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ON THE ESTRADA INDEX OF POINT ATTACHING STRICT k-QUASI TREE GRAPHS

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Abstract. Let $G = (V, E)$ be a finite and simple graph with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its eigenvalues. The Estrada index of G is $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. For a positive integer k, a connected graph G is called strict k-quasi tree if there exists a set U of vertices of size k such that $G \setminus U$ is a tree and this is as small as possible with this property. In this paper, we define point attaching strict k-quasi tree graphs and obtain the graph with minimum Estrada index among point attaching strict k-quasi tree graphs with k even cycles.

1. Introduction

Let $G = (V(G), E(G))$ be a finite and simple graph of order n, where by $V(G)$ and $E(G)$ we denote the set of vertices and edges, respectively. Let $A(G)$ be the adjacency matrix of G, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The Estrada index of G is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$

which was first proposed by Estrada in 2000 [6]. We refer reader to [7, 8, 15, 16] for multiple applications of Estrada index in various fields, for example in network science and biochemistry. The results for trees can be found in [3, 10, 13, 19]. Gutman approximated the Estrada index of cycles and paths in [9]. The unicyclic graphs with maximum and minimum Estrada index have been determined in [5]. Recently, the Esrada index of the cactus graphs in which every block is a triangle, has been characterized in [11, 12].

A connected graph G is called *quasi tree* if there exists $v_0 \in V(G)$ such that $G \setminus \{v_0\}$ is a tree. Lu in [14] has determined the Randić index of quasi trees. The Harary index

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of quasi tree graphs and generalized quasi tree graphs are presented in [18]. A *strict k-quasi tree graph* G is a connected graph which is not a tree, and k is the smallest positive integer such that there exists a k-element subset U of vertices for which $G \setminus U$ is a tree.

Let G be a connected graph constructed from pairwise disjoint connected graphs $G_1, G_2, \ldots, G_d$ as follows: select a vertex of G_1,, a vertex of G_2,, and identify these two vertices. Then continue in this manner inductively. More precisely, suppose that we have already used $G_1, G_2, \ldots, G_i$ in the construction, where $2 \leq i \leq d-1$. Then select a vertex in the already constructed graph (which may in particular be one of the already selected vertices) and a vertex G_{i+1}; and identify these two vertices. Note that the graph G constructed in this way has a tree-like structure, the G_i's being its building stones. We will briefly say that G is obtained by *point attaching* from $G_1, G_2, \ldots, G_d$ and that G_i's are the primary subgraphs of G [4].

A graph G is said to be point attaching strict k-quasi, if it is constructed from primary subgraphs $G_1, G_2, \ldots, G_d$ where each primary subgraph G_i is a strict k_i-quasi tree graph for each $1 \leq i \leq d$, and $k = \sum_{i=1}^{d} k_i$.

In this paper we study the Estrada index of point attaching strict k-quasi graphs.

2. Preliminaries

For $\ell \in \mathbb{N} \cup \{0\}$, let $S_\ell(G) = \sum_{i=1}^{n} \lambda_i^\ell$ be the ℓ^{th} spectral moment of G, which is equal to the number of closed walks of length ℓ in G [2]. For every graph G, we have $S_0(G) = n, S_1(G) = \mathbf{C}, S_2(G) = 2m, S_3(G) = 6\mathbf{D}$, and $S_4(G) = 2\sum_{i=1}^{n} d_i^2 - 2m + 8\mathbf{Q}$, where $n, \mathbf{C}, m, \mathbf{D}, \mathbf{Q}$ denote the number of vertices, the number of loops, the number of edges, the number of triangles and the number of quadrangles in G, respectively and $d_i = d_i(G)$ is the degree of vertex v_i in G [2]. Bearing in mind the Taylor expansion of e^x, we have the following equation for the Estrada index of graph G,

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i} = \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!} = \sum_{\ell=0}^{\infty} \frac{S_\ell(G)}{\ell!}. \tag{2.1}$$

It follows from Equation 2.1 that $EE(G)$ is a strictly monotonously increasing function of $S_\ell(G)$. Let G_1 and G_2 be two graphs. If $S_\ell(G_1) \leq S_\ell(G_2)$ holds for all positive integer ℓ, then $EE(G_1) \leq EE(G_2)$. Moreover, if the strict inequality $S_\ell(G_1) < S_\ell(G_2)$ holds for at least one value $\ell_0 \geq 0$, then $EE(G_1) < EE(G_2)$.

Recall that a sequence $a_0, a_1, \ldots, a_n$ of numbers is said to be *unimodal* if for some $0 \leq i \leq n$ we have $a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$, and this sequence is called *symmetric* if $a_i = a_{n-i}$ for $0 \leq i \leq n$ [17]. Thus a symmetric unimodal sequence $a_0, a_1, \ldots, a_n$ has its maximum at the middle term (n even) or middle two terms (n odd). Let A be the adjacency matrix of the graph G. It is well-known that the entry$(A^\ell)_{i,j}$ represents the number of walks of length ℓ from vertex v_i to vertex v_j [1]. Obviously, $(A^\ell)_{i,j} = (A^\ell)_{j,i}$ for undirected graphs.

Throughout this paper, $\Gamma(k)$ is a point attaching strict k-quasi tree graph with k even cycles (see Figure 1) and $M_\ell(G)$ denotes the set of closed walks of length ℓ in G, and we show that among all point attaching strict k-quasi tree graphs with k even cycles, $\Gamma(k)$ is the graph with minimum Estrada index.

3. The Number of Closed Walks of Length ℓ in $\Gamma(k)$

Let $M_\ell(k(c-1), i)$ denote the set of closed walks of length ℓ starting at the vertex v_i in $\Gamma(k)$ with k even cycles of length c and $|M_\ell(k(c-1), i)| = S_\ell(k(c-1), i)$ denote the number of closed walks of length ℓ starting at the vertex v_i in $\Gamma(k)$ (see Figure 1).

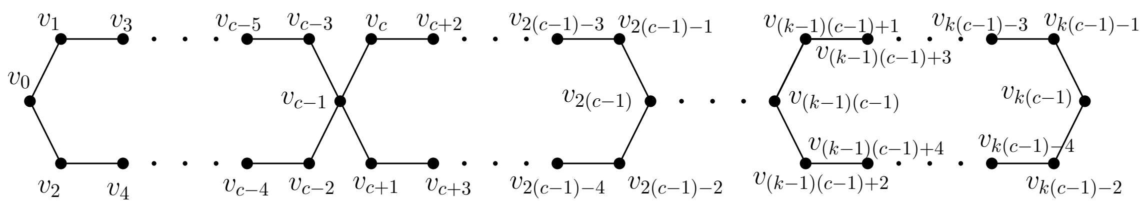

Figure 1. The graph $\Gamma(k)$.

Lemma 3.1. *The map $\varphi : V(\Gamma(k)) \longrightarrow V(\Gamma(k))$, given by $\varphi(v_i) = v_{k(c-1)-i}$ is an automorphism.*

Proof. One can easily see that φ is bijective. Let vertices v_i and v_j be adjacent. Then by definition of φ, we have the following cases.

(i) $\varphi(v_0) = v_{k(c-1)}$ and $\varphi(v_{k(c-1)}) = v_0$.

(ii) $i = t(c-1)$, $0 < t < k$. In this case $v_j \in \{v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\}$. Hence, $k(c-1) - i = k(c-1) - t(c-1) = (k-t)(c-1)$. This implies that $\varphi(v_i) = v_{(k-t)(c-1)}$.

We will only prove the case $v_j = v_{i-1}$. A similar argument can be used for other cases. If $v_j = v_{i-1}$, then $k(c-1) - j = k(c-1) - t(c-1) + 1 = (k-t)(c-1) + 1$. Hence $\varphi(v_j) = v_{(k-t)(c-1)+1}$ which is adjacent to $\varphi(v_i)$.

(iii) $i = t(c-1) + s$, $0 < t \leqslant k-1$, $1 \leqslant s \leqslant c-2$. In this case $v_j \in \{v_{i-2}, v_{i+2}\}$. Hence, $k(c-1) - j = k(c-1) - t(c-1) - s = (k-t)(c-1) - s$. This implies that $\varphi(v_i) = v_{(k-t)(c-1)-s}$.

If $v_j = v_{i-2}$, then $k(c-1) - t(c-1) - s + 2 = (k-t)(c-1) - s + 2$. Hence, $\varphi(v_j) = v_{(k-t)(c-1)-s+2}$ which is adjacent to $\varphi(v_i)$. The proof for case $v_j = v_{i+2}$ is similar. □

Corollary 3.1. *Let A be the adjacency matrix of the point attaching strict k-quasi tree graph $\Gamma(k)$. Then $(A^\ell)_{i,j} = (A^\ell)_{k(c-1)-i,k(c-1)-j}$ for $0 \leqslant i, j \leqslant k(c-1)$.*

Proof. This is an immediate consequence of Lemma 3.1. □

Lemma 3.2. *If $k \geqslant 2$ and t are integers and $0 \leqslant t \leqslant c-2$, then:*

$$\begin{aligned} S_\ell(k(c-1),t) &\leq S_\ell(k(c-1),t+(c-1)) \\ &\leq \cdots \leq S_\ell\left(k(c-1), t+\left(\left[\frac{k}{2}\right]-1\right)(c-1)\right) \\ &\leq S_\ell\left(k(c-1), t+\left[\frac{k}{2}\right](c-1)\right), \end{aligned}$$

where $\ell \geq c-1$. If $\ell \geq [\frac{k}{2}]$, then strict inequalities hold.

Proof. We prove every diagonal and the main diagonal of the matrix A^ℓ are unimodal. By Lemma 3.1, $(A^\ell)_{t,j} = (A^\ell)_{k(c-1)-t,k(c-1)-j}$. So we only need to show that the diagonals paralleling to the main diagonal increase for $t+j \leqslant k(c-1)$.

By induction on integer ℓ, we will show that for every $j \leqslant k(c-1)$ where $t+j+2c-2 \leqslant k(c-1)$, we have:

$$(A^\ell)_{t+c-1,j+c-1} \geqslant (A^\ell)_{t,j}.$$

By the definition of $\Gamma(k)$ we have $A_{t,j} = 1$ if and only if $A_{t+c-1,j+c-1} = 1$. Therefore, the result is hold for $\ell = 1$. Assume that the result holds for integer ℓ. There are four cases as follows.

Case 1: $t, j \equiv 0 \pmod{(c-1)}$.

Since the set of walks of length $\ell+1$ from v_t to v_j is in bijective correspondence with the set of walks of length ℓ from v_t to v_h adjacent to v_j, so

$$\begin{aligned} (A^{\ell+1})_{t+c-1,j+c-1} =& (A^\ell)_{t+c-1,j+c-2} + (A^\ell)_{t+c-1,j+c-3} + (A^\ell)_{t+c-1,j+c} \\ &+ (A^\ell)_{t+c-1,j+c+1}, \\ (A^{\ell+1})_{t,j} =& (A^\ell)_{t,j-1} + (A^\ell)_{t,j-2} + (A^\ell)_{t,j+1} + (A^\ell)_{t,j+2}. \end{aligned}$$

By the induction hypothesis, we have the following results:

$$\begin{aligned} (A^\ell)_{t+c-1,j+c-2} \geq& (A^\ell)_{t,j-1}, \\ (A^\ell)_{t+c-1,j+c} \geq& (A^\ell)_{t,j+1}, \quad \text{for } t+j+2 \leq k(c-1), \\ (A^\ell)_{t+c-1,j+c-3} \geq& (A^\ell)_{t,j-2}, \\ (A^\ell)_{t+c-1,j+c+1} \geq& (A^\ell)_{t,j+2}, \quad \text{for } t+j+2 \leq k(c-1). \end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$. In addition we will show that for $\ell \geq [k(c-1)/2]$ the strict inequalities hold.

For the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)$ and $(r-1)(c-1)$, $j \leq k(c-1)$. Then

$$(A^{\ell+1})_{r(c-1),c-1} = (A^\ell)_{r(c-1),c-2} + (A^\ell)_{r(c-1),c-3} + (A^\ell)_{r(c-1),c} + (A^\ell)_{r(c-1),c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),0} = (A^\ell)_{(r-1)(c-1),1} + (A^\ell)_{(r-1)(c-1),2}.$$

Note that, since $\Gamma(k)$ is symmetric we have,

$$\begin{aligned}(A^{\ell})_{r(c-1),c-2} =&(A^{\ell})_{r(c-1),c-3} > 0,\\ (A^{\ell})_{r(c-1),c} =&(A^{\ell})_{r(c-1),c+1} > 0,\\ (A^{\ell})_{r(c-1),1} =&(A^{\ell})_{r(c-1),2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$. So,

$$(A^{\ell+1})_{r(c-1),c-1} = 2(A^{\ell})_{r(c-1),c-2} + 2(A^{\ell})_{r(c-1),c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),0} = 2(A^{\ell})_{(r-1)(c-1),2}.$$

By the induction hypothesis, the following inequality holds:

$$(A^{\ell})_{r(c-1),c+1} \geq (A^{\ell})_{(r-1)(c-1),2}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1),c-1} > (A^{\ell+1})_{(r-1)(c-1),0}$. This causes the chain of strict inequalities

$$\begin{aligned}&(A^{\ell+2})_{r(c-1),2(c-1)} > (A^{\ell+2})_{(r-1)(c-1),c-1},\\ &(A^{\ell+3})_{r(c-1),3(c-1)} > (A^{\ell+3})_{(r-1)(c-1),2(c-1)}.\end{aligned}$$

Finally, we have

$$(A^{\ell+(k-r+1)})_{r(c-1),(k-r+1)(c-1)} > (A^{\ell+(k-r+1)})_{(r-1)(c-1),(k-r)(c-1)}.$$

Case 2: $t \equiv 0 \pmod{(c-1)}$ and $j \not\equiv 0 \pmod{(c-1)}$. Let $j \equiv 1 \pmod{(c-1)}$. Then

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^{\ell})_{t+c-1,j+c-2} + (A^{\ell})_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^{\ell})_{t,j-1} + (A^{\ell})_{t,j+2}.\end{aligned}$$

Similarly, by the induction hypothesis, we have

$$\begin{aligned}&(A^{\ell})_{t+c-1,j+c-2} \geq (A^{\ell})_{t,j-1},\\ &(A^{\ell})_{t+c-1,j+c+1} \geq (A^{\ell})_{t,j+2}, \quad \text{for } t+j+2 \leq k(c-1).\end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$.

In addition for the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)$ and $(r-1)(c-1)$. Then

$$\begin{aligned}(A^{\ell+1})_{r(c-1),c} =&(A^{\ell})_{r(c-1),c-1} + (A^{\ell})_{r(c-1),c+2} = (A^{\ell-1})_{r(c-1),c-2}\\ &+ (A^{\ell-1})_{r(c-1),c-3} + (A^{\ell-1})_{r(c-1),c} + (A^{\ell-1})_{r(c-1),c+1} + (A^{\ell})_{r(c-1),c+2}\end{aligned}$$

and

$$\begin{aligned}(A^{\ell+1})_{(r-1)(c-1),1} =&(A^{\ell})_{(r-1)(c-1),0} + (A^{\ell})_{(r-1)(c-1),3}\\ =&(A^{\ell-1})_{(r-1)(c-1),1} + (A^{\ell-1})_{(r-1)(c-1),2} + (A^{\ell})_{(r-1)(c-1),3}.\end{aligned}$$

Note that, since $\Gamma(k)$ is symmetric we have,

$$\begin{aligned}(A^{\ell-1})_{r(c-1),c-2} =&(A^{\ell-1})_{r(c-1),c-3} > 0,\\ (A^{\ell-1})_{r(c-1),c} =&(A^{\ell-1})_{r(c-1),c+1} > 0,\\ (A^{\ell-1})_{r(c-1),1} =&(A^{\ell-1})_{r(c-1),2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$.

So,

$$(A^{\ell+1})_{r(c-1),c} = 2(A^{\ell-1})_{r(c-1),c-2} + 2(A^{\ell-1})_{r(c-1),c} + (A^{\ell})_{r(c-1),c+2}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),1} = 2(A^{\ell-1})_{(r-1)(c-1),1} + (A^{\ell})_{(r-1)(c-1),3}.$$

By the induction hypothesis, the following inequalities hold:

$$(A^{\ell-1})_{r(c-1),c} \geq (A^{\ell-1})_{(r-1)(c-1),1},\ (A^{\ell})_{r(c-1),c+2} \geq (A^{\ell})_{(r-1)(c-1),3}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1),c} > (A^{\ell+1})_{(r-1)(c-1),1}$. This causes the chain of strict inequalities

$$\begin{aligned}&(A^{\ell+2})_{r(c-1),2(c-1)+1} > (A^{\ell+2})_{(r-1)(c-1),c},\\ &(A^{\ell+3})_{r(c-1),3(c-1)+1} > (A^{\ell+3})_{(r-1)(c-1),2(c-1)+1}.\end{aligned}$$

Finally, we have

$$(A^{\ell+k-r})_{r(c-1),(k-r+1)(c-1)+1} > (A^{\ell+k-r})_{(r-1)(c-1),(k-r)(c-1)+1}.$$

A similar argument can be used for the cases $j \equiv \{2, 3, \ldots, c-2\} \pmod{(c-1)}$.

Case 3: $t \not\equiv 0 \pmod{(c-1)}$ and $j \equiv 0 \pmod{(c-1)}$. Let $t \equiv 1 \pmod{(c-1)}$. Then

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^{\ell})_{t+c-1,j+c-2} + (A^{\ell})_{t+c-1,j+c-3} + (A^{\ell})_{t+c-1,j+c} + (A^{\ell})_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^{\ell})_{t,j-1} + (A^{\ell})_{t,j-2} + (A^{\ell})_{t,j+1} + (A^{\ell})_{t,j+2}.\end{aligned}$$

By the induction hypothesis, we have:

$$\begin{aligned}(A^{\ell})_{t+c-1,j+c-2} \geq&(A^{\ell})_{t,j-1},\\ (A^{\ell})_{t+c-1,j+c} \geq&(A^{\ell})_{t,j+1}, \quad \text{for } t+j+1 \leqslant k(c-1),\\ (A^{\ell})_{t+c-1,j+c-3} \geq&(A^{\ell})_{t,j-2},\\ (A^{\ell})_{t+c-1,j+c+1} \geq&(A^{\ell})_{t,j+2}, \quad \text{for } t+j+2 \leqslant k(c-1).\end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$.

For the strict inequality, let $1 \leq r \leq k$ be a fixed number, for two rows $r(c-1)+1$ and $(r-1)(c-1)+1$ we have

$$(A^{\ell+1})_{r(c-1)+1,c-1} = (A^{\ell})_{r(c-1)+1,c-2} + (A^{\ell})_{r(c-1)+1,c-3} + (A^{\ell})_{r(c-1)+1,c} + (A^{\ell})_{r(c-1)+1,c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,0} = (A^{\ell})_{(r-1)(c-1)+1,1} + (A^{\ell})_{(r-1)(c-1)+1,2}.$$

Note that since $\Gamma(k)$ is symmetric we have

$$\begin{aligned}(A^{\ell})_{r(c-1)+1,c-2} =&(A^{\ell})_{r(c-1)+1,c-3} > 0,\\ (A^{\ell})_{r(c-1)+1,c} =&(A^{\ell})_{r(c-1)+1,c+1} > 0,\\ (A^{\ell})_{r(c-1)+1,1} =&(A^{\ell})_{r(c-1)+1,2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$.

So,

$$(A^{\ell+1})_{r(c-1)+1,c-1} = 2(A^{\ell})_{r(c-1)+1,c-2} + 2(A^{\ell})_{r(c-1)+1,c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,0} = 2(A^{\ell})_{(r-1)(c-1)+1,2}.$$

By the induction hypothesis, the following inequality holds:

$$(A^{\ell})_{r(c-1)+1,c+1} \geq (A^{\ell})_{(r-1)(c-1)+1,2}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1)+1,c-1} > (A^{\ell+1})_{(r-1)(c-1)+1,0}$. This causes the chain of strict inequalities

$$\begin{aligned}&(A^{\ell+2})_{r(c-1)+1,2(c-1)} > (A^{\ell+2})_{(r-1)(c-1)+1,c-1},\\ &(A^{\ell+3})_{r(c-1)+1,3(c-1)} > (A^{\ell+3})_{(r-1)(c-1)+1,2(c-1)}.\end{aligned}$$

Finally, we have:

$$(A^{\ell+(k-r+1)})_{r(c-1)+1,(k-r+1)(c-1)-1} > (A^{\ell+(k-r+1)})_{(r-1)(c-1)+1,(k-r)(c-1)-1}.$$

A similar argument can be used for the cases $t \equiv \{2, 3, \ldots, c-2\} \pmod{(c-1)}$.

Case 4: $t \not\equiv 0 \pmod{(c-1)}$ and $j \equiv 1 \pmod{(c-1)}$. Let $t \equiv 1 \pmod{(c-1)}$, we have

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^{\ell})_{t+c-1,j+c-2} + (A^{\ell})_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^{\ell})_{t,j-1} + (A^{\ell})_{t,j+2}.\end{aligned}$$

By the induction hypothesis, the following inequality holds:

$$(A^{\ell})_{t+c-1,j+c-2} \geq (A^{\ell})_{t,j-1}, \quad (A^{\ell})_{t+c-1,j+c+1} \geq (A^{\ell})_{t,j+2}.$$

Hence, we have $(A^{\ell})_{t+c-1,j+c-1} \geq (A^{\ell})_{t,j}$. For the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)+1$ and $(r-1)(c-1)+1$. Then

$$\begin{aligned}(A^{\ell+1})_{r(c-1)+1,c} =&(A^{\ell})_{r(c-1)+1,c-1} + (A^{\ell})_{r(c-1)+1,c+2}\\ =&(A^{\ell-1})_{r(c-1)+1,c-2} + (A^{\ell-1})_{r(c-1)+1,c-3} + (A^{\ell-1})_{r(c-1)+1,c}\\ &+ (A^{\ell-1})_{r(c-1)+1,c+1} + (A^{\ell})_{r(c-1)+1,c+2}\end{aligned}$$

and

$$\begin{aligned}(A^{\ell+1})_{(r-1)(c-1)+1,1} =&(A^{\ell})_{(r-1)(c-1)+1,0} + (A^{\ell})_{(r-1)(c-1)+1,3}\\ =&(A^{\ell-1})_{(r-1)(c-1)+1,1} + (A^{\ell-1})_{(r-1)(c-1)+1,2} + (A^{\ell})_{(r-1)(c-1)+1,3}.\end{aligned}$$

Note that since $\Gamma(k)$ is symmetric, $(A^{\ell-1})_{r(c-1)+1,c-2} = (A^{\ell-1})_{r(c-1)+1,c-3} > 0$, $(A^{\ell-1})_{r(c-1)+1,c} = (A^{\ell-1})_{r(c-1)+1,c+1} > 0$ and $(A^{\ell-1})_{r(c-1)+1,1} = (A^{\ell-1})_{r(c-1)+1,2} > 0$, for $\ell \geq r(c-1)$. So,

$$(A^{\ell+1})_{r(c-1)+1,c} = 2(A^{\ell-1})_{r(c-1)+1,c-2} + 2(A^{\ell-1})_{r(c-1)+1,c} + (A^{\ell})_{r(c-1)+1,c+2}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,1} = 2(A^{\ell-1})_{(r-1)(c-1)+1,1} + (A^{\ell})_{(r-1)(c-1)+1,3}.$$

By the induction hypothesis, the following inequalities hold:

$$(A^{\ell-1})_{r(c-1)+1,c} \geqslant (A^{\ell-1})_{(r-1)(c-1)+1,1},\ (A^{\ell})_{r(c-1)+1,c+2} \geqslant (A^{\ell})_{(r-1)(c-1)+1,3}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1)+1,c} > (A^{\ell+1})_{(r-1)(c-1)+1,1}$. This causes the chain of strict inequalities

$$(A^{\ell+2})_{r(c-1)+1,2(c-1)+1} > (A^{\ell+2})_{(r-1)(c-1)+1,c},$$
$$(A^{\ell+3})_{r(c-1)+1,3(c-1)+1} > (A^{\ell+3})_{(r-1)(c-1)+1,2(c-1)+1}.$$

Finally,

$$(A^{\ell+k-r})_{r(c-1)+1,(k-r+1)(c-1)+1} > (A^{\ell+k-r})_{(r-1)(c-1)+1,(k-r)(c-1)+1}.$$

A similar argument can be used for $t \equiv r \in \{2, 3, \dots, c-2\} \pmod{(c-1)}$. □

The number of closed walks of length ℓ starting at the vertex v_t is equal to the entry (t,t) in matrix A^{ℓ}. Therefore,

$$S_{\ell}(k(c-1), t+(c-1)) = (A^{\ell})_{t+(c-1),t+(c-1)}.$$

By the induction hypothesis, we conclude that $S_{\ell}(k(c-1), t+(r-1)(c-1)) \leq S_{\ell}(k(c-1), t+r(c-1))$ for all $0 \leq t \leq c-1$ and $r \leq [\frac{k}{2}](c-1)$. Hence the strict inequality holds when $\ell \geq [\frac{k}{2}]$.

4. The Minimum Estrada Index of $\Gamma(k)$

Let G' be a point attaching strict k_1-quasi tree graph of even length c and $\delta \in V(G')$. For $k - k_1 = k_2$, let $G'(\lfloor \frac{k_2}{2} \rfloor, \lceil \frac{k_2}{2} \rceil)$ be the graph obtained from G' by attaching two graphs $\Gamma(\lfloor \frac{k_2}{2} \rfloor)$ and $\Gamma(\lceil \frac{k_2}{2} \rceil)$ at δ.

Let $N_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1); \delta)$ (respectively, $N_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1); \delta)$ be the set of (δ,δ)-walks of length ℓ in $G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1))$ (respectively, $G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1)$ starting and ending at the edges or only one edge in G' and let $N'_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1)); \delta)$ (respectively, $N'_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1); \delta))$ be the set of (δ,δ)-walks of length ℓ in $G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1))$ (respectively, $G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1)$ starting and ending at the edges or only one edge in union $\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil)$ (respectively, $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)$).

In the following let $G'(\lfloor\frac{k_2}{2}\rfloor(c-1), \lceil\frac{k_2}{2}\rceil(c-1)) := G(1)$ and let $G'(\lfloor\frac{k_2}{2}\rfloor(c-1)+c-1, \lceil\frac{k_2}{2}\rceil(c-1)-c+1) := G(2)$. By our definition, both graphs $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$ and $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ are isomorphic to $\Gamma(k_2)$, so they are denoted by $\Gamma(k_2)$.

Lemma 4.1. *If $\lfloor\frac{k_2}{2}\rfloor \geq 1$, then for positive integer ℓ,*

(i) $|N_\ell(G'(2);\delta)| \leq |N_\ell(G'(1));\delta)|$;

(ii) $|N'_\ell(G'(2);\delta)| \leq |N'_\ell(G'(1));\delta)|$.

Proof. Let $\omega \in N_\ell(G'(2);\delta)$, we may decompose ω into maximal sections in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ or in G'. Each of them is one of the following types.

(Type 1): a (δ,δ)- walk in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$.

(Type 2): a walk in $G'(2)$ with all edges in G'.

Similarly, we may decompose any $\omega \in N_\ell(G'(1);\delta)$ into maximal sections in G' or in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$. Each of them is one of the following types.

(Type 3): a (δ,δ)- walk in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$.

(Type 4): a walk in $G'(1)$ with all edges in G'.

Next, for any $\omega \in N_\ell(G'(2);\delta)$, we can replace the even indices by the odd indices that are in front of each other see Figure 2. Hence, from now on, ω is a (δ,δ)- walk with

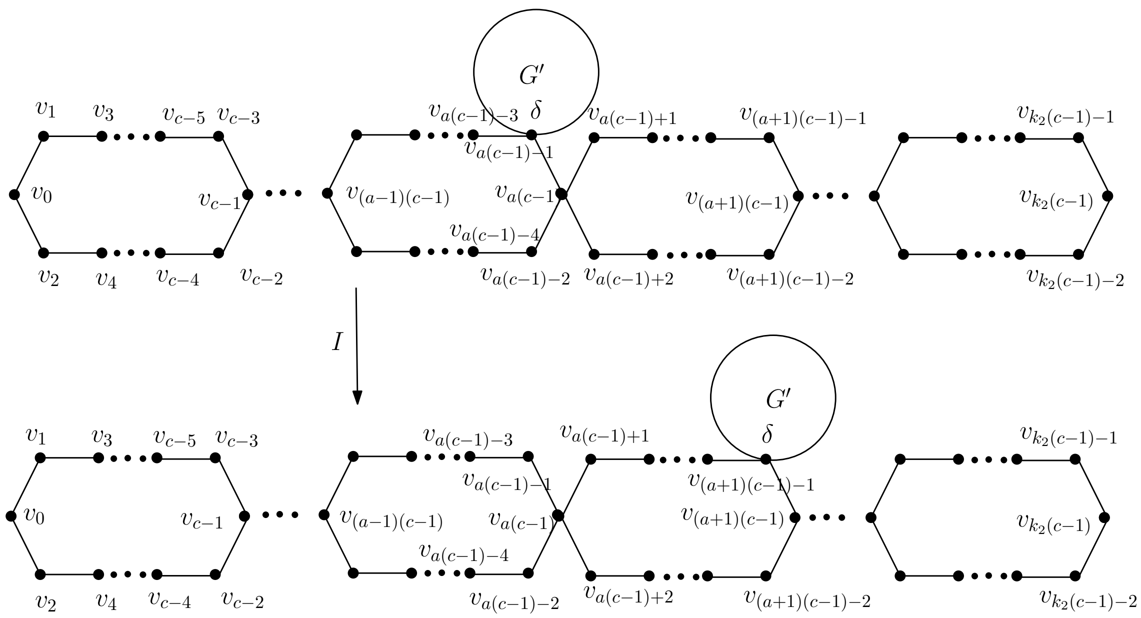

FIGURE 2. Transformation I.

only odd or even indices. So ω is a (δ,δ)- walk with odd indices. By Lemma 3.2 there is an injection mapping $\xi^1_{s'}$ that is a (δ,δ)- walk of length s' in $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ into a (δ,δ)- walk of length s' in $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$.

Let $\omega' = \omega_1\omega_2\omega_3\cdots \in N_\ell(\Gamma(k_2))$, where ω_i is a walk of length s'_i of type (1) or (2) for $i \geq 1$. Let $\xi^\star(\omega') = \xi^\star(\omega_1)\xi^\star(\omega_2)\cdots$, where $\xi^\star(\omega_i) = \xi^1_{s'_i}(\omega_i)$ and $\xi^\star(\omega_i) = \omega_i$ if ω_i

is of type 2 so $\xi^\star(\omega_i)$ for $i \geq 1$ is of type 3 or 4 and thus $\xi^\star(\omega') \in N_\ell(G'(1))$. Thus $|N_\ell(G'(2);\delta)| \leqslant |N_\ell(G'(1);\delta)|$. This prove (i). The proof for (ii) is similar. □

Theorem 4.1. *If $\lfloor \frac{k_2}{2} \rfloor \geq 1$, then $S_\ell(G'(2)) \leq S_\ell(G'(1))$. For $\ell \geq [\frac{k_2}{2}](c-1)$, the strict inequality holds.*

Proof. Let B_1 and B_2 be the sets of closed walks of length ℓ in $G'(1)$ and $G'(2)$ respectively, containing some edges in G'. Then $S_\ell(G'(2)) = S_\ell(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)) + |B_2|$ and $S_\ell(G'(1)) = S_\ell(\Gamma(\lfloor \frac{k_2}{2})\rfloor \cup \Gamma(\lceil \frac{k_2}{2} \rceil)) + |B_1|$. Since $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)$ and $\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil)$ are isomorphic to $\Gamma(k_2)$, we only need to prove that $|B_2| \leqslant |B_1|$ for all $\ell \geqslant 0$. Let B_{21} and B_{22} be two subsets of B_2 for which every closed walk starts at a vertex in $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1))$ and $V(G') - \{\delta\}$, respectively. Then $|B_2| = |B_{21}| + |B_{22}|$. Let B_{11} and B_{12} be two subsets of B_1 for which every closed walk starts at a vertex in $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil))$ and $V(G') - \{\delta\}$, respectively. Then $|B_1| = |B_{11}| + |B_{12}|$.

We may decompose any $\omega \in B_{21}$ into three parts $\omega_1\omega_2\omega_3$, where ω_1, ω_3 are walks in $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)$ and ω_2 is the longest walk of ω in $G'(2)$ starting and ending at the edges or only one edge in G'. By the choice of ω_2, we have that ω_2 is a (δ,δ)-walk. Let $B_{21}(\omega,\ell) = \{\omega \in B_{21} : \omega_2 \text{ is a } (\delta,\delta)\text{- walk}\}$. Thus $|B_{21}| = |B_{21}(\omega,\ell)|$. Let $B_{11}(\omega,\ell) = \{\omega \in B_{11} : \omega_2 \text{ is a } (\delta,\delta)\text{-walk}\}$. So $|B_{11}| = |B_{11}(\omega,\ell)|$.

Let $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)) := V(2)$. Then

$$\begin{aligned}
|B_{21}(\omega,\ell)| &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geq 0,\ell_2\geq 2}} \sum_{\beta\in V(2)} S_{\ell_1}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \beta, \delta\right) \\
&\quad \times |N_{\ell_2}(G'(2);\delta)| S_{\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \delta, \beta\right) \\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |N_{\ell_2}(G'(2);\delta)| \\
&\quad \times \sum_{\beta\in V(2)} S_{\ell_1}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \beta, \delta\right) \\
&\quad \times S_{\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \delta, \beta\right) \\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geq 0,\ell_2\geq 2}} |N_{\ell_2}(G'(2);\delta)| . S_{\ell_1+\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \delta\right).
\end{aligned}$$

Similarly,

$$|B_{21}(\omega,\ell)| = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |N_{\ell_2}(G'(1);\delta)| S_{\ell_1+\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2} \right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right); \delta\right).$$

By Lemma 4.1, we have $|N_{\ell_2}(G'(2);\delta)| \le |N_{\ell_2}(G'(1);\delta)|$ for all positive integers ℓ_2 and by Lemma 3.2, we have $S_t(\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1));\delta) \le S_t(\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil);\delta)$ for all positive integers t. Thus $|B_{21}(\omega,\ell)| \le |B_{11}(\omega,\ell)|$. Note that this inequality is strict for some positive integer $\ell_0 = t_0 + c - 1$ where $t_0 \ge \frac{k_2}{2}$. Also $|B_{21}| \le |B_{11}|$ for all positive integers ℓ, and it is strict for some positive integer ℓ_0.

By a similar argument as above, we can prove that $|B_{22}| \le |B_{12}|$. Thus $|B_2| \le |B_1|$ for all positive integers ℓ, and it is strict for some positive integer ℓ_0. $\square$

Lemma 4.2. *For all integer $\ell > c$, $k \ge 2$, we have*

$$S_\ell(k(c-1),2) \le S_\ell(k(c-1),4) \le \cdots \le S_\ell(k(c-1),c/2-2), S_\ell(k(c-1),c/2).$$

Proof. First, we show that every diagonal parallel to the main diagonal and the main diagonal are unimodal. Let H be the subgraph of $\Gamma(k)$ with vertex set $\{v_0, v_1, \ldots, v_c - 1\}$. By Lemma 3.1, we only need to show that the diagonals parallel to the main diagonal increase for $s + j \le c - 1$. Let s be an even integer. For the odd integer the proof is similar. Using induction on integer ℓ, we will prove that $(A^\ell)_{s+2,j+2} \ge (A^\ell)_{s,j}$ for all $0 \le s, j \le c-2$ with $s + j \le c - 1$.

Note that by the definition of $\Gamma(k)$, two vertices v_s and v_j are adjacent if and only if v_{s+2} and v_{j+2} are adjacent.

We have the following cases.

Case 1: $j \equiv 0 \pmod 2$ and $j \neq 0$. Then

$$\begin{aligned}(A^{\ell+1})_{s+2,j+2} =& (A^\ell)_{s+2,j} + (A^\ell)_{s+2,j+4},\\ (A^{\ell+1})_{s,j} =& (A^\ell)_{s,j-2} + (A^\ell)_{s,j+2}.\end{aligned}$$

By the induction hypothesis, we have the following results:

$$\begin{aligned}(A^\ell)_{s+2,j} \ge& (A^\ell)_{s,j-2},\\ (A^\ell)_{s+2,j+4} \ge& (A^\ell)_{s,j+2}, \quad \text{for } s+j+4 \le c-1.\end{aligned}$$

Hence, we have $(A^\ell)_{s+2,j+2} \ge (A^\ell)_{s,j}$. Since, there is a closed walk of length c starting from v_0 which is not including the edge $v_c v_{c+1}$, the inequality is strict for $\ell > c$.

Case 2: $j \equiv 1 \pmod 2$. The proof is similar to Case 1. $\square$

The number of closed walks of length ℓ starting at the even vertex v_s is equal to the entry (s,s) in matrix A^ℓ,

$$S_\ell(c-1,s) = (A^\ell)_{s,s}.$$

By induction hypothesis, we can conclude that $S_\ell(c-1,s) \le S_\ell(c-1,s+2)$ for every $0 < s < c-1$. Note that the strict inequality holds when $\ell \ge \frac{c}{2}$.

Let G be a point attaching strict k_1-quasi tree graph of even length c and $\alpha \in V(G)$ and let C_c be the cycle H of $\Gamma(k)$ with k_2 cycles where $k_1 + k_2 = k$. We decompose C_c into two paths denote by $P_{\frac{c}{2}}$ and $Q_{\frac{c}{2}}$, having common vertices in initial and final. Let $G(\frac{c}{2},\frac{c}{2})$ be the graph obtained from G by attaching $P_{\frac{c}{2}}$ and $Q_{\frac{c}{2}}$ at α in G.

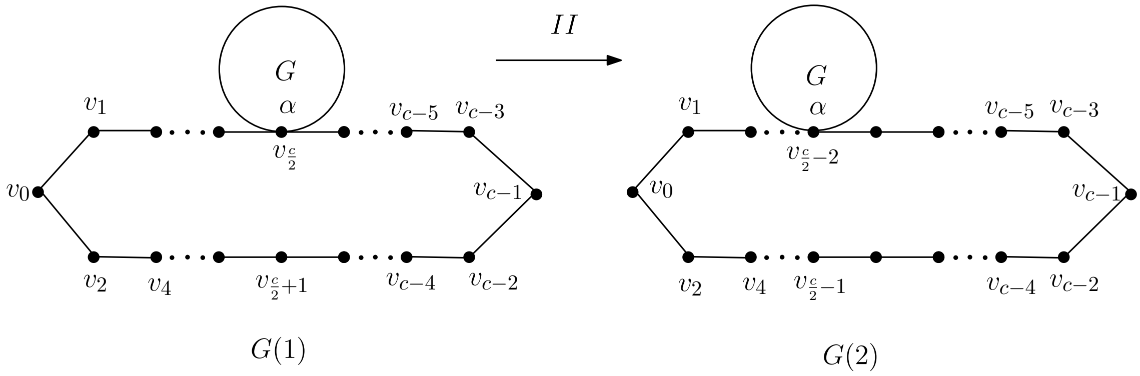

FIGURE 3. Transformation II.

Let $M_\ell(G(\frac{c}{2},\frac{c}{2});\alpha)$ (respectively $M_\ell(G(\frac{c}{2}+2,\frac{c}{2}-2);\alpha)$) be the set of (α,α)-walks of length ℓ in $G(\frac{c}{2},\frac{c}{2})$ (respectively $G(\frac{c}{2}+2,\frac{c}{2}-2)$) starting and ending at the edges or only one edge in G and let $M'_\ell(G(\frac{c}{2},\frac{c}{2});\alpha)$ (respectively $M'_\ell(G(\frac{c}{2}+2,\frac{c}{2}-2);\alpha)$) be the set of (α,α)-walks of length ℓ in $G(\frac{c}{2},\frac{c}{2})$ (respectively $G(\frac{c}{2}+2,\frac{c}{2}-2)$), starting and ending at the edges or only one edge in $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ (respectively $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$). In the following let $G(\frac{c}{2},\frac{c}{2}) := G(1)$ and $G(\frac{c}{2}+2,\frac{c}{2}-2) := G(2)$. By definition $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ are isomorphic to C_1, so we denoted them by C_1.

Lemma 4.3. *Let c be an even integer. If $\ell \geq \frac{c}{2}$, then*

(i) $|M_\ell(G(2);\alpha)| \leq |M_\ell(G(1));\alpha)|$;
(ii) $|M'_\ell(G(2);\alpha)| \leq |M'_\ell(G(1));\alpha)|$.

Proof. Let $\omega \in M_\ell(G(2);\alpha)$, we may decompose ω into maximal sections in union $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ or in G. Each of them is one of the following types.

(1) a (α,α)- walk in union $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$.
(2) a walk in $G(2)$ with all edges in G.

Similarly, we may decompose any $\omega \in M_\ell(G(1);\alpha)$ into maximal sections in union $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ or in G. Each of these maximal sections has one of the following types.

(3) a (α,α)-walk in union $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$.
(4) a walk in $G(1)$ with all edges in G.

Next, since $\Gamma(k)$ is symmetric, for any $\omega \in M_\ell(G(2);\alpha)$, we can replace the even indices with the odd indices that are in front of each other see Figure 3. Hence, from now on, ω is a (α,α)- walk with only odd or even indices. So without loss of generality ω is a (α,α)-walk with only odd indices. By definition, two unions $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ are isomorphic to C_1 and by Lemma 4.2 there exists an injection mapping η^1_ℓ from a (α,α)-walk of length ℓ in $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ into a (α,α)- walk of length ℓ in $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$. Let $\omega = \omega_1\omega_2\omega_3\cdots \in M_\ell(P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1})$, where ω_i is a walk of length ℓ_i of type (1) or (2) for $i \geq 1$. Let $\eta^\star(\omega) = \eta^\star(\omega_1)\eta^\star(\omega_2)\ldots$ where $\eta^\star(\omega_i) = \eta^1_{\ell_i}(\omega_i)$ and $\eta^\star(\omega_i) = \omega_i$ if ω_i is type (2) so $\eta^\star(\omega_i)$ for $i \geq 1$ is of type (3) or (4) and thus

$\eta^\star(\omega) \in M_\ell(G(1))$. Thus, $|M_\ell(G(2);\alpha)| \leq |M_\ell(G(1);\alpha)|$. This prove (i). The proof of (ii) is similar. □

Theorem 4.2. *Let c be an even integer. If $\frac{c}{2} \geq 3$, then $S_\ell(G(2)) \leq S_\ell(G(1))$. For $\ell > \frac{c}{2}$, the strict inequality holds.*

Proof. Let A_1 and A_2 be two sets of closed walks of length ℓ in $G(1)$ and $G(2)$, respectively, containing some edges in G. Then $S_\ell(G(2)) = S_\ell(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}) + |A_2|$ and $S_\ell(G(1)) = S_\ell(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}}) + |A_1|$.

By our definition, $P_{\frac{c}{2}} \cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}$ are isomorphic to C_1, and we need only to prove that $|A_2| \leq |A_1|$ for all $\ell \geq 0$.

Let A_{21} and A_{22} be two subsets of A_2 for which every closed walk starts at a vertex in $V(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1})$ and in $V(G) - \{\alpha\}$, respectively. Then $|A_2| = |A_{21}| + |A_{22}|$.

Let A_{11} and A_{12} be two subsets of A_1 for which every closed walk starts at a vertex in $V(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}})$ and in $V(G) - \{\alpha\}$, respectively. Then $|A_1| = |A_{11}| + |A_{12}|$.

We may decompose any $\omega \in A_{21}$ into three sections $\omega_1\omega_2\omega_3$, where ω_1, ω_3 are walks in $P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}$ and ω_2 is the longest walk of ω in $G(2)$ starting and ending at the edges in G. By the choice of ω_2, we have that ω_2 is a (α,α)-walk. Let $A_{21}(\omega,\ell) = \{\omega \in A_{21} : \omega_2 \text{ is a } (\alpha,\alpha)\text{-walk}\}$. So, we have $|A_{21}| = |A_{21}(\omega,\ell)|$.

Let $A_{11}(\omega,\ell) = \{\omega \in A_{11} : \omega_2 \text{ is a } (\alpha,\alpha)\text{-walk}\}$. So, we have $|A_{11}| = |A_{11}(\omega,\ell)|$.

Let $V(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}) := V(1)$. Let $t = |M_{\ell_2}(G(2);\alpha)|$. From this decomposition for $\omega \in A_{21}$ and by the definition of $A_{21}(\omega,\ell)$, we have

$$\begin{aligned} |A_{21}(\omega,\ell)| &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} \sum_{\beta\in V(1)} S_{\ell_1}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\beta,\alpha).t.S_{\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha,\beta) \\ &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} t. \sum_{\beta\in V(1)} S_{\ell_1}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\beta,\alpha).S_{\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha,\beta) \\ &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} .t.S_{\ell_1+\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha). \end{aligned}$$

Similarly,

$$|A_{21}(\omega,\ell)| = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |M_{\ell_2}(G(1);\alpha)|.S_{\ell_1+\ell_3}(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}};\alpha).$$

By Lemma 4.3, we have $|M_{\ell_2}(G(2);\alpha)| \leq |M_{\ell_2}(G(1);\alpha)|$ for all positive integers ℓ_2 and by Lemma 4.2, we have $S_t(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha) \leq S_t(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}};\alpha)$ for all positive integers t. Thus $|A_{21}(\omega,\ell)| \leq |A_{11}(\omega,\ell)|$. Note that this inequality is strict for some positive integer $\ell_0 = t_0 + c - 1$ where $t_0 \geq \frac{c}{2}$. Also $|A_{21}| \leq |A_{11}|$ for all positive integers ℓ, and it is strict for some positive integer ℓ_0.

By similar argument as above, we can prove that $|A_{22}| \leq |A_{12}|$. Thus $|A_2| \leq |A_1|$ for all positive integers ℓ, and it is strict for some positive integer ℓ_0. □

Corollary 4.1. *For graphs $G(1)$ and $G(2)$ we have $EE(G(1)) > EE(G(2))$.*

Proof. From Theorem 4.2, we have

$$EE(G(2)) = \sum_{\ell \geq 0} \frac{S_\ell(G(2))}{(\ell)!} < \sum_{\ell \geq 0} \frac{S_\ell(G(1))}{(\ell)!} = EE(G(1)). \qquad \square$$

The transformation from $G(1)$ to $G(2)$, depicted in Figure 3, is called transformation slowromancapi@ of $G(1)$.

Corollary 4.2. *For two graphs $G'(1)$ and $G'(2)$, we have $EE(G'(1)) > EE(G'(2))$.*

Proof. By Theorem 4.1, we have

$$EE(G'(2)) = \sum_{\ell \geq 0} \frac{S_\ell(G'(2))}{(\ell)!} < \sum_{\ell \geq 0} \frac{S_\ell(G'(1))}{(\ell)!} = EE(G'(1)). \qquad \square$$

The transformation from $G'(1)$ to $G'(2)$, depicted in Figure 2, is called transformation slowromancapi@ of $G'(1)$. Transformation slowromancapiii@ is similar to transformation slowromancapii@ which obtained by attaching $\alpha \in G$ at v_0. There is a closed walks in $M_c((c-1), 0)$ which is not including the edge $v_c v_{c+1}$. So there is a closed walk in $M_c((c-1), 1)$ not in $M_c((c-1), 0)$. Hence, transformation slowromancapiii@ strictly decreases the Estrada index for $\ell \geq c$.

Let G be a point attaching strict k-quasi tree graph with k even cycles of length c, obtained by attaching the subgraphs $G_1, G_2, \ldots, G_{\frac{\Delta}{2}}$ at u with the maximum degree Δ. By using transformations slowromancapi@, slowromancapii@ and slowromancapiii@, G_i s, $(1 \leq i \leq \frac{\Delta}{2})$ can be changed into the graphs Γ_i s. These transformations change G into G^* which is obtained by attaching Γ_i s at u. Each application of transformation strictly decreases its Estrada index. So we have $EE(G^*) < EE(G)$. Finally repeatedly applying transformation I, G^* can be changed into the graph $\Gamma(k)$ that is obtained from $\bigcup_{i=1}^{\frac{\Delta}{2}} \Gamma(k_i)$. So we have the following result.

Theorem 4.3. *Let G be a point attaching strict k-quasi tree graph with k even cycles. Then $EE(\Gamma(k)) \leq EE(G)$.*

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