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A CATEGORICAL CONNECTION BETWEEN CATEGORIES (m, n)-HYPERRINGS AND (m, n)-RING VIA THE FUNDAMENTAL RELATION Γ^*

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Abstract. Let R be an (m, n)-hyperring. The Γ^*-relation on R in the sense of Mirvakili and Davvaz [34] is the smallest strong compatible relation such that the quotient R/Γ^* is an (m, n)-ring. We use Γ^*-relation to define a fundamental functor, F from the category of (m, n)-hyperrings to the category of (m, n)-rings. Also, the concept of a fundamental (m, n)-ring is introduced and it is shown that every (m, n)-ring is isomorphic to R/Γ^* for a nontrivial (m, n)-hyperring R. Moreover, the notions of partitionable and quotientable are introduced and their mutual relationship is investigated. A functor G from the category of classical (m, n)-rings to the category of (m, n)-hyperrings is defined and a natural transformation between the functors F and G is given.

1. Introduction

The notion of n-ary groups (also called n-group or multiary group) is a generalization of that of groups. An n-ary group (G, f) is a pair of a set G and a map $f : G\times\cdots\times G \to G$, which is called an n-ary operation. The earliest work on these structures was done in 1904 by Krasner [24] and in 1928 by Dörnte [22]. Such n-ary groups have many applications to computer science, coding theory, topology, combinatorics and quantum physic (see [18–21, 36] and [38]). One of the applications is the entering into algebraic hyperstructures theory defined by Marty in [30]. This work is initiated by Davvaz and Vougiouklis [16] by defining n-ary hypergroups. By its generalization, (m, n)-hyperrings and (m, n)-hypermodules were introduced and studied in different contexts. Some of the studies can be seen in [2, 5, 11, 27–29, 32, 33] and [34].

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On the other hand, fundamental relations are one of important concepts in algebraic hyperstructures theory which classical algebraic structures will be obtained from algebraic hyperstructures by them. The relations have been studied and investigated on hypergroups in [23] and [25], on hyperrings in [1, 13, 15] and [42], and on hypermodules in [3] and [4]. After defining n-ary hyperstructures, fundamental relations were extended on them. This extension done on n-ary hypergroups in [12] and [16], on (m, n)-hyperrings in [34] and (m, n)-hypermodules in [5]. The Γ^*-relation in the sense of Mirvakili and Davvaz [34] is one of relations on an (m, n)-hyperring by which an (m, n)-ring is induced via the quotient.

In this paper, in Section 2, we give some basic preliminaries about (m, n)-rings and (m, n)-hyperrings. In Section 3, we define the concept of a fundamental (m, n)-ring and prove that every (m, n)-ring is isomorphic to R/Γ^* for a nontrivial (m, n)-hyperring R. In Section 4, we define the notion of quotiontable and partitionable (m, n)-hyperrings and study a relationship between them. Finally, in Section 5, we introduce the category of (m, n)-hyperrings, denoted by $(m, n) - \mathcal{H}_r$ and investigate functorial connections between the categories of (m, n)-hyperrings and (m, n)-rings via Γ^*-relation. Moreover, a natural transformation between these functors is characterized.

2. (m, n)-Rings and (m, n)-Hyperrings

In this section we recall some definitions about (m, n)-rings and (m, n)-hyperrings based on [9, 16] and [34] for development of our paper.

Let H be a nonempty set. A mapping $f : \underbrace{H \times \cdots \times H}_{n} \longrightarrow H$ $(\mathcal{P}^*(H))$, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H, is called an n-ary operation (hyperoperation). A pair (H, f) consisting of a set H and an n-ary operation (hyperoperation) f on H is called an n-ary groupoid (hypergroupoid). Note that for abbreviation, the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j and for $j < i$, x_i^j is the empty set. Also, $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \cdots = y_j = y$ the last expression will be written as $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$. If f is an n-ary operation (hyperoperation) and $t = l(n-1)+1$ for some $l \geq 1$, then t-ary operation (hyperoperation) $f_{(l)}$ is defined by

$$f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{l}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(l-1)(n-1)+2}^{l(n-1)+1}).$$

An n-ary operation (hyperoperation) f is called associative, if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right),$$

holds, for every $1 \leq i < j \leq n$ and all $x_1^{2n-1} \in H$. An n-ary groupoid (hypergroupoid) with the associative n-ary operation (hyperoperation) is called an n-ary semigroup (semihypergroup). An n-ary groupoid (hypergroupoid) (H, f) in which the equation $b = f(a_1^{i-1}, x_i, a_{i+1}^n)$ $(b \in f(a_1^{i-1}, x_i, a_{i+1}^n))$ has a solution $x_i \in H$, for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an n-ary quasigroup (quasihypergroup).

If (H, f) is an n-ary semigroup (semihypergroup) and an n-ary quasigroup (quasihypergroup), then (H, f) is called an n-ary group (hypergroup). An n-ary groupoid (hypergroupoid) (H, f) is commutative, if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$, then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

Definition 2.1. Let (H, f) be an n-ary group (hypergroup). A non-empty subset B of H is called an n-ary subgroup (subhypergroup) of (H, f), if $f(x_1^n) \in B$ ($f(x_1^n) \subseteq B$) for all $x_1^n \in B$, and the equation $b = f(b_1^{i-1}, x_i, b_{i+1}^n)$ ($b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$) has a solution $x_i \in B$, for all $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

Definition 2.2. An (m, n)-ring (hyperring) is an algebraic structure (R, f, g), which satisfies the following axioms:

(1) (R, f) is an m-ary group (hypergroup);
(2) (R, g) is an n-ary semigroup (semihypergroup);
(3) the n-ary operation (hyperoperation) g is distributive with respect to the m-ary operation (hyperoperation) f, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

We say that an (m, n)-ring (hyperring) (R, f, g) has an identity element if there exists $1 \in R$ such that $x = g(1^{(i)}, x, 1^{(n-i-1)})$ ($\{x\} = g(1^{(i)}, x, 1^{(n-i-1)})$) for all $0 \leq i \leq n-1$.

Example 2.1. Consider the ring $(\mathbb{Z}, +, \cdot)$ where "$+$" and "$\cdot$" are ordinary addition and multiplication on the set of all integers. It is easy to see that $\mathbb{Z}$ with $f(x, y, z) = x+y+z$ and $g(x, y, z) = x \cdot y \cdot z$ for all $x, y, z \in \mathbb{Z}$ will give rise to a $(3, 3)$-ring. Now, consider the following 3-ary hyperoperations on $\mathbb{Z}$ $h(x, y, z) = \{x, y, z, x+y, x+z, y+z, x+y+z\}$ and $k(x, y, z) = \{x \cdot y \cdot z\}$. Then, it can be seen that $(\mathbb{Z}, h, k)$ is a $(3, 3)$-hyperring.

Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two (m, n)-hyperrings. The mapping $\varphi : R_1 \to R_2$ is called a homomorphism from R_1 to R_2, if for all $x_1^m, y_1^n \in R_1$ we have

$$\varphi(f_1(x_1^m)) = f_2(\varphi(x_1), \ldots, \varphi(x_m)) \quad \text{and} \quad \varphi(g_1(y_1^n)) = g_2(\varphi(y_1), \ldots, \varphi(y_n)).$$

3. Fundamental (m, n)-Rings

Let (R, f, g) be an (m, n)-hyperring and ρ be an equivalence relation on R. If A and B are non-empty subsets of R, then $A\bar{\rho}B$ means that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $\nu \in B$, there exists $u \in A$ that $u\rho\nu$. We write $A\bar{\bar{\rho}}B$ if $a\rho b$ for any $a \in A$ and $b \in B$. The equivalence relation ρ is called compatible on (R, f), if $a_i\rho b_i$ for all $1 \leq i \leq m$ implies that $f(a_1^m)\bar{\rho}f(b_1^m)$. Moreover, it is called strongly compatible if $f(a_1^m)\bar{\bar{\rho}}f(b_1^m)$ when $a_i\rho b_i$ for $1 \leq i \leq m$.

Now assume that $\frac{R}{\rho} = \{\rho(r) \mid r \in R\}$, be the set of all equivalence classes of R with respect to ρ. Define m-ary and n-ary hyperoperations f/ρ and g/ρ on $\frac{R}{\rho}$ as follow:

$$f/\rho(\rho(a)_1^m) = \{\rho(c) \mid c \in f(\rho(a)_1^m)\} \quad \text{and} \quad g/\rho(\rho(a)_1^n) = \{\rho(c) \mid c \in g(\rho(a)_1^n)\}.$$

Based on [16], in [34], it was shown that $(R/\rho, f/\rho, g/\rho)$ is an (m,n)-hyperring (ring) if and only if ρ is (strongly) compatible relation on R. Mirvakili and Davvaz in [34] introduced the strongly compatible relation Γ^* on (m,n)-hyperrings as follows.

Let (R,f,g) be an (m,n)-hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, where $s = k(m-1)+1$, the relation $\Gamma_{k;l_1^s}$ is defined by

$$x\Gamma_{k;l_1^s}y \;\Leftrightarrow\; \{x,y\} \subseteq f_{(k)}(u_1,\dots,u_s),$$

where $u_i = g_{(l_i)}(x_{i1}^{it_i})$ for some $x_{i1}^{it_i} \in R$ with $t_i = l_i(n-1)+1$ such that $1 \leq i \leq s$. Now, set $\Gamma_k = \bigcup\limits_{l_1^s \in N} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup\limits_{k \in \mathbb{N}^*} \Gamma_k$. The results [34, Theorem 5.5 and 5.6] yield that the transitive closure of Γ, Γ^*, is a strongly compatible relation on R that is the smallest equivalence relation such that $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m,n)-ring. Hence, Γ^* is said to be a *fundamental* relation on R.

Lemma 3.1. *Let (R,f,g), (S,f',g') be (m,n)-hyperrings and $h : R \to S$ be a homomorphism. Then, for all $x,y \in R$,*

(i) $x\Gamma^*y$ implies $h(x)\Gamma^*h(y)$;
(ii) if h is an injection, then $h(x)\Gamma^*h(y)$ implies that $x\Gamma^*y$;
(iii) if h is a bijection, then $x\Gamma^*y$ if and only if $h(x)\Gamma^*h(y)$;
(iv) if h is a bijection, then $h(\Gamma^*(x)) = \Gamma^*(h(x))$.

Proof. (i) Let $x\Gamma^*y$. Then there exist $k, l_1^s \in \mathbb{N}$ and $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$ and $1 \leq i \leq s$ such that $\{x,y\} \subseteq f_{(k)}(u_1,\dots,u_s)$, where $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Since h is homomorphism, we have

$$\begin{aligned}
\{h(x),h(y)\} = h\{x,y\} &\subseteq h\Big(f_{(k)}(u_1,\dots,u_s)\Big)\\
&= f'_{(k)}\Big(h(u_1,\dots,u_s)\Big)\\
&= f'_{(k)}\Big(h\Big(g_{(l_1)}(x_{11}^{1t_1}),\dots,g_{(l_s)}(x_{s1}^{st_s})\Big)\Big)\\
&= f'_{(k)}\Big(g'_{(l_1)}\Big(h(x)_{11}^{1t_1}\Big),\dots,g'_{(l_s)}\Big(h(x)_{s1}^{st_s}\Big)\Big).
\end{aligned}$$

So, $h(x)\Gamma^*h(y)$.

(ii) For $x,y \in R$, since $h(x)\Gamma^*h(y)$, there exist $k, l_1^s \in \mathbb{N}$ and $z_{i1}^{it_i} \in S$, where $t_i = l_i(n-1)+1$ and $1 \leq i \leq s$ such that $\{h(x),h(y)\} \subseteq f'_{(k)}(u_1,\dots,u_s)$ for $u_i = g'_{(l_i)}(z_{i1}^{it_i})$. Now, for an injection $h : (R,f,g) \to (S,f',g')$ we have

$$\begin{aligned}
\{x,y\} = \Big\{h^{-1}(h(x)), h^{-1}(h(y))\Big\} &= h^{-1}\Big(\{h(x),h(y)\}\Big)\\
&\subseteq h^{-1}\Big(f'_{(k)}(u_1,\dots,u_s)\Big)\\
&= f_{(k)}\Big(g_{(l_1)}\Big(h^{-1}(z)_{11}^{1t_1}\Big),\dots,g_{(l_s)}\Big(h^{-1}(z)_{s1}^{st_s}\Big)\Big).
\end{aligned}$$

So, $x\Gamma^*y$.

(iii) It is clear by (i) and (ii).

(iv) Let $x \in R$. By (iii), we have

$$h(\Gamma^*(x)) = \bigcup_{y\in\Gamma^*(x)} h(y) = \bigcup_{x\Gamma^*y} h(y) = \bigcup_{h(x)\Gamma^*h(y)} h(y) = \Gamma^*(h(x)). \qquad \square$$

Corollary 3.1. *Let* (R_1, f_1, g_1) *and* (R_2, f_2, g_2) *be isomorphic* (m, n)*-hyperrings. Then* $R_1/\Gamma^* \cong R_2/\Gamma^*$.

Proof. Let $h : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$ be an isomorphism. Define $\eta : R_1/\Gamma^* \to R_2/\Gamma^*$ by $\eta\Big(\Gamma^*(x)\Big) = \Gamma^*\Big(h(x)\Big)$. By Lemma 3.1, η is well-defined, one to one and onto. Hence, η is an isomorphism, since h is a homomorphism. $\square$

Definition 3.1. An (m, n)-ring (R, f, g) is called a *fundamental* (m, n)*-ring* if there exists a non-trivial (m, n)-hyperring, say (S, f', g'), such that $(S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g)$.

Remark 3.1. It is needed to explain what a non-trivial (m, n)-hyperring is. An (m, n)-hyperring (S, f', g') is said to be *trivial* if $|f'(x_1^m)| = |g'(y_1^n)| = 1$ for all $x_1^m, y_1^n \in S$. For example, let (R, f, g) be an (m, n)-ring. Define m-ary and n-ary hyperoperations $f'(x_1^m) = \{f(x_1^m)\}$ and $g'(y_1^n) = \{g(y_1^n)\}$ for all $x_1^m, y_1^n \in R$. Then (R, f', g') is a trivial (m, n)-hyperring.

Lemma 3.2. *Let* (R, f, g) *be an* (m, n)*-ring with identity, then for any* (m, n)*-ring* S *with identity, there exist* m*-ary and* n*-ary hyperoperations "*f'*" and "*g'*" on* $R \times S$ *such that* $(R \times S, f', g')$ *is an* (m, n)*-hyperring.*

Proof. Let S be a non-zero (m, n)-ring with identity 1. Define m-ary and n-ary hyperoperations "f'" and "g'" on $R \times S$ as follows:

$$f'\Big((r_1, s_1), \ldots, (r_m, s_m)\Big) = \Big\{(f(r_1^m), s_1), \ldots, (f(r_1^m), s_m)\Big\},$$
$$g'\Big((r_1, s_1), \ldots, (r_n, s_n)\Big) = \Big\{(g(r_1^n), s_1), \ldots, (g(r_1^n), s_n)\Big\}.$$

(For abbreviation, $f'\Big((r_1, s_1), \ldots, (r_m, s_m)\Big)$ denoted by $f'\Big((r, s)_1^m\Big)$, similarly this is for g'). Clearly "f'" and "g'" are associative and "g'" is distributive with respect to "f'". Also, we have

$$\begin{aligned} f'\Big((r,s)_1^{i-1}, R\times S, (r,s)_{i+1}^m\Big) &= \bigcup_{(r',s')\in R\times S} f'\Big((r,s)_1^{i-1}, (r',s'), (r,s)_{i+1}^m\Big) \\ &= \bigcup_{(r',s')\in R\times S} \Big\{(f(r_1^{i-1}, r', r_{i+1}^m), s_1), \ldots, (f(r_1^{i-1}, r', r_{i+1}^m), s_{i-1}), \\ &\quad (f(r_1^{i-1}, r', r_{i+1}^m), s'), (f(r_1^{i-1}, r', r_{i+1}^m), s_{i+1}), \\ &\quad \ldots, (f(r_1^{i-1}, r', r_{i+1}^m), s_m)\Big\} \\ &= R \times S. \end{aligned}$$

Thus, $(R \times S, f', g')$ is an (m,n)-hyperring. $\square$

The (m,n)-hyperring $(R \times S, f', g')$ is called an *associated* (m,n)*-hyperring* to R (via S) and denoted by R_S.

Theorem 3.1. *Let (R, f, g) and (T, f, g) be isomorphic (m,n)-rings with identity. Then, for any (m,n)-ring S with identity, R_S and T_S are isomorphic (m,n)-hyperrings.*

Proof. Let $h : R \to T$ be an homomorphism. Define $\omega : (R \times S, f', g') \to (T \times S, f', g')$ by $\omega(r,s) = (h(r), s)$ for all $(r,s) \in R \times S$. Since h is an isomorphism, it is easy to see that ω is well-defined and a bijection. Now we verify that ω is a homomorphism.

$$\begin{aligned}
\omega\Big(f'\Big((r,s)_1^m\Big)\Big) &= \omega\Big(\Big\{(f(r_1^m), s_1), \ldots, (f(r_1^m), s_m)\Big\}\Big)\\
&= \Big\{\omega\Big(f(r_1^m), s_1\Big), \ldots, \omega\Big(f(r_1^m), s_m\Big)\Big\}\\
&= \Big\{\Big(h(f(r_1^m)), s_1\Big), \ldots, \Big(h(f(r_1^m)), s_m\Big)\Big\}\\
&= \Big\{\Big(f(h(r)_1^m), s_1\Big), \ldots, \Big(f(h(r)_1^m), s_m\Big)\Big\}\\
&= f'\Big((h(r), s)_1^m\Big)\\
&= f'\Big(\omega\Big((r,s)_1^m\Big)\Big).
\end{aligned}$$

Similarly, $\omega\Big(g'\Big((r,s)_1^n\Big)\Big) = g'\Big(\omega\Big((r,s)_1^n\Big)\Big)$. Thus, $(R \times S, f', g') \cong (T \times S, f', g')$. $\square$

Theorem 3.2. *Every (m,n)-ring is a fundamental (m,n)-ring.*

Proof. Let (R, f, g) be an (m,n)-ring. By Lemma 3.2, for any (m,n)-ring S, $(R \times S, f', g')$ is an (m,n)-hyperring. For any $r \in R$ and $(s, s') \in S \times S$ we have $\{(r,s), (r,s')\} = g'((r,s), (1,s')_1^{n-1})$, so $(r,s)\Gamma^*(r,s')$. Hence, $(r,s') \in \Gamma^*(r,s)$. Thus, $\Gamma^*(r,s) = \{(r,x) \mid x \in S\}$. Define the mapping $\theta : (R \times S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \to (R, f, g)$ by $\theta(\Gamma^*(r,s)) = r$. It is clear that θ is well-defined and one to one, since for any $(r,s), (r',s') \in R \times S$, $\Gamma^*(r,s) = \Gamma^*(r',s')$ if and only if $(r',s') \in \Gamma^*(r,s)$ if and only if $r = r'$ if and only if $\theta(\Gamma^*(r,s)) = \theta(\Gamma^*(r',s'))$. θ is a homomorphism. Let $(r,s)_1^m$, $(r,s)_1^n \in R \times S$. We have

$$\begin{aligned}
\theta\Big(f'/\Gamma^*(\Gamma^*(r,s)_1^m)\Big) =&\theta\Big(\Gamma^*(f(r_1^m), s_1)\Big) = \cdots = \theta\Big(\Gamma^*(f(r_1^m), s_m)\Big) = f(r_1^m)\\
=&f\Big(\theta(\Gamma^*(r,s))_1^m\Big)\theta\Big(g'/\Gamma^*(\Gamma^*(r,s)_1^n)\Big) = \theta\Big(\Gamma^*(g(r_1^n), s_1)\Big)\\
=&\cdots = \theta\Big(\Gamma^*(g(r_1^n), s_n)\Big) = g(r_1^n) = g\Big(\theta(\Gamma^*(r,s))_1^n\Big).
\end{aligned}$$

Since for any $r \in R$, $\theta(\Gamma^*(r,0)) = r$, then θ is onto. Thus, θ is an isomorphism. $\square$

Theorem 3.3. *Let (R, f, g) be an (m, n)-hyperring. Then there exist an (m, n)-ring S, m-ary and n-ary hyperoperations f' and g' on $R \times S$ such that (R, f, g) can be embedded in $(R \times S, f', g')$.*

Proof. Let (R, f, g) be an (m, n)-hyperring and set $S = (R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$. Define m-ary and n-ary hyperoperations f' and g' on $R \times R/\Gamma^*$, as following:

$$f'\Big((r, \Gamma^*(v))_1^m\Big) = \Big(f(r_1^m), \Gamma^*(f(v_1^m))\Big),$$
$$g'\Big((r, \Gamma^*(v))_1^n\Big) = \Big(g(r_1^n), \Gamma^*(g(v_1^n))\Big).$$

Let $(r, \Gamma^*(v))_1^m = (r', \Gamma^*(v'))_1^m$, then $r_j = r'_j$ and $\Gamma^*(v_j) = \Gamma^*(v'_j)$ for all $1 \le j \le m$. Since $\Gamma^*(v_j) = \Gamma^*(v'_j)$ for all $j = 1, \dots, m$, there exist k_j, $l_{1_j}^{s_j} \in \mathbb{N}$ and $x_{i_j 1}^{i_j t_{i_j}} \in R$, where $t_{i_j} = l_{i_j}(n-1)+1$ and $i_j = 1_j, \dots, s_j$, such that $\{v_j, v'_j\} \subseteq f_{(k_j)}(u_{1_j}, \dots, u_{s_j})$, where $u_{i_j} = g_{(l_j)}(x_{i_j 1}^{i_j t_{i_j}})$. Hence,

$$\begin{aligned}\{f(v_1^m), f(v'^m_1)\} &\subseteq \Big\{f(v_1^m), f(v_1, v'^m_2), f(v'_1, v_2, v'^m_3), \dots, f(v'^m_1)\Big\}\\ &\subseteq f\Big(f_{(k_1)}(u_{1_1}, \dots, u_{s_1}), \dots, f_{(k_m)}(u_{1_m}, \dots, u_{s_m})\Big)\end{aligned}$$

and

$$\begin{aligned}\{g(v_1^n), g(v'^n_1)\} &\subseteq \Big\{g(v_1^n), g(v_1, v'^n_2), g(v'_1, v_2, v'^n_3), \dots, g(v'^n_1)\Big\}\\ &\subseteq g\Big(f_{(k_1)}(u_{1_1}, \dots, u_{s_1}), \dots, f_{(k_n)}(u_{1_n}, \dots, u_{s_n})\Big).\end{aligned}$$

Thus, $\Gamma^*(f(v_1^m)) = \Gamma^*(f(v'^m_1))$ and $\Gamma^*(g(v_1^n)) = \Gamma^*(g(v'^n_1))$. So, $(f(r_1^m), \Gamma^*(f(v_1^m) = f(r'^m_1), \Gamma^*(f(v'^m_1))$ and $(g(r_1^n), \Gamma^*(g(v_1^n)) = (g(r'^n_1), \Gamma^*(g(v'^n_1)))$. Therefore, the m-ary and n-ary hyperoperations f' and g' are well-defined. Now, we show that $(R \times S, f', g')$ is an (m, n)-hyperring. Let $(r, \Gamma^*(v))_1^m \in R \times S$. Then for any $i, j \in \{1, \cdots, m\}$, since "$f$" is associative, it follows that:

$$\begin{aligned}&f'\Big((r, \Gamma^*(v))_1^{i-1}, f'((r, \Gamma^*(v))_i^{m+i-1}), (r, \Gamma^*(v))_{m+i}^{2m-1}\Big)\\ =&\Big(f(r_1^{i-1}, f(r_i^{m+i-1}), r_{m+i}^{2m-1}), \Gamma^*(f(v_1^{i-1}, f(v_i^{m+i-1}), v_{m+i}^{2m-1})\Big)\\ =&\Big(f(r_1^{j-1}, f(r_j^{m+j-1}), r_{m+j}^{2m-1}), \Gamma^*(f(v_1^{j-1}, f(v_j^{m+j-1}), v_{m+j}^{2m-1})\Big)\\ =&f'\Big((r, \Gamma^*(v))_1^{j-1}, f'((r, \Gamma^*(v))_j^{m+j-1}), (r, \Gamma^*(v))_{m+j}^{2m-1}\Big).\end{aligned}$$

So, f' is associative. Similarly, it can be shown that g' is associative on $R \times S$. Now, we verify the reproduction property. Since $f(r_1^{i-1}, R, r_{i+1}^m) = R$ and $R/\Gamma^* = \bigcup_{t \in R} \Gamma^*(t)$, so

$$
\begin{aligned}
&f'\Big((r,\Gamma^*(v))_1^i, R\times S, (r,\Gamma^*(v))_{i+1}^m\Big)\\
=&\bigcup_{(r',\Gamma^*(v'))\in R\times S} f'\Big((r,\Gamma^*(v))_1^i,(r',\Gamma^*(v')),(r,\Gamma^*(v))_{i+1}^m\Big)\\
=&\bigcup_{(r',\Gamma^*(v'))\in R\times S}\Big(f(r_1^i,r',r_{i+1}^m),\Gamma^*(f(v_1^i,s',v_{i+1}^m))\Big)\\
&= R\times\Gamma^*(R)=R\times S.
\end{aligned}
$$

To investigate distributivity law, let $(r',\Gamma^*(v'))_1^m\in R\times S$, $(r,\Gamma^*(v))_1^n\in R\times S$. Since g is distributive with respect to f, then

$$
\begin{aligned}
&g'\Big((r,\Gamma^*(v))_1^{i-1}, f'((r',\Gamma^*(v'))_1^m),(r,\Gamma^*(v))_{i+1}^n\Big)\\
=&\Big(g(r_1^{i-1},f({r'}_1^m),r_{i+1}^n),\Gamma^*(g(v_1^{i-1},f({v'}_1^m),v_{i+1}^n))\Big)\\
=&\Big(f(g(r_1^{i-1},{r'}_1,r_{i+1}^n),\dots,g(r_1^{i-1},{r'}_m,r_{i+1}^n)),\\
&\Gamma^*(f(g(v_1^{i-1},{v'}_1,v_{i+1}^n),\cdots,g(v_1^{i-1},{v'}_m,v_{i+1}^n)))\Big)\\
=&f'\Big(g'((r,\Gamma^*(v))_1^{i-1},(r',\Gamma^*(v'))_1,(r,\Gamma^*(v))_{i+1}^n),\dots,\\
&g'((r,\Gamma^*(v))_1^{i-1},(r',\Gamma^*(v'))_m,(r,\Gamma^*(v))_{i+1}^n)\Big).
\end{aligned}
$$

So, $(R\times S,f',g')$ is an (m,n)-hyperring. Now, define the mapping $\theta:(R,f,g)\to(R\times S,f',g')$, by $\theta(r)=(r,\Gamma^*(r))$. Let $r,r'\in R$. Then $r=r'$ if and only if $(r,\Gamma^*(r))=(r',\Gamma^*(r'))$ if and only if $\theta(r)=\theta(r')$. Let $r_1^m,r_1^n\in R$. Then

$$\theta(f(r_1^m))=(f(r_1^m),\Gamma^*(f(r_1^m))=f'((r,\Gamma^*(r))_1^m)=f'(\theta(r)_1^m)$$

and

$$\theta(g(r_1^n))=(g(r_1^n),\Gamma^*(g(r_1^n))=g'((r,\Gamma^*(r))_1^n)=g'(\theta(r)_1^n),$$

where $\theta(r)_1^k$ means $\theta(r_1),\dots,\theta(r_k)$ for $k=m$ or $k=n$. Thus, (R,f,g) can be embedded in $(R\times S,f',g')$. □

Theorem 3.4. *Let R and S be two sets such that $|R|=|S|$. If (R,f,g) is an (m,n)-hyperring, then there exist m-ary and n-ary hyperoperations "f'" and "g'" on "S", such that (R,f,g) and (S,f',g') are isomorphic (m,n)-hyperrings*

Proof. Since $|R|=|S|$, then there exists a bijection $\phi:R\to S$. For any $s_1^m,s_1^n\in S$, define the m-ary and n-ary hyperoperations "f'" and "g'" as follows:

$$f'(s_1^m)=\phi(f(r_1^m)),\quad g'(s_1^n)=\phi(g(r_1^n)).$$

First we prove that f' and g' are well-defined. Let $s_i={s'}_i$, where $s_i=\phi(r_i)$, $s'_i=\phi(r'_i)$ and $r_i,r'_i\in R$ for $i=1,\dots,m$. So, $s_i={s'}_i$ implies that $\phi(r_i)=\phi(r'_i)$. Since ϕ is bijection, then $r_i = r'_i$ for $i = 1, \ldots, m$ and so $f'(s_1^m) = \phi(f(r_1^m)) = \phi(f({r'}_1^m)) = f'({s'}_1^m)$, similarly $g'(s_1^n) = g'({s'}_1^n)$. Moreover, since

$$\phi(f(r_1^m)) = f'(\phi(r)_1^m), \qquad (3.1)$$
$$\phi(g(r_1^n)) = g'(\phi(r)_1^n),$$

ϕ is a homomorphism. Now, it is enough to show that (S, f', g') is an (m, n)-hyperring. Define the map $\theta : (R, f, g) \to (S, f', g')$ by $\theta(x) = \phi(x)$. Since ϕ is bijection then θ is a bijection. Now we show that θ is a homomorphism. Let $r_1^m \in R$. Then, by (3.1), $\theta(f(r_1^m)) = \phi(f(r_1^m)) = f'(\phi(r)_1^m) = f'(\theta(r)_1^m)$ and $\theta(g(r_1^n)) = \phi(g(r_1^n)) = g'(\phi(r)_1^n) = g'(\theta(r)_1^n)$. Thus, θ is an isomorphism and so (S, f', g') is an (m, n)-hyperring. □

Corollary 3.2. *Let (R, f, g) be an (m, n)-ring of infinite order. Then there exist m-ary and n-ary hyperoperations "f'" and "g'" on R such that (R, f, g) is a fundamental (m, n)-ring of itself, i.e., $(R/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g)$.*

Proof. For a given (m, n)-ring (R, f, g), consider the smallest associated (m, n)-hyperring $(R \times \mathbb{Z}_2, f', g')$. By Theorem 3.2, $\left(\frac{(R \times \mathbb{Z}_2, f', g')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right) \cong (R, f, g)$. Since R is infinite set, then $|R| = |R \times \mathbb{Z}_2|$ and, by Theorem 3.4, there exist m-ary and n-ary hyperoperations "f''" and "g''" on (R, f, g), such that (R, f'', g'') and $(R \times \mathbb{Z}_2, f', g')$, are isomorphic (m, n)-hyperrings. Now, we have

$$(R, f, g) \cong \left(\frac{(R \times \mathbb{Z}_2, f', g')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right) \cong \left(\frac{(R, f'', g'')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right).$$

Hence, (R, f, g) is a fundamental (m, n)-ring of itself. □

We recall the relation $\beta_f = \bigcup_{k \geq 1} \beta_k$ on an n-ary semihypergroup (R, f) defined by Davvaz and Vougiouklis in [16], where $x\beta_k y$ if and only if there exist $t = k(m-1)+1$ and $z_1^t \in R$ such that $\{x, y\} \subseteq f_{(k)}(z_1^t)$. It is well known that β_f is the smallest strongly compatible equivalence relation on n-ary semihypergroup (R, f) such that $(R/\beta_f, f/\beta_f)$ is an n-ary semigroup. Clearly, $\beta_f \subseteq \Gamma$ and so $\beta_f^* \subseteq \Gamma^*$.

Theorem 3.5. *Every finite (m, n)-ring is not its fundamental (m, n)-ring.*

Proof. Let (R, f, g) be a finite (m, n)-ring, $|R| = n$. If "f'" and "g'", are m-ary and n-ary hyperoperations on R, such that (R, f, g) is an (m, n)-hyperring, then there exist $x_1^m \in R$ such that $|f'(x_1^m)| \geq 2$. Hence, there are $a, b \in f(x_1^m)$. So $a\beta_f b$ and then $a\Gamma b$. Therefore, $a\Gamma^* b$ and $\Gamma^*(a) = \Gamma^*(b)$. Since $R/\Gamma^* = \{\Gamma^*(t) \mid t \in R\}$, then $|R/\Gamma^*| < n$. Thus, $(R, f, g) \ncong (R/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*)$. □

4. Embeddable (m, n)-Hyperring

In this section we introduce the concepts of partitionable and quotientable (m, n)-hyperrings and investigate the relation between them. Also, we give some results concerning about these concepts.

Definition 4.1. An (m,n)-hyperring (R,f_1,g_1) is said to be a partitionable (m,n)-hyperring if there exists an (m,n)-ring (S,f,g), an equivalence relation ρ on (S,f,g), non-trivial m-ary and n-ary hyperoperations f' and g' such that $(S/\rho, f', g') \cong (R, f_1, g_1)$.

Theorem 4.1. *Every (m,n)-hyperring is a partitionable (m,n)-hyperring.*

Proof. Let (R,f,g) be an (m,n)-hyperring. Then we consider three cases.

Case 1. Let R be finite and $|R| = n$. Define on $\mathbb{Z}$ the equivalence relation ρ by

$$x\rho y \Leftrightarrow x \equiv y \pmod{n}.$$

Clearly $|R| = |\mathbb{Z}/\rho|$. So, by Theorem 3.4, there exist m-ary and n-ary hyperoperations f' and g' on $\mathbb{Z}/\rho$, such that $(\mathbb{Z}/\rho, f', g')$ is an (m,n)-hyperring and $(R,f,g) \cong (\mathbb{Z}/\rho, f', g')$.

Case 2. Let R be infinite countable. Then $|R| = |\mathbb{Z}|$. Let $\mathcal{A} = \{A_i\}_{i\in\mathbb{Z}}$ be a partition of $\mathbb{Z}$ such that there exists an index $j \in \mathbb{Z}$ such that $|A_j| = 2$ and for any $j \neq i \in \mathbb{Z}$, $|A_i| = 1$. Clearly, the binary relation ρ on $\mathbb{Z}$, by

$$r\rho s \Leftrightarrow (\exists k \in \mathbb{Z}) \text{ s.t } \{r,s\} \subseteq A_k$$

is an equivalence relation on $\mathbb{Z}$ and clearly $|\mathbb{Z}| = |\mathcal{A}| = \left|\frac{\mathbb{Z}}{\rho}\right|$. Thus, by Theorem 3.4, there exist m-ary and n-ary hyperoperations "f'" and "g'" on $\mathbb{Z}/\rho$, such that $(\mathbb{Z}/\rho, f', g')$ is an (m,n)-hyperring and $(R, f_1, g_1) \cong (\mathbb{Z}/\rho, f', g')$.

Case 3. Let R be uncountable. Then $|R| = |\mathbb{R}|$ and similarly as in case 2 it can be concluded that R is a partitionable (m,n)-hyperring. $\square$

Let (R,f,g) be an (m,n)-ring. We say that (N,g) is a *normal* subgroup of n-semigroup (R,g), if $g(a_1^{i-1}, N, a_{i+1}^n) = g(a_{\sigma(1)}^{\sigma(i-1)}, N, a_{\sigma(i+1)}^{\sigma(n)})$, for all $a_1^n \in R$, $\sigma \in \mathbb{S}_n$ and $1 \leq i \leq n$. Also, for a normal subgroup N of (S,g), we set

$$S/N = \{g(x_1^{i-1}, N, x_{i+1}^n) \mid x_i \in S, 1 \leq i \leq n\}.$$

Definition 4.2. An (m,n)-hyperring (R,f,g) is called a quotientable (m,n)-hyperring if there exist an (m,n)-ring (S,h,k), non-trivial m-ary and n-ary hyperoperations f' and g' such that $(S/N, f', g') \cong (R,f,g)$, where N is a normal subgroup of the n-semigroup of (S,k).

Theorem 4.2. *Every (m,n)-hyperring is a quotientable (m,n)-hyperring.*

Proof. Let (R,f,g) be an (m,n)-hyperring and consider the following cases.

Case 1. Let R be finite and $|R| = n$. Consider $(\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{\bar{0}\}, \odot)$ and set $g(x_1^n) = \bigodot_{i=1}^{n} x_i$ for $x_1^n \in \mathbb{Z}_n$. Clearly, $N = \{\bar{1}\}$ is a normal subgroup of $(\mathbb{Z}_n^*, g)$ and $|R| = |\mathbb{Z}_n/N|$. Thus, by Theorem 3.4, there exist m-ary and n-ary hyperoperations f' and g' on $\mathbb{Z}_n/N$ such that $(\mathbb{Z}_n/N, f', g')$ is an (m,n)-hyperring and $(R,f,g) \cong (\mathbb{Z}_n/N, f', g')$.

Case 2. Let R be infinite countable and $|R| = |\mathbb{Z} \times \mathbb{Z}|$. Note that $(\mathbb{Z} \times \mathbb{Z}, f, g)$ is an (m,n)-ring such that $f((a,b)_1^m) = (a_1 + \cdots + a_m, b_1 + \cdots + b_m)$ and $g((a,b)_1^n) =$

$(a_1 \cdot a_2 \cdots a_n, b_1 \cdot b_2 \cdots b_n)$ for any $a_1^m, a_1^n, b_1^m, b_1^n \in \mathbb{Z}$, where " $+$ " and " $\cdot$ " are ordinary binary operations on $\mathbb{Z}$. Now, let $N = \{(-1,1),(1,1)\}$. Then N is a normal in $((\mathbb{Z}\times\mathbb{Z})^*, g)$. Clearly $|\mathbb{Z}\times\mathbb{Z}| = |(\mathbb{Z}\times\mathbb{Z})/N|$. Hence, by Theorem 3.4, there exist m-ary and n-ary hyperoperations f' and g' on $(\mathbb{Z}\times\mathbb{Z})/N$ such that $((\mathbb{Z}\times\mathbb{Z})/N, f', g')$ is an (m,n)-hyperring and $(R,f,g) \cong ((\mathbb{Z}\times\mathbb{Z})/N, f', g')$.

Case 3. Let R be uncountable. Then $|R| = |\mathbb{R}\times\mathbb{R}|$ and similarly as in case 2 we conclude that R is a quotientable (m,n)-hyperring. □

Theorem 4.3. *Every quotientable (m,n)-hyperring is a partitionable (m,n)-hyperring.*

Proof. Let (R, f_1, g_1) be a quotientable (m,n)-hyperring. Then, there exist an (m,n)-ring (S,f,g), non-trivial m-ary and n-ary hyperoperations f' and g' such that $(S/N, f', g') \cong (R, f_1, g_1)$, where N is a normal subgroup the n-semigroup (S,g). Define, the binary relation ρ on S as follows:

$$x\rho y \Leftrightarrow g(x, x_2^{i-1}, N, x_{i+1}^n) = g(y, x_2^{i-1}, N, x_{i+1}^n).$$

Clearly ρ is an equivalence relation on S and for any $s \in S$, $\rho(s) = g(s, x_2^{i-1}, N, x_{i+1}^n)$. Hence, (R, f_1, g_1) is a partitionable (m,n)-hyperring. □

Remark 4.1. Consider the (m,n)-hyperring $(\mathbb{Z}_3, f, g)$ with the m-ary and n-ary hyperoperations $f(x_1^m) = \mathbb{Z}_3$ and $g(y_1^n) = \mathbb{Z}_3$ for all $x_1^m, y_1^n \in \mathbb{Z}_3$. Define on $\mathbb{Z}$ the relation ρ by $\rho = \{(0,0),(2k,2k'),(2k+1,2k'+1)\}$ Clearly ρ is an equivalence relation and $|\mathbb{Z}_3| = |\frac{\mathbb{Z}}{\rho}|$. Hence, by Theorem 4.1, $(\mathbb{Z}_3, f, g)$ is a partitionable (m,n)-hyperring. But ρ is not a multiplicative normal n-subgroup of $\mathbb{Z}$. Thus, the converse of Theorem 4.3, is not valid.

Let (R, f_1, g_1) be an (m,n)-hyperring. Consider the canonical projection $\varphi : (R, f_1, g_1) \to (R/\Gamma^*, f_1/\Gamma^*, g_1/\Gamma^*)$ by $\varphi(r) = \Gamma^*(r)$. Also, by Theorem 4.2, there exist an (m,n)-ring (S,f,g), normal n-subgroup N such that $\theta : (R, f_1, g_1) \to (S/N, f', g')$ is an isomorphism. Hence, we have the following theorem.

Theorem 4.4. *Let (R, f_1, g_1) be a quotientable (m,n)-hyperring via an (m,n)-ring (S,f,g). Then there exists a unique homomorphism ψ, such that $\psi\theta = \varphi$.*

Proof. Since (R, f_1, g_1) is a quotientable (m,n)-hyperring via an (m,n)-ring (S,f,g), there exists a normal subgroup of the n-semigroup (S,g) such that $(S/N, f', g') \cong (R, f_1, g_1)$. Define $\psi : S/N \to R/\Gamma^*$ by $\psi(g(s_1^{i-1}, N, s_{i+1}^n)) = \Gamma^*(r)$ such that $\theta(r) = g(s_1^{i-1}, N, s_{i+1}^n)$ for any $s_1^n \in S$. Therefore $\psi = \varphi \circ \theta^{-1}$, so ψ is a homomorphism. Also, $\psi\theta(r) = (\varphi\circ\theta^{-1})(\theta(r)) = \varphi(r)$. Thus, the following diagram is commutative.

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & S/N \arrow[d, "\psi"] \\ & R/\Gamma^* \end{tikzcd}$$

Moreover, it is easy to see that ψ is unique. □

Corollary 4.1. *Let (R, f_1, g_1) be a quotientable (m, n)-hyperring via an (m, n)-ring (S, f, g). Then the following diagram is commutative.*

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "\varphi"'] & S/N \arrow[d, "\bar{\varphi}"] \\ R/\Gamma^* \arrow[r, "\bar{\theta}"] & (S/N)/\Gamma^* \end{tikzcd}$$

Proof. Define the maps $\bar{\theta} : R/\Gamma^* \to (S/N)/\Gamma^*$ by $\bar{\theta}(\Gamma^*(r)) = \Gamma^*(\theta(r))$ and $\bar{\varphi} : S/N \to (S/N)/\Gamma^*$ by $\bar{\varphi}(g(s_1^{i-1}, N, s_{i+1}^n)) = \Gamma^*(g(s_1^{i-1}, N, s_{i+1}^n))$. Since θ and φ are homomorphism, $\bar{\theta}$ and $\bar{\varphi}$ are so. Hence, for any $r \in R$

$$\bar{\varphi}\theta(r) = \bar{\varphi}\Big(g(s_1^{i-1}, N, s_{i+1}^n)\Big) = \Gamma^*\Big(g(s_1^{i-1}, N, s_{i+1}^n)\Big) = \Gamma^*(\theta(r)) = \bar{\theta}(\Gamma^*(r)) = \bar{\theta}\varphi(r).$$

□

5. Categorical Relations on (m, n)-Hyperrings and (m, n)-Rings

Now we introduce the category of (m, n)-hyperrings, denoted by $(m, n) - \mathcal{H}_r$. This category is defined as follows:

(i) the objects of $(m, n) - \mathcal{H}_r$ are (m, n)-hyperrings;

(ii) for the objects R and R' of $(m, n) - \mathcal{H}_r$, the set of all homomorphisms from R to R' are arrows and denoted by $h : R \to R'$.

In this section, we try to investigate the relation between two categories $(m, n) - \mathcal{H}_r$ and $(m, n) - \mathcal{R}_g$ (category of (m, n)-rings) and work on natural transformations between them. At first, we define an arrow $F : (m, n) - \mathcal{H}_r \to (m, n) - \mathcal{R}_g$ by $F(R) = R/\Gamma^*$, where (R, f, g) is an object of $(m, n) - \mathcal{H}_r$ and for any arrow $\nu : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$, we define:

$$F(\nu) : R_1/\Gamma^* \to R_2/\Gamma^* \text{ by } F(\nu)(\Gamma^*(x)) = \Gamma^*(\nu(x)), \quad \text{for every } x \in R_1.$$

By Corollary 3.1, F is well-defined. Hence, we have the following.

Theorem 5.1. *F is a covariant functor from $(m, n) - \mathcal{H}_r$ to $(m, n) - \mathcal{R}_g$.*

Proof. For any object (R, f, g) of $(m, n) - \mathcal{H}_r$, $F(R) = R/\Gamma^*$ is an (m, n)-ring and then $F(R)$ is an object in $(m, n) - \mathcal{R}_g$. Now, we show that $F(\nu)$ is an arrow in $(m, n) - \mathcal{R}_g$, for any arrow $\nu : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$. Let $\Gamma^*(x)_1^m,\ \Gamma^*(x)_1^n \in R_1/\Gamma^*$. Thus,

$$\begin{aligned} F(\nu)\Big(f_1/\Gamma^*\Big(\Gamma^*(x)_1^m\Big)\Big) &= F(\nu)\Big(\Gamma^*\Big(f_1(x_1^m)\Big)\Big) = \Gamma^*\Big(\nu\Big(f_1(x_1^m)\Big)\Big) \\ &= \Gamma^*\Big(f_2\Big(\nu(x_1), \ldots, \nu(x_m)\Big)\Big) \\ &= f_2/\Gamma^*\Big(\Gamma^*\Big(\nu(x_1)\Big), \ldots, \Gamma^*\Big(\nu(x_m)\Big)\Big) \\ &= f_2/\Gamma^*\Big(F(\nu)\Big(\Gamma^*(x_1)\Big), \ldots, F(\nu)\Big(\Gamma^*(x_m)\Big)\Big). \end{aligned}$$

Similarly, we have

$$F(\nu)\Big(g_1/\Gamma^*\Big(\Gamma^*(x)_1^n\Big)\Big) = g_2/\Gamma^*\Big(F(\nu)\Big(\Gamma^*(x_1)\Big),\ldots,F(\nu)\Big(\Gamma^*(x_n)\Big)\Big).$$

Also for the composition of two arrows $F(\nu)$ and $F(\omega)$ in $(m,n)-\mathcal{R}_g$, where $\nu : (R_1,f_1,g_1)\to(R_2,f_2,g_2)$ and $\omega : (R_2,f_2,g_2)\to(R_3,f_3,g_3)$, we have

$$F(\omega)\circ F(\nu) = F(\omega)(F(\nu)) = F(\omega)(\Gamma^*(\nu)) = \Gamma^*(\omega\circ\nu) = F(\omega\circ\nu).$$

Moreover, for $1_R : R\to R$ and $1_{F(R)} : R/\Gamma^*\to R/\Gamma^*$, we have

$$F(1_R)\Big(\Gamma^*(x)\Big) = \Gamma^*(1_R(x)) = \Gamma^*(x) = 1_{F(R)}(x).$$

Therefore, F is a covariant functor of $(m,n)-\mathcal{H}_r$ to $(m,n)-\mathcal{R}_g$. $\square$

Now, for $(m,n)-\mathcal{H}_r$, $(m,n)-\mathcal{R}_g$, any (m,n)-ring (R,f,g) and $S=\mathbb{Z}_2$, define a categorical arrow $U : (m,n)-\mathcal{R}_g\to(m,n)-\mathcal{H}_r$ by $U(R)=R_S$, which for any (m,n)-ring homomorphism $\nu : (R_1,f_1,g_1)\to(R_2,f_2,g_2)$ defined by

$$U(\nu)(x,y) = (\nu,1_S)(x,y) = (\nu(x),1_S(y)) = (\nu(x),y).$$

By Theorem 3.1, U is well-defined. Hence, we have the following theorem.

Theorem 5.2. *U is a covariant functor from $(m,n)-\mathcal{R}_g$ to $(m,n)-\mathcal{H}_r$.*

Proof. For any object (R,f,g) of $(m,n)-\mathcal{R}_g$ by Lemma 3.2, $U(R)=R\times S=R_S$ is an (m,n)-hyperring and so $U(R)$ is an object in $(m,n)-\mathcal{H}_r$. Consider any arrow $\nu : (R_1,f_1,g_1)\to(R_2,f_2,g_2)$ in $(m,n)-\mathcal{R}_g$. We show that $U(\nu)$ is an arrow in $(m,n)-\mathcal{H}_r$. Let $(r,s)_1^m$, $(r,s)_1^n\in R_1\times S$. Now, by Lemma 3.2,

$$\begin{aligned}
U(\nu)\Big(f_1'\Big((r,s)_1^m\Big)\Big) &= U(\nu)\Big(\{(f_1(r_1^m),s_1),\ldots,(f_1(r_1^m),s_m)\}\Big)\\
&= \Big\{U(\nu)\Big(f_1(r_1^m),s_1\Big),\ldots,U(\nu)\Big(f_1(r_1^m),s_m\Big)\Big\}\\
&= \Big\{\Big(\nu(f_1(r_1^m)),s_1\Big),\ldots,\Big(\nu(f_1(r_1^m)),s_m\Big)\Big\}\\
&= \Big\{\Big(f_2\Big(\nu(r_1),\ldots,\nu(r_m)\Big),s_1\Big),\ldots,\Big(f_2\Big(\nu(r_1),\ldots,\nu(r_m)\Big),s_m\Big)\Big\}\\
&= f_2'\Big(\Big(\nu(r_1),s_1\Big),\ldots,\Big(\nu(r_m),s_m\Big)\Big)\\
&= f_2'\Big(U(\nu)(r_1,s_1),\ldots,U(\nu)(r_m,s_m)\Big).
\end{aligned}$$

Similarly, we have $U(\nu)(g_1'\Big((r,s)_1^n\Big)) = g_2'(U(\nu)(r_1,s_1),\ldots,U(\nu)(r_n,s_n))$. Thus, $U(\nu) : R_1\times S\to R_2\times S$ is an (m,n)-hyperring homomorphism and so is an arrow in $(m,n)-\mathcal{H}_r$. Now, we investigate the composition property. Let ν and ω be arrows in $(m,n)-\mathcal{R}_g$. So,

$$U(\nu)\circ U(\omega)(r,s) = U(\nu)\Big(U(\omega)(r,s)\Big) = U(\nu)\Big(\omega(r),s\Big) = \Big(\nu\circ\omega(r),s\Big) = U(\nu\circ\omega)(r,s).$$

Moreover, consider $1_R : R \to R$ and $1_{U(R)} : U(R) \to U(R)$. For any $(r, s) \in R_S$

$$U(1_R)(r, s) = (1_R(r), s) = (r, s) = 1_{U(R)}(r, s).$$

Hence, U is a covariant functor of $(m, n) - \mathcal{R}_g$ to $(m, n) - \mathcal{H}_r$. □

Theorem 5.3. *The functor $U : (m, n) - \mathcal{R}_g \to (m, n) - \mathcal{H}_r$ is a faithful functor.*

Proof. Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be objects in $(m, n) - \mathcal{R}_g$, $\nu_1, \nu_2 : R_1 \to R_2$ be parallel arrows of $(m, n) - \mathcal{R}_g$ and $U(\nu_1) = U(\nu_2)$. So, for any $(r, s) \in R_{1S}$, $U(\nu_1)(r, s) = U(\nu_2)(r, s)$ and so $\nu_1 = \nu_2$. Thus, U is a faithful functor. □

Theorem 5.4. *On objects of $(m, n) - \mathcal{R}_g$, $F \circ U = 1$.*

Proof. For any object (R, f, g) in $(m, n) - \mathcal{R}_g$, we have

$$(F \circ U)(R, f, g) = F(R_S, f', g') = (R_S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g),$$

by Theorem 3.2. □

Theorem 5.5. *For functors $1, F \circ U : (m, n) - \mathcal{R}_g \to (m, n) - \mathcal{R}_g$ there exists a natural transformation $\mu : 1 \to F \circ U$.*

Proof. For two functors 1 and $F \circ U$ of $(m, n) - \mathcal{R}_g$ to $(m, n) - \mathcal{R}_g$, define a map $\mu : 1 \to F \circ U$ as follows:

$$\mu : 1(R) \to (F \circ U)(R) \text{ by } \mu(r) = \Gamma^*(r, 0).$$

Now, for any (m, n)-ring homomorphism $\nu : (R, f, g) \to (R', f', g')$, consider the following diagram.

$$\begin{tikzcd} 1(R) \arrow[r, "\mu_R"] \arrow[d, "1(\nu)"'] & (F \circ U)(R) \arrow[d, "F\circ U(\nu)"] \\ 1(R') \arrow[r, "\mu_{R'}"] & (F \circ U)(R') \end{tikzcd}$$

For any $r \in R$, we have

$$\begin{aligned} \Big((F \circ U)(\nu) \circ \mu_R\Big)(r) = F \circ U(\nu)\Big(\mu_R(r)\Big) &= F \circ U(\nu)\Big(\Gamma^*(r, 0)\Big) \\ &= \Gamma^*\Big(\nu(r), 0\Big) \\ &= \mu_{R'}\Big(\nu(r)\Big) \\ &= \mu_{R'}\Big(1(\nu)(r)\Big) = \Big(\mu_{R'} \circ 1(\nu)\Big)(r). \end{aligned}$$

So, μ is a natural transformation. □

Theorem 5.6. *For functors 1 and $U \circ F$ from $(m, n) - \mathcal{H}_r$ to $(m, n) - \mathcal{H}_r$, there exists a transformation $\theta : 1 \to U \circ F$ such that is natural.*

Proof. For two functors $1, U\circ F : (m,n)-\mathcal{H}_r \to (m,n)-\mathcal{H}_r$, define a map $\theta : 1 \to U\circ F$ as $\theta : 1(R) \to (U\circ F)(R)$ by $\theta(r) = \Big(\Gamma^*(r), 0\Big)$. Now, for any (m,n)-hyperring homomorphism $\nu : (R,f,g) \to (R',f',g')$, consider the following diagram.

$$\begin{array}{ccc} 1(R) & \xrightarrow{\theta_R} & (U\circ F)(R) \\ {\scriptstyle 1(\nu)}\downarrow & & \downarrow{\scriptstyle U\circ F(\nu)} \\ 1(R') & \xrightarrow{\theta_{R'}} & (U\circ F)(R') \end{array}$$

For any $r \in R$, we have

$$\begin{aligned}\Big((U\circ F)(\nu)\circ\theta_R\Big)(r) = U\circ F(\nu)\Big(\theta_R(r)\Big) &= U\circ F(\nu)\Big(\Gamma^*(r),0\Big)\\ &= \Big(\Gamma^*\Big(\nu(r)\Big),0\Big)\\ &= \theta_{R'}\Big(\nu(r)\Big)\\ &= \theta_{R'}\Big(1(\nu)(r)\Big)\\ &= \Big(\theta_{R'}\circ 1(\nu)\Big)(r).\end{aligned}$$

Therefore, θ is a natural transformation. □

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