$$

and $g_b \hookrightarrow ((g_b \hookrightarrow g_a) \hookrightarrow g_a) = g_1$. Since

$$(g_a \hookrightarrow g_a)(x) = g_a(x) \to g_a(x) = (x \to a) \to (x \to a) = 1 = x \to 1 = g_1(x)$$

and

$$\begin{aligned}
(g_a \hookrightarrow g_1)(x) &= g_a(x) \to g_1(x) = (x \to a) \to (x \to 1) \\
&= x \to (a \to 1) = x \to 1 = g_1(x),
\end{aligned}$$

for all $x \in X$, we have $g_a \hookrightarrow g_a = g_1$ and $g_a \hookrightarrow g_1 = g_1$. Assume that $g_a \to g_b = g_1$ and $g_b \to g_a = g_1$. Then

$$(x \to a) \to (x \to b) = g_a(x) \to g_b(x) = (g_a \hookrightarrow g_b)(x) = g_1(x) = x \to 1 = 1$$

and

$$(x \to b) \to (x \to a) = g_b(x) \to g_a(x) = (g_b \hookrightarrow g_a)(x) = g_1(x) = x \to 1 = 1,$$

for all $x \in X$. It follows that $g_a(x) = x \to a = x \to b = g_b(x)$ for all $x \in X$. Hence, $g_a = g_b$. Therefore, $(\mathcal{R}(X), \hookrightarrow, g_1)$ is a dual BCK-algebra. $\square$

Define an order "$\leq$" on $\mathcal{R}(X)$ as follows:

$$(\forall g_a, g_b \in \mathcal{R}(X))(g_a \leq g_b \Leftrightarrow (g_a \hookrightarrow g_b)(x) = g_1(x) \text{ for all } x \in X.$$

It is clear that if X is a positive implicative equality algebra, then $(\mathcal{R}(X), \leq)$ is a partially ordered set.

Proposition 4.4. *If X is a positive implicative equality algebra, then the following assertions are true in $\mathcal{R}(X)$:*

(1) $g_a \hookrightarrow g_b \leq (g_b \hookrightarrow g_c) \hookrightarrow (g_a \hookrightarrow g_c)$;
(2) $g_a \leq (g_a \hookrightarrow g_b) \hookrightarrow g_b$;
(3) $g_a \leq g_a$;
(4) $g_a \leq g_b$ *and* $g_b \leq g_a$ *imply* $g_a = g_b$;
(5) $g_a \leq g_1$;
(6) $f_a \leq f_b \Rightarrow f_b \hookrightarrow f_c \leq f_a \hookrightarrow f_c,\ f_c \hookrightarrow f_a \leq f_c \hookrightarrow f_b$;
(7) $f_a \hookrightarrow (f_b \hookrightarrow f_c) = f_b \hookrightarrow (f_a \hookrightarrow f_c)$;
(8) $f_a \leq f_b \hookrightarrow f_c \Rightarrow f_b \leq f_a \hookrightarrow f_c$;
(9) $f_a \hookrightarrow f_b \leq (f_c \hookrightarrow f_a) \hookrightarrow (f_c \hookrightarrow f_b)$;
(10) $f_a \leq f_b \hookrightarrow f_a$.

Proof. It is easy by routine calculations. □

5. Conclusions and Future Works

In this paper, the notion of (right) left mapping on equality algebras is introduced, some properties of it are investigated and it is proved that the set of all right mappings on positive implicative equality algebra makes a dual BCK-algebra. Also, we studied that under which condition the kernel of (right) left mapping is a filter. The notion of $\rightarrow$-endomorphism is introduced and it is proved that the set of all $\rightarrow$-endomorphisms on equality algebra is a commutative semigroup with zero element. Moreover, the relation between left mapping and $\rightarrow$-endomorphism and a characterization of positive implicative equality algebra are investigated.

In future work, by using the notion of (right) left mapping on equality algeras and the set of fixed point of that, we can introduce the notion of (right) left stabilizer on equality algebra and by using this notion we can define a basis of a topology on equality algebra. Also, we can introduce the notion of derivation on equality algebra and extend it.

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