Kragujevac Journal of Mathematics
Volume 50(5) (2026), Pages 827–845.

PICTURE FUZZY ORDERING AND $\mathbb{D}^*$-BASED LATTICES

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Abstract. In this paper, some fundamental concepts related to fuzzy relations, fuzzy lattices, intuitionistic fuzzy relations, and intuitionistic fuzzy lattices are extended to the picture fuzzy setting. Also, the structure of the set $\mathbb{D}^*$ of membership values of the picture fuzzy set that plays the role of a prototype for the picture fuzzy set was studied, and some of its basic properties were discussed. Furthermore, we have introduced the concepts of picture fuzzy filters in a crisp lattice, crisp filters in a picture fuzzy lattice, and picture fuzzy filters in a picture fuzzy lattice, and some of their properties, subtle differences, and extensions in terms of picture fuzzy sets are proved. As well as giving many characterizations of picture fuzzy filters in a picture fuzzy lattice. Finally, we present the necessary and sufficient requirements for a picture fuzzy subset to be a picture fuzzy prime filter.

1. Introduction

Many problems in daily life contain various levels of uncertainty. Since existing standard mathematical tools may not model such uncertainties, new ones are needed. Fuzzy sets [35] and intuitionistic fuzzy sets [9], were introduced to deal with uncertainty, are some of the well-known mathematical tools for the aforesaid purpose.

Zadeh introduced the ideas of fuzzy sets and fuzzy relations initially, followed by Goguen [26, 35, 36]. Many authors have investigated various approaches to fuzzy lattices, fuzzy filters, and related concepts, see [2–5, 19, 33]. Birkhoff introduced filters in 1935 [12] and Cartan in 1937 [17, 18]. They are basic in algebra and play a major role in the study of fuzzy logic. From a logical perspective, filters correspond to

Key words and phrases. Picture fuzzy set, picture fuzzy ordering, picture fuzzy lattice, picture fuzzy filter, picture fuzzy prime filter.

2020 *Mathematics Subject Classification.* Primary: 03E72, 06D72, 06B75. Secondary: 06B05, 06B10.

DOI

Received: February 07, 2023.

Accepted: May 17, 2024.

collections of provable formulae. The filters theory, which has numerous applications in mathematics such as logic and topology, seems to have received more attention in recent years. In [32], Venugopalan presented the idea of a fuzzy ordered set (foset) and defined the upper bound $U(A)$ and lower bound $L(A)$ of a set A. He also defined the supremum and infimum of a set A. In 1990, Bo and Wangming proposed the ideas of fuzzy sublattices and fuzzy ideals of a lattice [14]. As a fuzzy algebra, a fuzzy lattice was defined by Ajmal and Thomas [2] and fuzzy sublattices were described by the same authors. Using their α-cuts, Chon defined and characterized fuzzy lattices in 2009 and established several fundamental properties of fuzzy lattices [19]. Mezzomo [28] used a fuzzy partial order relation to describe fuzzy lattices. Additionally, he defined filters and α-filters of fuzzy lattices and he used their α-cuts to describe them.

In 1983, Atanassov first introduced intuitionistic fuzzy sets (IFSs) to address the issue of non-membership [8]. This concept was followed by the introduction in 1984, of a generalized intuitionistic fuzzy set, known as "intuitionistic L-fuzzy set" [11]. It has been shown to be particularly useful for dealing with ambiguity. Based on the intuitionistic fuzzy set concept of Atanassov, Burillo and Bustince [15] introduced the notion of intuitionistic fuzzy relation. Specifically, they presented the intuitionistic fuzzy order relation as a logical extension of the fuzzy order relation introduced by Zadeh [36] before. Many authors have studied the concept of intuitionistic fuzzy order, intuitionistic fuzzy lattice, intuitionistic fuzzy filter and intuitionistic fuzzy ideal [2, 6, 31, 37].

Although these sets can model many problems, there are much more problems and uncertainties in real life that they fail to model. For instance, in voting for an election, the decisions of the electorate may be split into three types: yes, no, and abstain. To model this problem and the problems similar to it, Cuong and Kreinovich put forward in 2013 [21] a new concept called "picture fuzzy sets". This idea is an intriguing development of both "fuzzy sets" and "intuitionistic fuzzy sets". The idea of an element's positive, negative, and neutral membership degrees with a sum less than or equal to one is the main contribution of Cuong Bui Cong and Vladik Kreinovich. This gives an unusual but great idea of what a mathematician and a lot of logic are like.

Not only does the resulting notion have a beautiful mathematical structure with connections to various fields of mathematics, but it also has a broad range of applications outside mathematics, for example in decision-making [7, 25, 34], Medical Diagnosis [23], investment risk [13] and other applications [1, 30].

The picture fuzzy set is one of the most reliable techniques to handle the uncertainties in the data throughout the decision-making process, where an intuitionistic fuzzy set may not yield satisfactory outcomes. It is an effective mathematical tool for dealing with uncertain real-life issues. It can function extremely effectively in ambiguous situations that call for responses of the yes, no, abstain, and rejection types. Fetanat and Tayebi are doing research to try to combine a new decision support system (DSS) with a picture fuzzy set based combined compromise solution (PFS-CoCoSo) [24].

Since $\mathbb{D}^*$ is a complete lattice, it is possible to use membership degrees with more freedom by interpreting picture fuzzy sets as $\mathbb{D}^*$-fuzzy sets.

The rest of this paper is structured as follows. Section 2 presents some essential concepts relevant to fuzzy sets, intuitionistic fuzzy set theory, picture fuzzy sets, and the structure set $\mathbb{D}^*$ (the set of membership values of a picture fuzzy set). In Section 3, we define the picture fuzzy relation and study its main properties. Section 4 extends the notion of fuzzy lattices and intuitionistic fuzzy lattices studied in [19] to picture fuzzy cases. As a consequence, we extend some results of [19] to picture fuzzy cases. Section 5 introduces the concept of a picture fuzzy sub-lattice, a picture fuzzy filter in a lattice. We achieve this by generalizing some existing notions and results in Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5, 33]) to the picture fuzzy case. In Section 6, we extend the notion of a crisp filter and fuzzy filter in a fuzzy lattice [29] to a crisp filter and a picture fuzzy filter in a picture fuzzy lattice and we give more characterizations of them. Section 7 focuses on prime filters and picture fuzzy prime filters of a picture fuzzy lattice. Finally, we present some concluding remarks in Section 7.

2. Preliminaries

This section provides a brief introduction to fuzzy sets, intuitionistic fuzzy sets and picture fuzzy sets.

Definition 2.1 ([35])**.** Suppose that X is a non-empty set. A fuzzy set E in X is given by $E = \{(x, \mu_E(x)) \mid x \in X\}$, with $\mu_E : X \to [0,1]$ represents the degree of membership of x in E.

Definition 2.2 ([9])**.** Suppose that X is a non-empty set. An intuitionistic fuzzy set E on X is given by $E = \{\langle x, \mu_E(x), \nu_E(x)\rangle \mid x \in X\}$, with $\mu_E : X \to [0,1]$ and $\nu_E : X \to [0,1]$ denote respectively the degree of membership and the degree of non-membership of x in E. The functions μ_E and ν_E should satisfy the condition: $\mu_E(x) + \nu_E(x) \leq 1$, for any $x \in X$.

Many authors have discussed related concepts such as fuzzy relations, intuitionistic fuzzy relations, fuzzy lattices, intuitionistic fuzzy lattices, etc. (see [2–5, 9, 10, 16, 19, 26, 29, 33, 35, 36]).

Definition 2.3 ([21])**.** Suppose that X is a non-empty set. A picture fuzzy set E on X is given by $E = \{\langle x, \mu_E(x), \eta_E(x), \nu_E(x)\rangle \mid x \in X\}$, where $\mu_E(x), \eta_E(x), \nu_E(x) \in [0,1]$ denote respectively the degree of positive membership of x in E, degree of neutral membership of x in E and degree of negative membership of x in E. μ_E, η_E and ν_E satisfy the condition $\mu_E(x) + \eta_E(x) + \nu_E(x) \leq 1$, for any $x \in X$.

The quantity $\pi(x) = 1 - (\mu_E(x) + \eta_E(x) + \nu_E(x))$ is called the degree of refusal membership of x in E.

According to [20, 27], consider the set $\mathbb{D}^*$ defined by:

$$\mathbb{D}^* = \left\{a = (a_1, a_2, a_3) \in [0,1]^3 \mid a_1 + a_2 + a_3 \leq 1\right\}.$$

This set plays the role of a prototype of a picture fuzzy set, and the study of this set allows us to perform picture fuzzy sets operations using these of $\mathbb{D}^*$.

Note that for $a \in \mathbb{D}^*$, a_1, a_2 and a_3 refer to the first, second and third components of a, i.e., $a = (a_1, a_2, a_3)$.

Obviously, for each picture fuzzy subset $E = \{\langle x, \mu_E(x), \eta_E(x), \nu_E(x)\rangle \mid x \in X\}$, corresponds to a $\mathbb{D}^*$-fuzzy subset, i.e., a mapping $E : X \to \mathbb{D}^*$ given by $E(x) = (\mu_E(x), \eta_E(x), \nu_E(x)) \in \mathbb{D}^*$.

Definition 2.4 ([20, 27])**.** For all $a, b \in \mathbb{D}^*$, we define the order relation $\preceq$ on $\mathbb{D}^*$ by $a \preceq b$ if and only if ($a_1 < b_1$ and $a_3 \geq b_3$) or ($a_1 = b_1$ and $a_3 > b_3$) or ($a_1 = b_1$ and $a_3 = b_3$ and $a_2 \leq b_2$), for all $a, b \in \mathbb{D}^*$.

Note that $(\mathbb{D}^*, \preceq)$ is a bounded lattice with top element $1_{\mathbb{D}^*} = (1, 0, 0)$ and bottom element $0_{\mathbb{D}^*} = (0, 0, 1)$. And for each $a, b \in \mathbb{D}^*$, $a \curlywedge b$ and $a \curlyvee b$ are given by

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - (a_1 \wedge b_1) - (a_3 \vee b_3), a_3 \vee b_3), & \text{otherwise,} \end{cases}$$

$$a \curlyvee b = \begin{cases} b, & \text{if } a \preceq b, \\ a, & \text{if } b \preceq a, \\ (a_1 \vee b_1, 0, a_3 \wedge b_3), & \text{otherwise.} \end{cases}$$

Concerning this definition it is worth pointing out the following aspect.

- If $a_1 \neq 0$, then $a_3 \neq 1$.
- $a = (a_1, a_2, a_3) \succ 0_{\mathbb{D}^*}$ equivalent ($a_1 > 0$) or ($a_2 > 0$) or ($a_1 = 0$ and $a_3 < 1$).
- The components of non-comparable elements $a, b \in \mathbb{D}^*$ verify that ($a_1 > b_1$ and $a_3 > b_3$) or ($a_1 < b_1$ and $a_3 < b_3$). We write $a \parallel b$.

Following that, we will discuss some fundamental properties for the order of $\mathbb{D}^*$ that will be useful in the sequel.

Proposition 2.1. *Let $a, b, c, d \in \mathbb{D}^*$. Then,*

(1) $a \curlywedge b \preceq a$, $a \curlywedge b \preceq b$;
(2) $a \preceq a \curlyvee b$, $b \preceq a \curlyvee b$;
(3) $a \curlywedge b \preceq a \curlyvee b$;
(4) $a \succ 0_{\mathbb{D}^*}$ *and* $b \succ 0_{\mathbb{D}^*}$ *if and only if* $a \curlywedge b \succ 0_{\mathbb{D}^*}$;
(5) $a \succeq c$ *and* $b \succeq c$ *if and only if* $a \curlywedge b \succeq c$;
(6) $a \succ 0_{\mathbb{D}^*}$ *or* $b \succ 0_{\mathbb{D}^*}$ *if and only if* $a \curlyvee b \succ 0_{\mathbb{D}^*}$;
(7) *if* $a \succeq c$ *or* $b \succeq c$, *then* $a \curlyvee b \succeq c$;
(8) $a \preceq c$ *and* $b \preceq c$ *if and only if* $a \curlyvee b \preceq c$;
(9) *if* $b \preceq c$, *then* $a \curlyvee b \preceq a \curlyvee c$ *and* $a \curlywedge b \preceq a \curlywedge c$;
(10) $(a \curlyvee b) \curlywedge (c \curlyvee d) \succeq (a \curlywedge c) \curlyvee (a \curlywedge d) \curlyvee (b \curlywedge c) \curlyvee (b \curlywedge d)$.

Proof. Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{D}^*$.

(4) Suppose that $a \succ 0_{\mathbb{D}^*}$, $b \succ 0_{\mathbb{D}^*}$. Then,

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3), & \text{otherwise.} \end{cases}$$

The result is clear if $a \curlywedge b = a$ or $a \curlywedge b = b$, it remains to prove that the property is true in the case $a \curlywedge b = (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3)$.

Since $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$, it follows that

$$\begin{cases} a_1 > 0 \text{ and } a_3 < 1 & (1) \\ \text{or} & \\ a_1 = 0 \quad \text{and} \quad a_3 < 1 & (2) \end{cases} \quad \text{and} \quad \begin{cases} b_1 > 0 \text{ and } b_3 < 1 & (3) \\ \text{or} & \\ b_1 = 0 \text{ and } b_3 < 1 & (4). \end{cases}$$

Then we distinguish four cases.

Case 01: If we have (1) and (3), i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), then $a_1 \wedge b_1 > 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 02: If we have (1) and (4), i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 03: If we have (2) and (3), i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 04: If we have (2) and (4), i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $a \curlywedge b \succ 0_{\mathbb{D}^*}$ and $a = 0_{\mathbb{D}^*}$. Then $a \curlywedge b = 0_{\mathbb{D}^*} \curlywedge b = 0_{\mathbb{D}^*}$, for each $b \in \mathbb{D}^*$. This is a contradiction. Thus, $a \curlywedge b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$.

(6) Suppose that $a \succ 0_{\mathbb{D}^*}$. Since $a \curlyvee b \succeq a$, then $a \curlyvee b \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $a \curlyvee b \succ 0_{\mathbb{D}^*}$, $a = 0_{\mathbb{D}^*}$ and $b = 0_{\mathbb{D}^*}$. Then, $a \curlyvee b = 0_{\mathbb{D}^*} \curlyvee 0_{\mathbb{D}^*} = 0_{\mathbb{D}^*}$. This is a contradiction. Thus, $a \curlyvee b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ or $b \succ 0_{\mathbb{D}^*}$.

The fact that $\mathbb{D}^*$ is a lattice, the rest of the properties are clear. □

3. Picture Fuzzy Relations

Here, we recall the definition of a picture fuzzy relation and investigate its main properties. We denote by $\mathbb{D}_0^* = \mathbb{D}^* - \{0_{\mathbb{D}^*}\}$.

Definition 3.1 ([20])**.** Suppose that X is a non-empty set. A picture fuzzy relation $\mathcal{R} : X \times X \to \mathbb{D}^*$ is defined by $\mathcal{R}(x,y) = (\mu_{\mathcal{R}}(x,y), \eta_{\mathcal{R}}(x,y), \nu_{\mathcal{R}}(x,y))$, for all $x, y \in X$, where $\mu_{\mathcal{R}} : X \times X \to [0,1]$, $\eta_{\mathcal{R}} : X \times X \to [0,1]$ and $\nu_{\mathcal{R}} : X \times X \to [0,1]$ satisfying the condition $0 \leq \mu_{\mathcal{R}}(x,y) + \eta_{\mathcal{R}}(x,y) + \nu_{\mathcal{R}}(x,y) \leq 1$, for every $(x,y) \in X \times X$.

In the sequel, $PFR(X)$ denotes the set of all the picture fuzzy relations on X.

Definition 3.2. Suppose that X is a non-empty set and $\mathcal{R}, \mathcal{P} \in PFR(X)$. Using Definition 2.4, we can define the following.

(i) The picture fuzzy inclusion by $\mathcal{R} \subseteq \mathcal{P}$ if and only if, for all $x, y \in X$,
$(\mu_{\mathcal{R}}(x,y), \eta_{\mathcal{R}}(x,y), \nu_{\mathcal{R}}(x,y)) \preceq (\mu_{\mathcal{P}}(x,y), \eta_{\mathcal{P}}(x,y), \nu_{\mathcal{P}}(x,y))$.

(ii) The picture fuzzy intersection $\mathcal{R} \cap \mathcal{P}$ by $(\mathcal{R} \cap \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y)$, for all $x, y \in X$.

(iii) The picture fuzzy union $\mathcal{R} \cup \mathcal{P}$ by $(\mathcal{R} \cup \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlyvee \mathcal{P}(x,y)$, for all $x, y \in X$.

(iv) The support of $\mathcal{R}$ by $S(\mathcal{R}) = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}\}$.

(v) The kernel of $\mathcal{R}$ by $\ker(\mathcal{R}) = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) = 1_{\mathbb{D}^*}\}$.

(vi) For all $\alpha \in \mathbb{D}_0^*$, we define the α-cut of $\mathcal{R}$ by $\mathcal{R}_\alpha = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) \succeq \alpha\}$.

Definition 3.3. Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. We say that $\mathcal{R}$ is

(i) reflexive if and only if $\mathcal{R}(x,x) = (1,0,0)$ for all $x \in X$;

(ii) perfect antisymmetric, if for every $x, y \in X$ with $x \neq y$ and $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(y,x) = 0_{\mathbb{D}^*}$;

(iii) transitive if and only if for all $x, y, z \in X$, $\mathcal{R}(x,z) \succeq \mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z)$.

Remark 3.1. The following statement is equivalent to the definition of perfect antisymmetry: for all $x, y \in X$, if $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$, then $x = y$.

Definition 3.4. Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. Then, $\mathcal{R}$ is called a picture fuzzy ordering, or a partial picture fuzzy ordering, if it is reflexive, perfect antisymmetric and transitive.

A picture fuzzy poset (PF-poset, for short) is a set with a picture fuzzy partial order relation.

Example 3.1. Let $X = \{x_1, x_2, x_3\}$ and let $\mathcal{R} \in PFR(X)$ be given by

$\mathcal{R}$	x_1	x_2	x_3
x_1	$(1.00, 0.00, 0.00)$	$(0.30, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_2	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_3	$(0.00, 0.30, 0.20)$	$(0.00, 0.00, 0.00)$	$(1.00, 0.00, 0.00)$

It is obvious that $(X, \mathcal{R})$ is a picture fuzzy ordering.

Definition 3.5. A picture fuzzy ordering $\mathcal{R}$ is linear (or total) on X if for any $x, y \in X$, either $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ or $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$.

A linearly PF-poset $(X, \mathcal{R})$ or a picture fuzzy chain is a PF-poset $(X, \mathcal{R})$ in which $\mathcal{R}$ is linear.

Lemma 3.1. *Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. If $\mathcal{R}$ is a picture fuzzy ordering relation on X, then $S(\mathcal{R})$ and $\ker(\mathcal{R})$ are order relations on X.*

Proof. Suppose that $(X, \mathcal{R})$ is a PF-poset. The reflexivity of $S(\mathcal{R})$ is direct. Since $\mathcal{R}(x,x) = (1,0,0) \succ 0_{\mathbb{D}^*}$ for all $x \in X$, then $(x,x) \in S(\mathcal{R})$.

For the antisymmetry, suppose that $(x,y), (y,x) \in S(\mathcal{R})$, i.e., $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$. Then, from the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$.

Concerning the transitivity, suppose that $(x, y), (y, z) \in S(\mathcal{R})$, that is, $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$. Since $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$, using Proposition 2.1 (4), we obtain $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ for all $x, z \in X$, thus $(x, z) \in S(\mathcal{R})$.

Therefore, $S(\mathcal{R})$ is a partial order relation on X.

Similarly, we obtain the same result for $\ker(\mathcal{R})$. □

Remark 3.2. The fact that $S(\mathcal{R})$ is a partial order relation on X does not imply that $\mathcal{R}$ is a picture fuzzy ordering relation on X.

Example 3.2. Let $X = a, b$. Consider the relation $\mathcal{R}$ defined on X by

$\mathcal{R}$	a	b
a	$(0.10, 0.30, 0.00)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.50, 0.03, 0.20)$

, its support is given by

$S(\mathcal{R})$	a	b
a	1	0
b	0	1

.

It is not difficult to see that $S(\mathcal{R})$ is a partial order relation on X but $\mathcal{R}$ is not a picture fuzzy ordering relation on X.

Proposition 3.1. *Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. $\mathcal{R}$ is a picture fuzzy ordering relation if and only if all cuts $\mathcal{R}_\alpha$ are order relations on X, for any $\alpha \in \mathbb{D}_0^*$.*

Proof. Let $\alpha \in \mathbb{D}_0^*$. Suppose $(X, \mathcal{R})$ is a PF-poset and let $x \in X$. Since $\mathcal{R}(x, x) = (1, 0, 0)$, then $\mathcal{R}(x, x) \succeq \alpha$, for all $\alpha \in \mathbb{D}_0^*$, so $(x, x) \in \mathcal{R}_\alpha$. Thus $\mathcal{R}_\alpha$ is reflexive.

Suppose that $(x, y), (y, x) \in \mathcal{R}_\alpha$, then $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, x) \succeq \alpha$. This implies that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. From the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$. Thus, $\mathcal{R}_\alpha$ is antisymmetric.

Suppose that $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Then, $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$.

From the transitivity of $\mathcal{R}$, we obtain $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \succeq \alpha$, this implies that $(x, z) \in \mathcal{R}_\alpha$. Thus, $\mathcal{R}_\alpha$ is transitive.

Hence, if $\mathcal{R}$ is a picture fuzzy ordering relation, then all cuts $\mathcal{R}_\alpha$ are order relations on X.

Conversely, assume that for all $\alpha \in \mathbb{D}_0^*$, $\mathcal{R}_\alpha$ is a partial ordering relation on X.

If $\alpha = 1_{\mathbb{D}^*} \in \mathbb{D}_0^*$, then $(x, x) \in \mathcal{R}_{1_{\mathbb{D}^*}}$ for all $x \in X$, i.e., $\mathcal{R}(x, x) = 1_{\mathbb{D}^*}$. Thus, $\mathcal{R}$ is reflexive.

Suppose that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. Then, there exist $\alpha, \beta \in \mathbb{D}_0^*$ such that $\mathcal{R}(x, y) = \alpha$ and $\mathcal{R}(y, x) = \beta$. Put $\gamma = \alpha \curlywedge \beta$.

It is obvious that $\mathcal{R}(x, y) \succeq \gamma$ and $\mathcal{R}(x, y) \succeq \gamma$, that is, $(x, y) \in \mathcal{R}_\gamma$ and $(y, x) \in \mathcal{R}_\gamma$. From the antisymmetry of $\mathcal{R}_\gamma$ we obtain $x = y$. Thus, $\mathcal{R}$ is perfect antisymmetric.

Let $x, y, z \in X$, and put $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$.

If $\alpha = 0_{\mathbb{D}^*}$, it is obvious that $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \preceq \mathcal{R}(x, z)$.

If $\alpha \succ 0_{\mathbb{D}^*}$, then we have $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$, that is, $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Using the transitivity of $\mathcal{R}_\alpha$ we obtain $(x, z) \in \mathcal{R}_\alpha$, i.e., $\mathcal{R}(x, z) \succeq \alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$. Thus, $\mathcal{R}$ is transitive. Hence, $\mathcal{R}$ is a picture fuzzy ordering. □

4. Picture Fuzzy Lattices

In the following, we first extend the notions of fuzzy lattice and intuitionistic fuzzy lattice studied in [2–5, 19, 22, 33], to picture fuzzy cases. As a consequence, we extend some results in this direction.

Definition 4.1. Suppose that $(X, \mathcal{R})$ is a PF-poset and E is a non-empty subset of X. An element $u \in X$ is an upper bound of E if for all $x \in E$, $\mathcal{R}(x, u) \succ 0_{\mathbb{D}^*}$. An upper bound u_0 of E is the least upper bound of E if for any upper bound u of $E, \mathcal{R}(u_0, u) \succ 0_{\mathbb{D}^*}$. An element $l \in X$ is a lower bound of E if for all $x \in E$, $\mathcal{R}(l, x) \succ 0_{\mathbb{D}^*}$. A lower bound l_0 of E is the greatest lower bound of E if for any lower bound l of E, $\mathcal{R}(l, l_0) \succ 0_{\mathbb{D}^*}$.

$x \sqcup y$ and $x \sqcap y$ denote respectively the least upper bound and the greatest lower bound of $\{x, y\}$.

Remark 4.1. Note that the least upper bound and the greatest lower bound of any picture fuzzy subset are unique when they exist. (The uniqueness comes from the perfect antisymmetry of $\mathcal{R}$).

Definition 4.2. A PF-poset $(X, \mathcal{R})$ is a picture fuzzy lattice (PFL, for short) if and only if for all $x, y \in X$, $x \sqcup y$ and $x \sqcap y$ exist.

Example 4.1. In Example 3.1, $(X, \mathcal{R})$ is a PFL. Indeed, $x_1 \sqcap x_2 = x_1$, $x_1 \sqcap x_3 = x_3$ and $x_2 \sqcap x_3 = x_3$. Also, $x_1 \sqcup x_2 = x_2$, $x_1 \sqcup x_3 = x_1$ and $x_2 \sqcup x_3 = x_2$.

The boundaries' remainders are obtained using commutativity and idempotence.

The proofs of the following two propositions are straightforward.

Proposition 4.1. *For a PFL $(X, \mathcal{R})$, let $x, y, z \in X$. Then,*

(1) $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$;
(2) $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$;
(3) $\mathcal{R}(z, x) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(z, y) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(z, x \sqcap y) \succ 0_{\mathbb{D}^*}$;
(4) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcup y = y$;
(5) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcap y = x$;
(6) *if* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$, *then* $\mathcal{R}(x \sqcap y, x \sqcap z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(x \sqcup y, x \sqcup z) \succ 0_{\mathbb{D}^*}$.

Proposition 4.2. *For a PFL $(X, \mathcal{R})$, let $x, y, z \in X$. Then,*

(1) $x \sqcup x = x$, $x \sqcap x = x$;
(2) $x \sqcup y = y \sqcup x$, $x \sqcap y = y \sqcap x$;
(3) $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$;
(4) $(x \sqcup y) \sqcap x = x$, $(x \sqcap y) \sqcup x = x$.

We now turn to a characterization of the relationship between a PFL and its level sets.

Proposition 4.3. *For a PF-poset $(X, \mathcal{R})$. If $(X, \mathcal{R}_\alpha)$ are lattices for all $\alpha \in \mathbb{D}_0^*$, then $(X, \mathcal{R})$ is a PFL.*

Proof. For a PF-poset $(X, \mathcal{R})$, assume that $(X, \mathcal{R}_\alpha)$ are crisp lattices. Let $\alpha \in \mathbb{D}_0^*$. For all $x, y \in X$, there exists $u_0 \in X$, such that $(x, u_0) \in \mathcal{R}_\alpha$, $(y, u_0) \in \mathcal{R}_\alpha$, and $(u_0, u) \in \mathcal{R}_\alpha$, for every upper bound u of $\{x, y\}$. Then, there exists u_0 such that $\mathcal{R}(x, u_0) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(y, u_0) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(u_0, u) \succ 0_{\mathbb{D}^*}$ for all upper bound u of $\{x, y\}$. Hence there exists a least upper bound u_0 of $\{x, y\}$ on $(X, \mathcal{R})$. In a similar way, there exists a greatest lower bound l_0 of $\{x, y\}$ on $(X, \mathcal{R})$. Thus, $(X, \mathcal{R})$ is a PFL. □

Remark 4.2. If $(X, \mathcal{R})$ is a PFL, then $(X, \mathcal{R}_\alpha)$ may not be a crisp lattice. Indeed.

Example 4.2. Let $(X, \mathcal{R})$ be a PFL, where $X = \{a, b, c, d\}$ and $\mathcal{R}$ defined by the following table

$\mathcal{R}$	a	b	c	d
a	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
b	$(0.30, 0.00, 0.40)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
c	$(0.50, 0.20, 0.10)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
d	$(0.70, 0.00, 0.20)$	$(0.4., 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(1.00, 0.00, 0.00)$

Consider the relation $\mathcal{R}_{(0.5,0.2,0.3)}$

$\mathcal{R}_{(0.5,0.2,0.3)}$	a	b	c	d
a	1	0	0	0
b	0	1	0	0
c	1	0	1	0
d	1	0	0	1

$(X, \mathcal{R}_{0.5,0.2,0.3})$ is a poset.

It is not difficult to see that $\{b, c\}$ has neither the least upper bound nor the greatest lower bound. So, $(X, \mathcal{R}_{0.5,0.2,0.3})$ is a poset but not a crisp lattice.

5. Picture Fuzzy Filters in a Lattice

This section introduces picture fuzzy sub-lattice and picture fuzzy filter in a crisp lattice by inspiring those of fuzzy and intuitionistic fuzzy case. We accomplish this by generalizing some existing results in the Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5, 33]) to the picture fuzzy case.

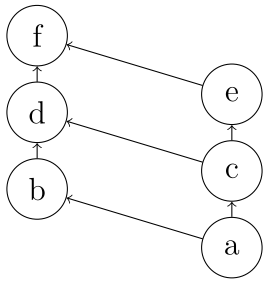
In the following, the symbol $\preceq$ indicates the picture fuzzy ordering defined on the set $\mathbb{D}^*$ as seen in Definition 2.4 and $\succeq$ is its dual.

Definition 5.1. For a crisp lattice $(X, \leqslant, \wedge, \vee)$, let E be a picture fuzzy subset on X. Then, E is called a picture fuzzy sublattice of $(X, \leqslant, \wedge, \vee)$, if for all $x, y \in X$

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;
(ii) $E(x \vee y) \succeq E(x) \curlywedge E(y)$.

Example 5.1. Let $X = \{a, b, c, d, e, f\}$ and let $(X, \leqslant)$ be the lattice given by the following table and represented by the given Hasse diagram .

$\mathcal{R}$	a	b	c	d	e	f
a	1	1	1	1	1	1
b	0	1	0	1	0	1
c	0	0	1	1	1	1
d	0	0	0	1	0	1
e	0	0	0	0	1	1
f	0	0	0	0	0	1



The picture fuzzy subset E defined on X by

X	$E(x)$
a	$(0.10, 0.50, 0.40)$
b	$(0.20, 0.10, 0.40)$
c	$(0.10, 0.20, 0.30)$
d	$(0.20, 0.00, 0.40)$
e	$(0.10, 0.20, 0.30)$
f	$(0.40, 0.40, 0.10)$

is a picture fuzzy sublattice of $(X, \leqslant)$.

Definition 5.2. Let $(X, \leq, \wedge, \vee)$ be a crisp lattice. A picture fuzzy subset E of X is called a picture fuzzy filter (PFF, for short) of X, if for all $x, y \in X$

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;
(ii) $E(x \vee y) \succeq E(x) \curlyvee E(y)$.

Example 5.2. Consider the lattice $(X, \leqslant)$ given in Example 5.1. The picture fuzzy subset E on X defined by

X	$E(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
c	$(0.10, 0.30, 0.50)$
d	$(0.10, 0.20, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

is a PFF of $(X, \leqslant)$.

Remark 5.1. Every PFF is a picture fuzzy sublattice. However, the opposite is not correct.

Example 5.3. The PFF E given in Example 5.2 is a picture fuzzy sublattice.

But the picture fuzzy sublattice E given in Example 5.1 is not a PFF, since $E(c \vee b) = (0.2, 0, 0.4)$, $E(c) \curlyvee E(b) = (0.2, 0, 0.3)$ and $(0.2, 0, 0.4) \not\succeq (0.2, 0, 0.3)$.

Proposition 5.1. *Let $(X, \leq)$ be a crisp lattice. If E, F are PFFs (resp. sublattices) of $(X, \leqslant)$, then $E \cap F$ is also a PFF (resp. sublattice) of $(X, \leqslant)$.*

Proof. Let E and F be two PFFs of $(X, \leqslant)$.

We have $(E \cap F)(x) = E(x) \curlywedge F(x)$. Then,

$$
\begin{aligned}
(E \cap F)(x \wedge y) &= E(x \wedge y) \curlywedge F(x \wedge y) \\
&\succeq (E(x) \curlywedge E(y)) \curlywedge (F(x) \curlywedge F(y)) \\
&= (E(x) \curlywedge F(x)) \curlywedge (E(y) \curlywedge F(y)) \\
&= (E \cap F)(x) \curlywedge (E \cap F)(y), \\
(E \cap F)(x \vee y) &= E(x \vee y) \curlywedge F(x \vee y) \\
&\succeq (E(x) \curlyvee E(y)) \curlywedge (F(x) \curlyvee F(y)) \\
&\succeq (E(x) \curlywedge F(x)) \curlyvee (E(x) \curlywedge F(y)) \\
&\quad \curlyvee (E(y) \curlywedge F(x)) \curlyvee (E(y) \curlywedge F(y)) \\
&\succeq (E(x) \curlywedge F(x)) \curlyvee (E(y) \curlywedge F(y)) \\
&= (E \cap F)(x) \curlyvee (E \cap F)(y).
\end{aligned}
$$

Hence, $E \cap F$ is a PFFs of $(X, \leqslant)$.

The same argument can be applied to the picture fuzzy sublattice case. □

Remark 5.2. The union of two PFFs (resp. sublattices) of $(X, \leqslant)$ need not be a PFF (resp. sublattice) of $(X, \leqslant)$.

Example 5.4. Consider the lattice $(X, \leqslant)$ given in Example 5.1. Let E be the PFF given in Example 5.2 and consider the PFF E^* on X by

X	$E^*(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.10, 0.40, 0.50)$
c	$(0.00, 0.00, 1.00)$
d	$(0.30, 0.10, 0.40)$
e	$(0.00, 0.00, 1.00)$
f	$(0.30, 0.20, 0.40)$

Then,

X	$(E \cup E^*)(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.10, 0.40, 0.50)$
c	$(0.10, 0.30, 0.50)$
d	$(0.30, 0.10, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

Since $(E \cup E^*)(c \wedge b) = 0_{\mathbb{D}^*}$, $(E \cup E^*)(c) \curlywedge (E \cup E^*)(b) = (0.1, 0.3, 0.5)$ and $(0, 0, 1)$ $\not\succeq (0.1, 0.3, 0.5)$, this implies that $E \cup E^*$ is not a PFF.

Proposition 5.2. *Let $(X, \leq)$ be a crisp lattice. If E is a PFF of $(X, \leqslant)$, then $S(E)$ and $\ker(E)$ are crisp filters on $(X, \leqslant)$.*

Recall that $S(E) = \{x \in X \mid E(x) \succ 0_{\mathbb{D}^*}\}$, $\ker(E) = \{x \in X \mid E(x) = 1_{\mathbb{D}^*}\}$.

Proof. Suppose that E is a PFF of $(X, \leqslant)$.

(i) Let $x \in S(E)$ and $y \in X$ such that $x \leqslant y$, it follows that $E(x) \succ 0_{\mathbb{D}^*}$ and $x \vee y = y$. Since $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succ 0_{\mathbb{D}^*}$, then $y \in S(E)$.

(ii) Let $x, y \in S(E)$. We prove that $x \wedge y \in S(E)$. $x, y \in S(E)$ implies $E(x) \succ 0_{\mathbb{D}^*}$ and $E(y) \succ 0_{\mathbb{D}^*}$. Since $E(x \wedge y) \succeq E(x) \curlywedge E(y)$, then according to Proposition 2.1(4), $E(x \wedge y) \succ 0_{\mathbb{D}^*}$. Hence, $x \wedge y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Theorem 5.1. *Let $(X, \leq)$ be a crisp lattice. A picture fuzzy subset E of X is a PFF if and only if its α-cuts are filters of $(X, \leqslant)$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_0^*$.*

Proof. Suppose that E is a PFF on $(X, \leqslant)$ and prove that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^*$.

(i) Let $x \in E_\alpha$ and $y \in X$ such that $x \leqslant y$. Then, $E(x) \succeq \alpha$ and $x \vee y = y$. It follows that $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succeq \alpha$. Hence, $y \in E_\alpha$.

(ii) Let $x, y \in E_\alpha$. Then, it holds that $E(x) \succeq \alpha$ and $E(y) \succeq \alpha$. Since $E(x \wedge y) \succeq E(x) \curlywedge E(y) \succeq \alpha$, thus $x \wedge y \in E_\alpha$.

Conversely, suppose that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^*$, and show that E is a PFF on $(X, \leqslant)$.

(i) Let $x, y \in X$ and let $\alpha \in \mathbb{D}_0^*$. Put $E(x) \curlywedge E(y) = \alpha$. We have $E(x) \succeq E(x) \curlywedge E(y) = \alpha$ and $E(y) \succeq E(x) \curlywedge E(y) = \alpha$, that is, $x, y \in E_\alpha$. Since E_α is a filter, then $x \wedge y \in E_\alpha$. This implies that $E(x \wedge y) \succeq \alpha = E(x) \curlywedge E(y)$.

(ii) Let $x, y \in X$ and let $\alpha, \beta \in \mathbb{D}_0^*$ such that $E(x) = \alpha$ and $E(y) = \beta$. Then $x \in E_\alpha$ and $y \in E_\beta$. Since E_α and E_β are filters, it follow that $x \vee y \in E_\alpha$ and $x \vee y \in E_\beta$, i.e., $E(x \vee y) \succeq \alpha$ and $E(x \vee y) \succeq \beta$.
Hence, $E(x \vee y) \succeq \alpha \curlyvee \beta = E(x) \curlyvee E(y)$.

□

6. Filters in a Picture Fuzzy Lattice

This section extends the notion of a crisp filter and fuzzy filter in a fuzzy lattice [29] to a crisp filter and a PFF in a PFL as well as providing more characterizations of them.

Definition 6.1. For a PFL $(X, \mathcal{R}, \sqcap, \sqcup)$, let E be a non-empty subset of X. E is a crisp filter on $(X, \mathcal{R}, \sqcap, \sqcup)$ if the following conditions are satisfied.

(F1) If $x \in X, y \in E$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$, then $x \in E$.

(F2) If $x, y \in E$, then $x \sqcap y \in E$.

Definition 6.2. For a PFL $(X, \mathcal{R}, \sqcap, \sqcup)$, let E be a picture fuzzy subset of X. E is a PFF on $(X, \mathcal{R}, \sqcap, \sqcup)$ if it satisfies the following conditions:

(PFF1) $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, for all $x, y \in X$;

(PFF2) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ implies $E(x) \preceq E(y)$, for all $x, y \in X$.

Proposition 6.1. *Let* $(X, \mathcal{R})$ *be a picture fuzzy lattice. If* E *and* F *are two PFFs of* $(X, \mathcal{R})$ *, then* $E \cap F$ *is a PFF of* $(X, \mathcal{R})$ *.*

Proof. Suppose that E and F are two PFFs of $(X, \mathcal{R})$. Then, for all $x, y \in X$,

$$\begin{aligned}(E \cap F)(x \sqcap y) &= E(x \sqcap y) \curlywedge F(x \sqcap y) \\ &\succeq (E(x) \curlywedge E(y)) \curlywedge (F(x) \curlywedge F(y)) \\ &= (E(x) \curlywedge F(x)) \curlywedge (E(y) \curlywedge F(y)) \\ &= (E \cap F)(x) \curlywedge (E \cap F)(y).\end{aligned}$$

On the other hand, if $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $E(x) \preceq E(y)$ and $F(x) \preceq F(y)$, this implies that $E(x) \curlywedge F(x) \preceq E(y) \curlywedge F(y)$. That is, $(E \cap F)(x) \preceq (E \cap F)(y)$. □

Remark 6.1. The union of two PFFs is not always a PFF, as demonstrated in the following example.

Example 6.1. Let $X = \{0, a, b, 1\}$ and assume that $(X, \mathcal{R})$ is the lattice given by the following table

$\mathcal{R}$	0	a	b	1
0	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.50)$	$(0.30, 0.10, 0.50)$	$(0.30, 0.00, 0.40)$
a	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.30, 0.00, 0.40)$
b	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.40)$
1	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$

.

We define the two PFFs E_1 and E_2 on $(X, \mathcal{R})$ by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

.

Then,

X	$(E_1 \cap E_2)(x)$	$(E_1 \cup E_2)(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$	$(0.20, 0.40, 0.30)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

.

It is obvious that $E_1 \cap E_2$ is a PFF of $(X, \mathcal{R})$, but $E_1 \cup E_2$ is not a PFF. Indeed, $(E_1 \cup E_2)(a \sqcap b) = 0_{\mathbb{D}^*}$, $(E_1 \cup E_2)(a) \curlywedge (E_1 \cup E_2)(b) = (0.1, 0.6, 0.3)$ and $(0, 0, 1) \not\succeq (0.1, 0.6, 0.3)$.

Proposition 6.2. *For a PFL* $(X, \mathcal{R})$*, let* E *be a picture fuzzy subset on* X*. If* E *is a PFF of* $(X, \mathcal{R})$ *, then* $S(E)$ *and* $\ker(E)$ *are crisp filters of* $(X, \mathcal{R})$ *.*

Proof. Suppose that E is a PFF of $(X, \mathcal{R})$ and let $x, y \in X$.

(F1) If $x \in S(E)$ and $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ implies $E(y) \succeq E(x) \succ 0_{\mathbb{D}^*}$. Thus, $y \in S(E)$.

(F2) If $x, y \in S(E)$, then $E(x) \succ 0_{\mathbb{D}^*}$ and $E(y) \succ 0_{\mathbb{D}^*}$. Since $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, according to Proposition 2.1 (4), $E(x \sqcap y) \succ 0_{\mathbb{D}^*}$, i.e., $x \sqcap y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Proposition 6.3. *For a PFL* $(X, \mathcal{R})$*, let* E *be a picture fuzzy subset on* X*.* E *is a PFF of* $(X, \mathcal{R})$ *if and only if its* α*-cuts are crisp filters of* $(X, \mathcal{R})$*.*

Proof. Let $x, y \in X$ and let $\alpha \in \mathbb{D}_0^*$. Suppose that E is a PFF of $(X, \mathcal{R})$.

(F1) If $x \in E_\alpha$ and $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $E(x) \succeq \alpha$ and $E(y) \succeq E(x) \succeq \alpha$. Hence, $y \in E_\alpha$.

(F2) If $x, y \in E_\alpha$, then $E(x) \succeq \alpha$ and $E(y) \succeq \alpha$, then $E(x \sqcap y) \succeq E(x) \curlywedge E(y) \succeq \alpha$, That is, $x \sqcap y \in E_\alpha$.

Conversely, suppose that E_α are crisp filters of $(X, \mathcal{R})$, for all $\alpha \in \mathbb{D}_0^*$.

(PFF1) Put $E(x) \curlywedge E(y) = \beta$. It is obvious that, if $E(x) \curlywedge E(y) = 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$. When β is greater than $0_{\mathbb{D}^*}$, we have $E(x) \succeq \beta$ and $E(y) \succeq \beta$ it follows that $x, y \in E_\beta$. Then $x \sqcap y \in E_\beta$. Hence, $E(x \sqcap y) \succeq \beta = E(x) \curlywedge E(y)$.

(PFF2) Suppose that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and put $E(x) = \gamma$. The case $\gamma = 0_{\mathbb{D}^*}$ is trivially. When γ is greater than $0_{\mathbb{D}^*}$, then, $x \in E_\gamma$ implies $y \in E_\gamma$. Hence, $E(y) \succeq \gamma = E(x)$. □

Now, we give some characterizations of PFFs of a PFL.

Theorem 6.1. *E is a PFF of a PFL $(X, \mathcal{R})$ if and only if it satisfies* (PFF1) *and*

(PFF3) $E(x) \succeq E(x \sqcap y) \curlywedge E(y)$*, for all* $x, y \in X$*.*

Proof. Suppose that E is a PFF of $(X, \mathcal{R})$. It suffices to prove (PFF3).

Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, then according to (PFF1) and (PFF2), $E(x) \succeq E(x \sqcap y) \succeq E(x \sqcap y) \curlywedge E(y)$.

Conversely, suppose that (PFF1) and (PFF3) are satisfied. If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y = x$ implies $E(y) \succeq E(x \sqcap y) \curlywedge E(x) = E(x)$. □

Theorem 6.2. *E is a PFF of a PFL $(X, \mathcal{R})$ if and only if it satisfies*

(PFF4) $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$ *implies* $E(z) \succeq E(x) \curlywedge E(y)$*, for all* $x, y, z \in X$*.*

Proof. Let $x, y, z \in X$. Suppose that E is a PFF of $(X, \mathcal{R})$.

If $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$, then according to (PFF1) and (PFF2), $E(z) \succeq E(x \sqcap y) \succeq E(x) \curlywedge E(y)$.

Conversely, suppose that (PFF4) is satisfied. Then the following hold.

(PFF1) Since $\mathcal{R}(x \sqcap y, x \sqcap y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$.

(PFF2) If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x \sqcap x, y) \succ 0_{\mathbb{D}^*}$. It follows that $E(y) \succeq E(x) \curlywedge E(x)$, that is, $E(y) \succeq E(x)$. □

Theorem 6.3. *E is a PFF of a PFL $(X, \mathcal{R})$ if and only if it satisfies* (PFF1) *and*

(PFF5) $E(x \sqcup y) \succeq E(x)$*, for all* $x, y \in X$*.*

Proof. Let $x, y \in X$. Suppose that E is a PFF of $(X, \mathcal{R})$. It suffices to prove (PFF5). Since $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcup y) \succeq E(x)$.

Conversely, suppose that (PFF1) and (PFF5) are satisfied. If $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $x \sqcup y = y$. Hence, $E(x) \preceq E(x \sqcup y) = E(y)$. □

Theorem 6.4. *E is a PFF of a PFL $(X,\mathcal{R})$ if and only if it satisfies*

(PFF6) *for all $x,y \in X, E(x \sqcap y) = E(x) \curlywedge E(y)$.*

Proof. Suppose that E is a PFF of $(X,\mathcal{R})$.

In view of the definition of a PFF, it suffices to show that $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Let $x,y \in X$. Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \preceq E(x)$ and $E(x \sqcap y) \preceq E(y)$. Hence $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Conversely, suppose that $E(x \sqcap y) = E(x) \curlywedge E(y)$, for all $x,y \in X$.

- $E(x \sqcap y) = E(x) \curlywedge E(y)$ implies (PFF1).
- If $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y = x$. Thus $E(x) = E(x \sqcap y) = E(x) \curlywedge E(y)$, that is, $E(x) \preceq E(y)$. □

7. Picture Fuzzy Prime Filters in a Picture Fuzzy Lattice

Prime filters, as well as picture fuzzy prime filters of a PFL, are the topic of this section's discussion.

Definition 7.1. Suppose that $(X,\mathcal{R},\sqcap,\sqcup)$ is a PFL and E is a crisp filter of $(X,\mathcal{R})$. Then E is called a crisp prime filter if for all $x,y \in X$, $x \sqcup y \in E$ imply that $x \in E$ or $y \in E$.

Definition 7.2. Suppose that $(X,\mathcal{R},\sqcap,\sqcup)$ is a PFL and E is a PFF of $(X,\mathcal{R})$. E is called a picture fuzzy prime filters (PFPF, for short) if for any $x,y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

Remark 7.1. The intersection of two PFPFs of $(X,\mathcal{R})$ does not be necessarily a PFPF of $(X,\mathcal{R})$.

Example 7.1. Consider the lattice $(X,\mathcal{R})$ given in Example 6.1.

Let E_1, E_2 be two PFPFs on $(X,\mathcal{R})$ defined by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.20, 0.40, 0.30)$	$(0.10, 0.50, 0.20)$

.

Then,

X	$(E_1 \cap E_2)(x)$
0	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
1	$(0.10, 0.60, 0.30)$

.

It is easy to check that $E_1 \cap E_2$ is a PFF. But $(E_1 \cap E_2)(a \sqcup b) = (E_1 \cap E_2)(1) = (0.10, 0.60, 0.30)$. In the other hand, $(E_1 \cap E_2)(a) \curlyvee (E_1 \cap E_2)(b) = (0,0,1) \curlyvee (0,0,1) = (0,0,1) \neq (0.10, 0.60, 0.30)$. Hence, the PFF $E_1 \cap E_2$ is not prime.

Proposition 7.1. *Suppose that* $(X, \mathcal{R})$ *is a PFL. If* E *is a PFPF on* $(X, \mathcal{R})$*, then* $S(E)$ *and* $\ker(E)$ *are crisp prime filters on* $(X, \mathcal{R})$*.*

Proof. Suppose that E is a PFPF on $(X, \mathcal{R})$. From Proposition 6.2, it holds that $S(E)$ is a filter on $(X, \mathcal{R})$. We then demonstrate that $S(E)$ is prime.

Let $x, y \in X$. If $x \vee y \in S(E)$, then $E(x \vee y) \succ 0_{\mathbb{D}^*}$. Since $E(x \vee y) = E(x) \curlyvee E(y)$, this implies from Proposition 2.1 (6) that $E(x) \succ 0_{\mathbb{D}^*}$ or $E(y) \succ 0_{\mathbb{D}^*}$. Hence, either $x \in S(E)$ or $y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Theorem 7.1. *Suppose that* $(X, \mathcal{R})$ *is a PFL and let* E *be a picture fuzzy subset on* X*. If for all* $\alpha \in \mathbb{D}_0^*$*,* E_α *are prime filters on* $(X, \mathcal{R})$*, then* E *is a PFPF on* $(X, \mathcal{R})$*.*

Proof. Suppose that E_α are prime filters on $(X, \mathcal{R})$, for all $\alpha \in \mathbb{D}_0^*$. From Proposition 6.3, E is a PFF on $(X, \mathcal{R})$. It remains to show the primality of E, i.e., for all $x, y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

Put $E(x \sqcup y) = \alpha$, then $x \sqcup y \in E_\alpha$. Since E_α is a prime filter, this implies that $x \in E_\alpha$ or $y \in E_\alpha$. Hence, $E(x) \curlyvee E(y) \succeq \alpha = E(x \sqcup y)$.

In contrast, since $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}$ these imply from (PFF1) that $E(x) \preceq E(x \sqcup y)$ and $E(y) \preceq E(x \sqcup y)$. Hence, $E(x) \curlyvee E(y) \preceq E(x \sqcup y)$. □

Remark 7.2. Unlike the fuzzy case, the converse implication in Proposition 7.1 is not true.

Example 7.2. Let $(X, \mathcal{R})$ be the lattice given as follows

$\mathcal{R}$	0	a	b	c	1
0	$1_{\mathbb{D}^*}$	$(0.20, 0.01, 0.70)$	$(0.40, 0.20, 0.30)$	$(0.50, 0.10, 0.30)$	$1_{\mathbb{D}^*}$
a	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.40, 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(0.70, 0.00, 0.20)$
b	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$(0.30, 0.00, 0.40)$
c	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.50, 0.20, 0.10)$
1	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$

The following table represented two picture fuzzy subsets: E_1 and its support, and E_2 and its kernel

X	$E_1(x)$	$E_2(x)$	$S(E_1)(x)$	$\ker(E_2)(x)$
0	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	0	0
a	$(0.10, 0.60, 0.30)$	$0_{\mathbb{D}^*}$	1	0
b	$(0.10, 0.40, 0.10)$	$(0.10, 0.30, 0.20)$	1	0
c	$(0.20, 0.10, 0.30)$	$1_{\mathbb{D}^*}$	1	1
1	$(0.20, 0.60, 0.10)$	$1_{\mathbb{D}^*}$	1	1

It is easy to see that $S(E_1)$ and $\ker(E_2)$ are crisp prime filters on $(X, \mathcal{R})$, but E_1 and E_2 are not PFPFs on $(X, \mathcal{R})$.

8. Conclusion and Future Work

After refining the $(\mathbb{D}^*, \preceq)$ laws associated with $\preceq$ so that $\mathbb{D}^*$ is a complete lattice, and investigating the algebraic structure of $\mathbb{D}^*$ (A study made in a paper that will appear in TWS), it is customary to study picture fuzzy lattices, picture fuzzy sub-lattice and picture fuzzy filter in a crisp lattice by inspiring those of fuzzy and intuitionistic fuzzy case in this paper. Also, we have studied prime filters and picture fuzzy prime filters of a picture fuzzy lattice.

In future work, we plan to characterize principal picture fuzzy filters (resp. picture fuzzy ideals) on a picture fuzzy lattice. Also, we intend to introduce the notion for picture fuzzy homomorphism and picture fuzzy isomorphism and do some picture fuzzy isomorphism theorems of picture fuzzy lattices. In particular, we characterize some quotients of picture fuzzy lattice classes by their picture fuzzy ideals. Also, we study t-picture fuzzy lattices (picture fuzzy lattices w.r.t. a triangular norm) and picture fuzzy t-filters.

Acknowledgements. The authors would like to thank the Editor and the anonymous referees for their constructive comments, which helped to improve this paper. This research work is supported by the The General Direction of Scientific Research and Technological Development (DGRSDT)- Algeria.

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