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EXISTENCE AND STABILITY OF NONLOCAL INITIAL VALUE PROBLEMS INVOLVING GENERALIZED KATUGAMPOLA DERIVATIVE

ARIF S. BAGWAN[1] AND DEEPAK B. PACHPATTE[2]

Abstract. In this paper, the existence results for the solutions to nonlocal initial value problems involving generalized Katugampola derivative are established. Some fixed point theorem techniques are used to derive the existence results. In the sequel, we investigate the generalized Ulam-Hyers-Rassias stability corresponding to our problem. Some examples are given to illustrate our main results.

1. Introduction

In recent decades, the theory of continuous fractional calculus and their applications have remains a centre of attraction in many mathematical research. Indeed, fractional differential equations (FDEs) have grabbed desired attention by many authors. One can see [1–5, 7–13, 20, 21, 23, 26, 27, 33, 34] and references therein. Several definitions of fractional derivatives and integrals have been introduced during the theoretical development of fractional calculus. See [1, 2, 5, 7, 8, 16, 20–22, 25, 27] and references therein.

Initially, Hilfer et al. [16,17] have proposed linear differential equations involving new fractional operator. They applied operational method to solve such FDEs. Further, Furati et al. [14, 15] investigated non-linear problems and discussed existence and non-existence results for FDEs with Hilfer derivative operator. Benchohra et al. [6, 7], U. N. Katugampola [20, 21], D. B. Dhaigude et al. [8, 9], Kou et al. [23], J. Wang et al. [32, 33] and many more authors, see [1, 2, 5, 19, 29, 31] and references therein, have established the existence results for FDEs with several fractional derivative operators.

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Recently, D. S. Oliveira et al. [27] in their article proposed a new fractional differential operator: Hilfer-Katugampola frational derivative (also known as generalized Katugampola derivative). Further, they established the existence and uniqueness results for the FDEs with generalized Katugampola derivative.

The theory of Ulam stability is also evolved as one of the most interesting field of research. Initially, Ulam [30] established the results on the stability of functional equations. Thereafter, remarkable interest have been shown by authors towards the study of Ulam-Hyers stability and Ulam-Hyers-Rassias stability for various FDEs, see [1, 6, 7, 18, 24, 31, 33] and references therein.

In this paper, we studied the existence and stability of nonlocal initial value problem (IVP) involving generalized Katugampola derivative of the form:

$$ {}^{\rho}D_{a^+}^{\mu,\nu}u(t) = f(t,u(t)), \quad \mu \in (0,1), \nu \in [0,1], t \in (a,b], \tag{1.1} $$

$$ {}^{\rho}I_{a^+}^{1-\beta}u(a) = \sum_{i=1}^{m} \lambda_i u(\kappa_i), \quad \mu \leq \beta = \mu + \nu(1-\mu) < 1, \kappa_i \in (a,b], \tag{1.2} $$

where f is a given function such that $f : (a,b] \times \mathbb{R} \to \mathbb{R}$, $0 < \rho$. The operator ${}^{\rho}D_{a^+}^{\mu,\nu}$ is the generalized Katugampola fractional derivative of order μ and type ν and the operator ${}^{\rho}I_{a^+}^{1-\beta}u(a)$ is the Katugampola fractional integral of order $1-\beta$, with $a > 0$, κ_i, $i = 1, 2, \ldots, m$, are prefixed points satisfying $a < \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m < b$.

Furthermore, the paper is arranged as follows. In Section 2, we recall some basic definitions, important results and preliminary facts. We establish the equivalent mixed type Volterra integral equation for the IVP (1.1)–(1.2). In Section 3, we present existence of solution using the Krasnoselskii fixed point theorem. Further, we present the generalized Ulam-Hyers-Rassias stability to our problem. An illustrative example is given at the end of the main results.

2. Preliminary Results

In this section, we provide some basic definitions of generalized fractional integrals and derivatives, some important results and preliminary facts that are very useful to us in the sequel.

Let $0 < a < b < \infty$ be a finite interval on $\mathbb{R}^+$ and $C[a,b]$ be the Banach space of all continuous functions $h : [a,b] \to \mathbb{R}$ with the norm

$$ \|h\|_C = \max\{|h(t)| : t \in [a,b]\}. $$

For $0 \leq \beta < 1$ and the parameter $\rho > 0$ we define the weighted space of continuous functions h on $(a,b]$ by

$$ C_{\beta,\rho}[a,b] = \left\{ h : (a,b] \to \mathbb{R} : \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\beta} h(t) \in C[a,b] \right\}, $$

with the norm

$$\|h\|_{C_{\beta,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\beta h(t) \right\|_C = \max_{t\in[a,b]} \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\beta h(t) \right|.$$

It is obvious that $C_{0,\rho}[a,b] = C[a,b]$.

Let $\delta_\rho = \left(t^{\rho-1} \frac{d}{dt} \right)$. We define the Banach space of continuously differentiable functions h on $[a,b]$ by

$$C^1_{\delta_\rho,\beta}[a,b] = \{h : [a,b] \to \mathbb{R} : \delta_\rho h \in C_{\beta,\rho}[a,b]\},$$

with the norms

$$\|h\|_{C^1_{\delta_\rho,\beta}} = \|h\|_C + \|\delta_\rho h\|_{C_{\beta,\rho}}$$

and

$$\|h\|_{C^1_{\delta_\rho,\beta}} = \max\{|\delta_\rho h(t)| : t \in [a,b]\}.$$

Note that $C^0_{\delta_\rho,\beta}[a,b] = C_{\beta,\rho}[a,b]$.

Definition 2.1 (Katugampola fractional integral [20, 27])**.** Let $\mu, c \in \mathbb{R}$, with $\mu > 0$, $u \in Z^p_c(a,b)$, where $Z^p_c(a,b)$ is the space of Lebesgue measurable functions with complex values. The left-sided Katugampola fractional integral of order μ is defined by

$$\left({}^\rho I^\mu_{a^+} u\right)(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^t \frac{x^{\rho-1} u(x)}{(t^\rho - x^\rho)^{1-\mu}} dx, \quad t > a. \tag{2.1}$$

Definition 2.2 (Katugampola fractional derivative [21, 27])**.** Let $\mu, \rho \in \mathbb{R}$ be such that $\mu \notin \mathbb{N}$, $0 < \mu, \rho$. The left-sided Katugampola fractional derivative of order μ is defined by

$$\left({}^\rho D^\mu_{a^+} u\right)(t) = \delta^n_\rho \left({}^\rho I^{n-\mu}_{a^+} u\right)(t) = \frac{\rho^{1-n+\mu}}{\Gamma(n-\mu)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{x^{\rho-1} u(x)}{(t^\rho - x^\rho)^{1-n+\mu}} dx, \tag{2.2}$$

where $n = [\mu] + 1$ is such that $[\mu]$ is the integer part of μ.

Definition 2.3 (Generalized Katugampola fractional derivative [27])**.** Let $0 < \mu \le 1$ and $0 \le \nu \le 1$. The generalized Katugampola fractional derivative (of order μ and type ν) with respect to t is defined by

$$\begin{aligned} \left({}^\rho D^{\mu,\nu}_{a^+} u\right)(t) &= \left\{ \pm {}^\rho I^{\nu(1-\mu)}_{a\pm} \left(t^{\rho-1} \frac{d}{dt} \right)^1 {}^\rho I^{(1-\nu)(1-\mu)}_{a\pm} u \right\}(t) \\ &= \left\{ \pm {}^\rho I^{\nu(1-\mu)}_{a\pm} \delta_\rho {}^\rho I^{(1-\nu)(1-\mu)}_{a\pm} u \right\}(t), \end{aligned} \tag{2.3}$$

where $\rho > 0$, $u \in C_{1-\beta,\rho}[0,1]$ and I is Katugampola fractional integral defined in (2.1).

Remark 2.1. ([27]). For $\beta = \mu + \nu(1-\mu)$, the generalized Katugampola fractional derivative operator ${}^{\rho}D_{a^+}^{\mu,\nu}$ can be expressed as

$$ {}^{\rho}D_{a^+}^{\mu,\nu} = {}^{\rho}I_{a^+}^{\nu(1-\mu)}\delta_{\rho}{}^{\rho}I_{a^+}^{1-\beta} = {}^{\rho}I_{a^+}^{\nu(1-\mu)\rho}D_{a^+}^{\beta}. \tag{2.4} $$

Lemma 2.1 ([27]). *Let $\mu > 0$, $0 \leq \beta < 1$ and $u \in C_{\beta,\rho}[a,b]$. Then*

$$ \left({}^{\rho}D_{a^+}^{\mu}{}^{\rho}I_{a^+}^{\mu}u\right)(t) = u(t), \quad \text{for all } t \in (a,b]. $$

Lemma 2.2 (Semigroup property [27]). *Let $\mu > 0$, $\nu > 0$, $1 \leq q \leq \infty$, $a, b \in (0,\infty)$ such that $a < b$ and $\rho, s \in \mathbb{R}$, $s \leq \rho$. Then the following property holds true*

$$ \left({}^{\rho}I_{a^+}^{\mu}{}^{\rho}I_{a^+}^{\nu}u\right)(t) = \left({}^{\rho}I_{a^+}^{\mu+\nu}u\right)(t), $$

for all $u \in Z_s^q(a,b)$.

Lemma 2.3 ([27]). *Let $t > a$ and for $\mu \geq 0$ and $\nu > 0$, we have*

$$ \left[{}^{\rho}D_{a^+}^{\mu}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1}\right](t) = 0, \quad 0 < \mu < 1, $$

$$ \left[{}^{\rho}I_{a^+}^{\mu}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\nu-1}\right](t) = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\nu-1}. $$

Lemma 2.4 ([27]). *Let $\mu > 0$, $0 \leq \beta < 1$ and $a, b \in (0,\infty)$ such that $a < b$ and $u \in C_{\beta,\rho}[a,b]$. Then*

$$ \left({}^{\rho}I_{a^+}^{\mu}u\right)(a) = \lim_{t\to a^+}\left({}^{\rho}I_{a^+}^{\mu}u\right)(t) = 0, $$

and ${}^{\rho}I_{a^+}^{\mu}u$ is continuous on $[a,b]$ if $\beta < \mu$.

Lemma 2.5 ([27]). *Let $\mu \in (0,1)$, $\nu \in [0,1]$ and $\beta = \mu + \nu - \mu\nu$. If $u \in C_{1-\beta}^{\beta}[a,b]$ then*

$$ {}^{\rho}I_{a^+}^{\beta}{}^{\rho}D_{a^+}^{\beta}u = {}^{\rho}I_{a^+}^{\mu}{}^{\rho}D_{a^+}^{\mu,\nu}u $$

and

$$ {}^{\rho}D_{a^+}^{\beta}{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}D_{a^+}^{\nu(1-\mu)}u. $$

Lemma 2.6 ([27]). *Let $\mu \in (0,1)$, $0 \leq \beta < 1$. If $u \in C_{\beta}[a,b]$ and ${}^{\rho}I_{a^+}^{1-\mu}u \in C_{\beta}^{1}[a,b]$, then for all $t \in (a,b]$*

$$ \left({}^{\rho}I_{a^+}^{\mu}{}^{\rho}D_{a^+}^{\mu}u\right)(t) = -\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1}\frac{\left({}^{\rho}I_{a^+}^{1-\beta}u\right)(a)}{\Gamma(\mu)} + u(t). $$

Lemma 2.7 ([27]). *Let $u \in L^1(a,b)$. If ${}^{\rho}D_{a^+}^{\nu(1-\mu)}u$ exists on $L^1(a,b)$, then*

$$ {}^{\rho}D_{a^+}^{\mu,\nu\rho}I_{a^+}^{\mu}u = {}^{\rho}I_{a^+}^{\nu(1-\mu)\rho}D_{a^+}^{\nu(1-\mu)}u. $$

Lemma 2.8 ([27]). *Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a function where $f(\cdot, u(\cdot)) \in C_{1-\beta}[a, b]$. A function $u \in C_{1-\beta}^{\beta}[a, b]$ is a solution of fractional IVP*

$$\begin{aligned} D_{a^+}^{\mu,\nu} u(t) &= f(t, u(t)), \quad \mu \in (0,1), \, \nu \in [0,1], \\ I_{a^+}^{1-\beta} u\left(a^+\right) &= u_0, \quad \beta = \mu + \nu - \mu\nu, \end{aligned}$$

if and only if u satisfies the integral equation of Volterra type:

$$u(t) = \frac{u_0(t-a)^{\beta-1}}{\Gamma(\beta)} + \frac{1}{\Gamma(\mu)} \int_a^t (t-x)^{\mu-1} f(x, u(x))\, dx.$$

Definition 2.4 (Volterra integral equation). A linear Volterra integral equation of the second kind has the form of

$$u(t) = u_0(t) + \int_a^t K(t, x) u(x)\, dx,$$

where K is a kernel.

Theorem 2.1 (Krasnoselskii fixed point theorem [28]). *Let E be a nonempty closed, bounded and convex subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Further, assume that F and G are two operators defined on E which map E into $\mathcal{B}$ such that*

(a) *$F(x) + G(y) \in E$ for all $x, y \in E$;*
(b) *F is a contraction;*
(c) *G is continuous and compact.*

Then $F + G$ has a fixed point in E.

Using the above fundamental results, the following theorem yields the equivalence between the IVP (1.1)–(1.2) and an improved mixed type Volterra integral equation.

Theorem 2.2. *Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a function such that for any $u \in C_{1-\beta}[a, b]$ $f(\cdot, u(\cdot)) \in C_{1-\beta}[a, b]$, where $\beta = \mu + \nu - \mu\nu$, with $0 < \mu \le 1$, $0 \le \nu \le 1$. Function $u \in C_{1-\beta}^{\beta}[a, b]$ is a solution of IVP (1.1)–(1.2) if and only if it satisfies the following mixed type Volterra integral equation*

$$\begin{aligned} u(t) =& \frac{K}{\Gamma(\mu)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \\ &+ \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx, \end{aligned} \tag{2.5}$$

where $K = \left\{\Gamma(\beta) - \sum\limits_{i=1}^{m} \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1}\right\}^{-1}$.

Proof. Let $u \in C^{\beta}_{1-\beta}[a,b]$ be a solution of IVP (1.1)–(1.2). Then by the Lemma 2.8 the solution of IVP (1.1)–(1.2) can be written as

$$u(t) = \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{1}{\Gamma(\mu)} \int\limits_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx. \tag{2.6}$$

Now, substitute $t = \kappa_i$ in the above equation

$$u(\kappa_i) = \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{1}{\Gamma(\mu)} \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.$$

Multiplying by λ_i the both hand sides, we get

$$\lambda_i u(\kappa_i) = \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{\lambda_i}{\Gamma(\mu)} \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.$$

Thus, we have

$$\begin{aligned}
{}^\rho I^{1-\beta}_{a^+} u(a) =& \sum_{i=1}^m \lambda_i u(\kappa_i), \\
=& \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} \sum_{i=1}^m \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \\
&+ \frac{1}{\Gamma(\mu)} \sum_{i=1}^m \lambda_i \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx,
\end{aligned}$$

which implies

$$\left({}^\rho I^{1-\beta}_{a^+} u\right)(a) = \frac{\Gamma(\beta)}{\Gamma(\mu)} K \sum_{i=1}^m \lambda_i \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx. \tag{2.7}$$

Substituting (2.7) in (2.6) we get (2.5), which proved that u also satisfies integral equation (2.5) when it satisfies IVP (1.1)–(1.2). This proved the necessity. Now, we prove the sufficiency by applying ${}^\rho I^{1-\beta}_{a^+}$ to both hand sides of the integral equation (2.5), we have

$$\begin{aligned}
{}^\rho I^{1-\beta}_{a^+} u(t) =& {}^\rho I^{1-\beta}_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{K}{\Gamma(\mu)} \sum_{i=1}^m \lambda_i \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \\
&+ {}^\rho I^{1-\beta}_{a^+} {}^\rho I^{\mu}_{a^+} f(x, u(x)).
\end{aligned}$$

By using Lemma 2.2, Lemma 2.1 and Lemma 2.3, we have

$${}^\rho I^{1-\beta}_{a^+} u(t) = \frac{\Gamma(\beta)}{\Gamma(\mu)} K \sum_{i=1}^m \lambda_i \int\limits_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx + {}^\rho I^{1-\nu(1-\mu)}_{a^+} f(t, u(t)).$$

Since $1-\nu(1-\mu)>1-\beta$, by taking the limit as $t\to a$ and using Lemma 2.4, we have

$$\left({}^{\rho}I_{a^+}^{1-\beta}u\right)(a)=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx. \tag{2.8}$$

Now, substituting $t=\kappa_i$ in (2.5), we have

$$\begin{aligned}u(\kappa_i)=&\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&+\frac{1}{\Gamma(\mu)}\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx.\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i u(\kappa_i)=&\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&+\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\=&\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&\times\left\{K\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}+1\right\}\\=&\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{i=1}^{m}\lambda_i\int\limits_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx.\end{aligned}\tag{2.9}$$

It follows from (2.8) and (2.9), that

$${}^{\rho}I_{a^+}^{1-\beta}u(a)=\sum_{i=1}^{m}\lambda_i u(\kappa_i).$$

It follows from Lemma 2.3 and Lemma 2.5 and by applying ${}^{\rho}D_{a^+}^{\beta}$ to both hand sides of (2.5) that

$${}^{\rho}D_{a^+}^{\beta}u(t)={}^{\rho}D_{a^+}^{\nu(1-\mu)}f(t,u(t)). \tag{2.10}$$

Since $u\in C_{1-\beta}^{\beta}[a,b]$ and by the definition of $C_{1-\beta}^{\beta}[a,b]$, we have ${}^{\rho}D_{a^+}^{\beta}u\in C_{1-\beta}^{\beta}[a,b]$. Then ${}^{\rho}D_{a^+}^{\nu(1-\mu)}f={}^{\rho}D^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}[a,b]$. It is obvious that for any $f\in C_{1-\beta}[a,b]$, ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}[a,b]$, then ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}^{1}[a,b]$. Thus, f and ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f$ satisfy both the conditions of Lemma 2.6.

Now, it follows from Lemma 2.6, by applying ${}^{\rho}I_{a^+}^{\nu(1-\mu)}$ on both sides of (2.10), that

$$\left({}^{\rho}D_{a^+}^{\mu,\nu}u\right)(t) = -\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu(1-\mu)-1} \frac{{}^{\rho}I_{a^+}^{1-\nu(1-\mu)} f(a)}{\Gamma(\nu(1-\mu))} + f(t, u(t)). \tag{2.11}$$

By Lemma 2.4, it implies that ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)} f(a) = 0$. Hence, (2.11) reduces to

$$\left({}^{\rho}D_{a^+}^{\mu,\nu}u\right)(t) = f(t, u(t)).$$

This completes the proof. □

3. Main Result

In the sequel, let us introduce the following hypothesis.

[$Q1$] Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any $u \in C_{1-\beta}[a, b]$ $f(\cdot, u(\cdot)) \in C_{1-\beta}^{\nu(1-\mu)}[a, b]$. For all $u, v \in \mathbb{R}$ there exists a positive constant $J > 0$ such that

$$|f(t, u) - f(t, v)| \le J|u - v|.$$

[$Q2$] The constant

$$\sigma := \frac{JB(\mu,\beta)}{\Gamma(\mu)} \left\{ |K| \sum_{i=1}^{m} \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1} + \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu} \right\} < 1, \tag{3.1}$$

where K is defined in the Theorem 2.2.

Now, we will establish our main existence result for IVP (1.1)–(1.2) using Krasnoselskii fixed point theorem.

Theorem 3.1. *Assume that the hypothesis* [$Q1$] *and* [$Q2$] *are satisfied. Then IVP* (1.1)–(1.2) *has at least one solution in* $C_{1-\beta}^{\beta}[a, b]$.

Proof. According to Theorem 2.2, it is sufficient to prove the existence result for the mixed type integral equation (2.5).

Now, define the operator Δ by

$$\begin{aligned}(\Delta u)(t) = &\frac{K}{\Gamma(\mu)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \\ &+ \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.\end{aligned} \tag{3.2}$$

It is obvious that the operator Δ is well defined and maps $C_{1-\beta}[a, b]$ into $C_{1-\beta}[a, b]$. Let $\hat{f}(x) = f(x, 0)$ and

$$\eta := \frac{B(\mu,\beta)}{\Gamma(\mu)} \left\{ |K| \sum_{i=1}^{m} \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1} + \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu} \right\} \left\|\hat{f}\right\|_{C_{1-\beta}}. \tag{3.3}$$

Consider a ball $B_s := \left\{u \in C_{1-\beta}[a, b] : \|u\|_{C_{1-\beta}} \le s\right\}$, with $\frac{\eta}{1-\sigma} \le s, \sigma < 1$.

Now, let us subdivide the operator Δ into two operators F and G on B_s as follows:

$$(Fu)(t) = \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx$$

and

$$(Gu)(t) = \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.$$

The proof is divided into following steps.

Step I. For every $u, v \in B_s$, $Fu + Gv \in B_s$. For the operator F

$$(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta} = \frac{K}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx, \quad t \in (a, b],$$

we have

$$\begin{aligned}\left|(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta}\right| \leq& \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} |f(x, u(x))|\, dx \\ \leq& \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \left(|f(x, u(x)) - f(x, 0)|\right. \\ & \left. + |f(x, 0)|\right) dx \\ \leq& \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \left(J|u(x)| + \left|\hat{f}(x)\right|\right) dx.\end{aligned}$$

Here we use the fact that

$$\begin{aligned}\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} |u(x)|\, dx \leq& \left\{\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\beta-1} x^{\rho-1} dx\right\} \\ & \times \|u(x)\|_{C_{1-\beta}} \\ =& \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1} B(\mu, \beta)\, \|u(x)\|_{C_{1-\beta}}.\end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}\left|(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta}\right| \leq& \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \left\{\left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1} B(\mu, \beta)\right. \\ & \left. \times \left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}}\right)\right\},\end{aligned}$$

which gives
(3.5)
$$\|Fu\|_{C_{1-\beta}} \leq \frac{|K|\, B(\mu,\beta)}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \left\{ \left(\frac{{\kappa_i}^{\rho} - a^{\rho}}{\rho} \right)^{\mu+\beta-1} \left(J\|u(x)\|_{C_{1-\beta}} + \left\| \hat{f}(x) \right\|_{C_{1-\beta}} \right) \right\}.$$

For $t \in (a,b]$ and the operator G

$$(Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} = \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx,$$

we have

$$\begin{aligned} \left| (Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \right| \leq & \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} \left| f(x, u(x)) \right| dx \\ \leq & \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \\ & \times \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} \left(J\,|u(x)| + \left| \hat{f}(x) \right| \right) dx. \end{aligned}$$

Again, by using (3.4), we have

$$\begin{aligned} \left| (Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \right| \leq & \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \left\{ \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\mu+\beta-1} \right. \\ & \left. \times B(\mu,\beta) \left(J\|u(x)\|_{C_{1-\beta}} + \left\| \hat{f}(x) \right\|_{C_{1-\beta}} \right) \right\} \\ \leq & \frac{B(\mu,\beta)}{\Gamma(\mu)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{\mu} \left(J\|u(x)\|_{C_{1-\beta}} + \left\| \hat{f}(x) \right\|_{C_{1-\beta}} \right), \end{aligned}$$

which gives

$$\|(Gu)\|_{C_{1-\beta}} \leq \frac{B(\mu,\beta)}{\Gamma(\mu)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{\mu} \left(J\|u(x)\|_{C_{1-\beta}} + \left\| \hat{f}(x) \right\|_{C_{1-\beta}} \right). \tag{3.6}$$

Combining (3.5) and (3.6) for every $u, v \in B_s$ we have

$$\|Fu + Gv\|_{C_{1-\beta}} \leq \|Fu\|_{C_{1-\beta}} + \|(Gv)\|_{C_{1-\beta}} \leq \sigma s + \eta \leq s,$$

which implies that $Fu + Gv \in B_s$.

Step II. The operator F is contraction mapping.

For any $u, v \in B_s$ and the operator F

$$\{(Fu)(t) - (Fv)(t)\} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta}$$

$$=\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\left[f(x,u(x))-f(x,v(x))\right]dx$$

we have

$$\begin{aligned}\left|\{(Fu)(t)-(Fv)(t)\}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta}\right| \leq&\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\\ &\times\left|f(x,u(x))-f(x,v(x))\right|dx\\ \leq&\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\\ &\times J\left|u(x)-v(x)\right|dx\\ \leq&\frac{J|K|}{\Gamma(\mu)}B(\mu,\beta)\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\\ &\times\|u-v\|_{C_{1-\beta}},\end{aligned}$$

which gives

$$\|Fu-Fv\|_{C_{1-\beta}}\leq\frac{J|K|}{\Gamma(\mu)}B(\mu,\beta)\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\|u-v\|_{C_{1-\beta}}\leq\sigma\|u-v\|_{C_{1-\beta}}.$$

Hence, by the hypothesis $[Q2]$ the operator F is a contraction mapping.

Step III. The operator G is compact and continuous.

Since the function $f\in C_{1-\beta}[a,b]$, it is obvious from the definition of $C_{1-\beta}[a,b]$ that the operator G is continuous.

From the equation (3.6) of Step I clearly, G is uniformly bounded on B_s. Next we prove the compactness.

For any $a<t_1<t_2\leq b$ we have

$$\begin{aligned}\left|(Gu)(t_1)-(Gu)(t_2)\right|=&\left|\frac{1}{\Gamma(\mu)}\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\right.\\ &\left.-\frac{1}{\Gamma(\mu)}\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\right|\\ \leq&\frac{\|f\|_{C_{1-\beta}}}{\Gamma(\mu)}\left|\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}x^{\rho-1}dx\right.\\ &\left.-\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}x^{\rho-1}dx\right|\\ \leq&\frac{\|f\|_{C_{1-\beta}}B(\mu,\beta)}{\Gamma(\mu)}\left|\left(\frac{{t_1}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}-\left(\frac{{t_2}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\right|\end{aligned}$$

tending to zero as $t_2 \to t_1$, whether $\mu + \beta - 1 \geq 0$ or $\mu + \beta - 1 < 0$. Thus, G is equicontinuous. Hence, by Arzel-Ascoli Theorem, the operator G is compact on B_s.

It follows from Krasnoselskii fixed point theorem that the IVP (1.1)–(1.2) has at least one solution $u \in C_{1-\beta}[a,b]$. Using the Lemma 2.7 and repeating the process of proof in Theorem 2.2, one can show that this solution is actually in $C^{\beta}_{1-\beta}[a,b]$. This completes the proof. ☐

3.1. **Ulam-Hyers-Rassias stability.** In this section, we discuss the Ulam stability results for the solution of IVP (1.1)–(1.2).

Definition 3.1 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be Ulam-Hyers stable if there exists a real number $\psi > 0$ such that for every $\varepsilon > 0$ and for each solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu;\nu}_{a^+}u\right)(t) - f\left(t, u\left(t\right)\right)\right| \leq \varepsilon, \quad t \in (a,b]\,, \tag{3.7}$$

there exists $v \in C_{\beta,\rho}$, a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \varepsilon\psi, \quad t \in (a,b]\,.$$

Definition 3.2 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be generalized Ulam-Hyers stable if there exists a continuous function $\psi_f : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi_f(0) = 0$ such that for every solution $u \in C_{\beta,\rho}$ of the inequality (3.7) there exists $v \in C_{\beta,\rho}$, a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \psi_f(\varepsilon)\,, \quad t \in (a,b]\,.$$

Definition 3.3 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be Ulam-Hyers-Rassias stable with respect to $\Psi \in C_{\beta,\rho}((a,b], \mathbb{R}_+)$ if there exists a real number $0 < \psi_\theta$ such that for every $0 < \varepsilon$ and for every solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu,\nu}_{a^+}u\right)(t) - f\left(t, u\left(t\right)\right)\right| \leq \varepsilon\Psi(t)\,, \quad t \in (a,b]\,, \tag{3.8}$$

there exists $v \in C_{\beta,\rho}$ a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \varepsilon\psi_\theta\Psi(t)\,, \quad t \in (a,b]\,.$$

Definition 3.4 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be generalized Ulam-Hyers-Rassias stable with respect to $\Psi \in C_{\beta,\rho}((a,b], \mathbb{R}_+)$ if there exists a real number $0 < \psi_\theta$ such that for every solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu;\nu}_{a^+}u\right)(t) - f\left(t, u\left(t\right)\right)\right| \leq \Psi(t)\,, \quad t \in (a,b]\,, \tag{3.9}$$

there exists $v \in C_{\beta,\rho}$ a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \psi_\theta\Psi(t)\,, \quad t \in (a,b]\,.$$

Remark 3.1 ([1]). Clearly

(a) from Definition 3.1 follows Definition 3.2;
(b) from Definition 3.3 follows Definition 3.4;
(c) from Definition 3.3 for $\Psi(\cdot) = 1$ follows Definition 3.2.

Now, we establish the results on generalized Ulam-Hyers-Rassias stability of the IVP (1.1)–(1.2).

Theorem 3.2. *Assume that* [$Q1$] *and following hypothesis hold.*

[$Q3$] *There exists* $\omega_\theta > 0$ *such that for each* $t \in (a, b]$ *we have*

$$^{\rho}I^{\mu}_{a^+}\Psi(t) \leq \omega_\theta \Psi(t).$$

[$Q4$] *There exists a function* $p \in C[(a,b],[0,\infty)]$ *such that for each* $t \in (a,b]$

$$|f(t,u(t))| \leq \frac{p(t)\Psi(t)}{1+|u|}|u|.$$

Then the solution of IVP (1.1)–(1.2) *satisfies the generalized Ulam-Hyers-Rassias stability with respect to* Ψ.

Proof. Let u be a solution of the inequality (3.9) and let v be a solution of IVP (1.1)–(1.2). Then we have

$$\begin{aligned}
v(t) =& \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,v(x))\,dx \\
&+ \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,v(x))\,dx \\
=& \Phi_v + \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,v(x))\,dx,
\end{aligned}$$

where

$$\Phi_v = \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,v(x))\,dx.$$

On the other hand, if $\sum_{i=1}^{m}\lambda_i u(\kappa_i) = \sum_{i=1}^{m}\lambda_i v(\kappa_i)$ and $^{\rho}I^{1-\beta}_{a^+}u(a) = {}^{\rho}I^{1-\beta}_{a^+}v(a)$, then $\Phi_u = \Phi_v$. Indeed,

$$\begin{aligned}
|\Phi_u - \Phi_v| \leq& \frac{|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1} \\
&\times |f(x,u(x)) - f(x,v(x))|\,dx \\
\leq& \frac{|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}J|u-v|\,dx \\
\leq& \frac{J|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i{}^{\rho}I^{1-\beta}_{a^+}|u(\kappa_i) - v(\kappa_i)| \\
=& 0.
\end{aligned}$$

Hence, $\Phi_u = \Phi_v$. Then we have

$$v(t) = \Phi_u + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, v(x))\, dx.$$

From inequality (3.9) and $[Q3]$ for each $t \in (a, b]$ we have

$$\left| u(t) - \Phi_u - \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \right| \leq {}^\rho I_{a^+}^\mu \Psi(t) \leq \omega_\theta \Psi(t).$$

Set $\tilde{p} = \sup_{t\in(a,b]} p(t)$. From the hypothesis $[Q3]$ and $[Q4]$ for each $t \in (a, b]$ we have

$$\begin{aligned}
|u(t) - v(t)| \leq & \left| u(t) - \Phi_u - \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \right| \\
& + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} |f(x, u(x)) - f(x, v(x))|\, dx \\
\leq & \omega_\theta \Psi(t) + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} 2\tilde{p} \Psi(x)\, dx \\
\leq & \omega_\theta \Psi(t) + 2\tilde{p} \left({}^\rho I_{a^+}^\mu \Psi\right)(t) \\
\leq & (1 + 2\tilde{p})\, \omega_\theta \Psi(t) \\
:= & \psi_\theta \Psi(t).
\end{aligned}$$

Thus, the IVP (1.1)–(1.2) is generalized Ulam-Hyers-Rassias stable with respect to Ψ. This completes the proof. ☐

Following theorem will be useful in the progress of our next result.

Theorem 3.3 ([1])**.** *Let (Ω, d) be a generalized complete metric space and a strictly contractive operator $\Phi : \Omega \to \Omega$, with a Lipschitz constant $E < 1$. If there exists a non negative integer j such that $d(\Phi^{j+1}u, \Phi^{j+1}u) < \infty$ for some $u \in \Omega$, then the following propositions hold true:*

- **A:** *$\{\Phi^j u\}_{n\in\mathbb{N}}$ converges to a fixed point u^* of Φ;*
- **B:** *u^* is a unique fixed point of Φ in $\Omega^* = \{v \in \Omega : d(\Phi^* u, v) < \infty\}$;*
- **C:** *if $v \in \Omega^*$, then $d(v, u^*) \leq \frac{1}{1-E} d(v, \Phi u)$.*

Let $Z = Z(I, \mathbb{R})$ be the metric space with the metric

$$d(u, v) = \sup_{t\in(a,b]} \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta} |u(t) - v(t)|}{\Psi(t)}.$$

Theorem 3.4. *Assume that $[Q3]$ and the following assumption hold.*

[Q5] *There exists* $\phi \in C((a,b],[0,\infty))$ *such that for every* $u, v \in \mathbb{R}$ *and for each* $t \in (a,b]$, *we have*

$$\left|f(t,u) - f(t,v)\right| \leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta} \phi(t)\, \Psi(t) \left|u - v\right|.$$

If

$$E := \left(\frac{G^\rho - a^\rho}{\rho}\right)^{1-\beta} \phi^* \omega_\theta < 1,$$

where $\phi^* = \sup_{t\in(a,b]} \phi(t)$, *then there exists a unique solution* u_0 *of the IVP* (1.1)–(1.2) *and IVP* (1.1)–(1.2) *is generalized Ulam-Hyers-Rassias stable. Moreover,*

$$\left|u(t) - u_0(t)\right| \leq \frac{\Psi(t)}{1-E}.$$

Proof. Let the operator $\Delta : C_{\beta,\rho} \to C_{\beta,\rho}$ be defined in (3.2). By using Theorem 3.3, we have

$$\begin{aligned}
\left|(\Delta u)(t) - (\Delta v(t))\right| \leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \left|f(x, u(x)) - f(x, v(x))\right| dx \\
\leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \phi(x)\, \Psi(x) \\
& \times \left| \left(\frac{x^\rho - a^\rho}{\rho}\right)^{1-\beta} u(x) - \left(\frac{x^\rho - a^\rho}{\rho}\right)^{1-\beta} v(x) \right| dx \\
\leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \phi^*(x)\, \Psi(x) \left\|u - v\right\|_{C_{1-\beta}} dx \\
\leq & \phi^* \left({}^\rho I_{a+}^{\mu}\right) \Psi(t) \left\|u - v\right\|_C \\
\leq & \phi^* \omega_\theta \Psi(t) \left\|u - v\right\|_C.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left| \left(\frac{t^\rho - x^\rho}{\rho}\right)^{1-\beta} (\Delta u)(t) - \left(\frac{t^\rho - x^\rho}{\rho}\right)^{1-\beta} (\Delta v(t)) \right| \leq & \left(\frac{G^\rho - a^\rho}{\rho}\right)^{1-\beta} \phi^* \omega_\theta \\
& \times \Psi(t) \left\|u - v\right\|_C.
\end{aligned}$$

Thus, we have

$$d(\Delta u, \Delta v) = \sup_{t\in(a,b]} \frac{\left\|(\Delta u)(t) - (\Delta v(t))\right\|_C}{\Psi(t)} \leq E \|u - v\|_C.$$

This completes the theorem. $\square$

3.2. **Examples.**

Example 3.1. Consider the following IVP:

$$
\begin{aligned}
&{}^{\rho}D_{0^+}^{\mu,\nu}u(t) = \frac{|u(t)|}{50e^{t+5}(1+|u(t)|)}, \quad t \in (0,1], && (3.10)\\
&{}^{\rho}I_{0^+}^{1-\beta}u(0) = 5u\left(\frac{1}{2}\right) + 3u\left(\frac{3}{4}\right), \quad \beta = \mu + \nu(1-\mu), && (3.11)
\end{aligned}
$$

where $\mu = \frac{1}{2}$, $\nu = \frac{2}{3}$ and $\beta = \frac{5}{6}$. Set $f(t,u) = \frac{|u|}{50e^{t+5}(1+|u|)}$, $t \in (0,1]$.

It is obvious that the function f is continuous. For any $u, v \in \mathbb{R}$ and $t \in (0,1]$

$$|f(t,u) - f(t,v)| \leq \frac{1}{50e^5}|u-v|.$$

Thus, the condition $[Q1]$ of Theorem 3.1 is satisfied, with $J = \frac{1}{50e^5}$. Moreover, with some elementary computation for $\rho > 0$ we have

$$|K| = \left|\left\{\Gamma\left(\frac{5}{6}\right) - \left[5\left(\frac{(1/2)^\rho - 0^\rho}{\rho}\right)^{-1/6} + 3\left(\frac{(3/4)^\rho - 0^\rho}{\rho}\right)^{-1/6}\right]\right\}^{-1}\right| < 1$$

and

$$
\begin{aligned}
\sigma = &\frac{1}{50e^5} \cdot \frac{B(1/2, 5/6)}{\Gamma(1/2)} \left\{|K|\left[5\left(\frac{(1/2)^\rho - 0^\rho}{\rho}\right)^{1/3} + 3\left(\frac{(3/4)^\rho - 0^\rho}{\rho}\right)^{1/3}\right]\right.\\
&\left. + \left(\frac{1^\rho - 0^\rho}{\rho}\right)^{1/2}\right\} < 1.
\end{aligned}
$$

Hence, the condition $[Q2]$ of Theorem 3.1 is satisfied.

It follows, from Theorem 3.1, that the IVP (3.10)–(3.11) has at least one solution in $C_{1/6}[0,1]$.

Now, let $\Psi(t) = \frac{1}{t^{2\rho-4}}$ and $p(t) = \frac{1}{50e^{t+5}}$, then

$$|f(t,u(t))| \leqslant \frac{1}{50e^{t+5}} \cdot \frac{1}{t^{2\rho-4}} \cdot \frac{|u(t)|}{(1+|u(t)|)}.$$

Thus, the condition $[Q4]$ of Theorem 3.2 is satisfied and with the obvious elementary computation, we have

$${}^{\rho}I_{0^+}^{\mu}\Psi(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_a^t \frac{x^{\rho-1}\Psi(x)}{(t^\rho - x^\rho)^{1-\mu}}dx \leq \frac{1}{\rho^\mu\Gamma(\mu)}B\left(\mu, \frac{4}{\rho} - 1\right)\Psi(t) \leq \omega_\theta\Psi(t).$$

Hence, the condition $[Q4]$ of Theorem 3.2 is satisfied with $\omega_\theta = \frac{1}{\rho^\mu\Gamma(\mu)}B\left(\mu, \frac{4}{\rho} - 1\right)$. It follows from the Theorem 3.2 that the IVP (3.10)–(3.11) is generalized Ulam-Hyers-Rassias stable.

4. Conclusion

We have investigated the sufficient conditions for the existence of solutions to the nonlocal initial value problems involving generalized Katugampola derivative. We have used Krasnoselskii fixed point theorem to develop the existence results. Further, we established some conditions for the generalized Ulam-Hyers-Rassias stability corresponding to the considered problem. Finally, as an application, a suitable example is given to demonstrate our main results.

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[1]Department of First Year Engineering,
Pimpri Chinchwad College of Engineering,
Nigdi, Pune-411044, India
Email address: arif.bagwan@gmail.com

[2]Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad-431004, India
Email address: pachpatte@gmail.com