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EXISTENCE AND ULAM STABILITY OF SOLUTION FOR SOME BACKWARD IMPULSIVE DIFFERENTIAL EQUATIONS ON BANACH SPACES

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Abstract. We consider backward impulsive differential equations with local and nonlocal conditions in Banach spaces. We establish less restrictive sufficient conditions for the existence and uniqueness of solutions than those found in earlier literature. The analysis primarily employs basic results from fixed point theory as the main tool. Furthermore, we investigate the Ulam stability of the solutions.

1. Introduction

Fractional calculus is a branch of mathematics that generalizes differentiation and integration to non-integer and complex orders. This extends the traditional framework of calculus, which is limited to integer orders. It provides powerful tools for describing and analyzing systems that exhibit non-local, nonlinear, and memory-dependent behavior. Applications span many fields, contributing to the understanding of complex phenomena in science and engineering.

Impulsive differential equations are used to model systems that experience sudden, instantaneous changes-known as impulses-at specific moments in time. These abrupt changes, such as collisions or environmental shocks, are often represented using impulse functions like the Dirac delta function. These equations combine fractional derivatives with impulse effects and are applied in mechanics, physics, engineering, biology, and control theory. They are crucial for modeling real-world phenomena,

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including earthquakes, chemical reactions, electrical circuits, and population dynamics. These equations help analyze systems with abrupt changes, such as intermittent control actions in engineering or environmental impacts on populations in biology.

The study of existence and Ulam stability for backward impulsive differential equations has been extensively explored in recent years. For fundamental theoretical foundations, landmark texts such as Deimling (1985) [7], Lakshmikantham et al. (1989) [10], Benchohra et al. (2006) [6], and Zhou (2014) [15] provide comprehensive treatments of functional analysis, impulsive systems, and fractional calculus. Building upon these foundations, early investigations into Ulam-type stability for impulsive ordinary differential equations were conducted by Wang et al. (2012) [13], while Benchohra and Lazreg (2014) [5] enriched the field by examining existence and stability for nonlinear implicit fractional differential equations with delay. Subsequent work expanded these stability analyses to broader contexts: Abbas et al. (2015) [1], Anguraj and Kanjanadevi (2017) [2], and Belbali and Benbachir (2021) [3] investigated global Ulam-Hyers stability and random impulses, while Liu et al. (2018) [11] introduced Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay equations. More recently, Benahmed et al. (2025) [4] further extended these stability insights for nonlinear impulsive differential equations. Specifically concerning backward systems in Banach spaces, key advancements were established by Hao and Liu (2017) [9] and Mahmoudi and Kessi (2021) [12]. Furthermore, the fixed-point methodology underlying our approach is well-supported in the literature; for instance, Ghanmi and Horrigue (2019) [8] treated a coupled system of fractional equations using Schauder's fixed-point theorem, while Wang et al. (2019) [14] established existence and uniqueness results for a single fractional boundary value problem.

In this paper, using some well-known fixed point theorems, we study the existence of solutions and their Ulam stability for the following backward impulsive fractional differential equations in Banach spaces

$$\begin{cases} {}^{CH}{}_tD_b^{\alpha}u(t) = f(t,u(t)), & t \in J=[1,T],\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k\left(u(t_k^-)\right), & k=1,\ldots,m, \\ u(T) = u_T, \end{cases} \tag{1.1}$$

where ${}^{CH}{}_tD_b^{\alpha}$ denotes the Caputo-Hadamard fractional derivative, with $0<\alpha<1$, and $1=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=T$, and where $\Delta u|_{t=t_k} = u\left(t_k^+\right) - u\left(t_k^-\right)$ represents the jump of the function u at t_k. The functions $I_k:\mathbb{R}\to\mathbb{R}$, for $k=1,2,\ldots,m$, are given, and $f: J\times\mathbb{R}\to\mathbb{R}$ is a nonlinear real-valued function.

Our approach consists in converting the initial value problem (1.1) into an equivalent integral equation and applying fixed point theorems such as those of Schaefer, Banach, or Krasnoselskii. Furthermore, we prove the existence of a unique solution-or at least one solution-to this problem with nonlocal conditions.

Consider the following nonlocal problem:

$$\begin{cases} {}^{CH}{}_tD_b^\alpha u(t) = f(t, u(t)), \quad t \in J = [1, T], t \neq t_k, \quad k = 1, \ldots, m, \\ \Delta u|_{t=t_k} = I_k \left(u \left(t_k^- \right) \right), \quad k = 1, \ldots, m, \\ u(T) - g(u) = u_T, \end{cases} \tag{1.2}$$

where f and I_k, for $k = 1, \ldots, m$, are defined as in the previous paragraph, and $g : PC(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function, where $PC(J, \mathbb{R})$ denotes the set of piecewise continuous functions, which will be defined later.

Our main contribution is to extend the space of interest to include continuous functions that are not Lipschitz, but whose primitives are Lipschitz continuous. For example, the function $f(x) = \sqrt{x}$ on $[0, 1]$ is not Lipschitz on $[0, 1]$, but its primitive

$$F(x) = \int_0^x \sqrt{t}\, dt = \frac{2}{3} x^{3/2}$$

is Lipschitz on $[0, 1]$. Similarly, the function $f(x) = |x|^{-1/2}$ on $[-1, 1] \setminus \{0\}$ is not Lipschitz, but its primitive is Lipschitz on $[-1, 1]$.

2. Preliminary Results

A normed space V is a Banach space if, whenever $\{x_n\}$ is a sequence of points in V such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ whenever $n, m > N$, then there exists $x \in V$ such that

$$\lim_{n \to +\infty} \|x_n - x\| = 0.$$

The function space we consider in this work is $C(K)$. Let K be a compact metric space. Then,

$$C(K) = \{f : K \to \mathbb{R} \mid f \text{ is continuous}\}.$$

The sup-norm on $C(K)$ is defined by

$$\|f\| = \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|.$$

Definition 2.1. A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous on the interval $[a, b]$ if:

(a) there exists a finite partition of the interval

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

such that f is continuous on each open subinterval (x_{i-1}, x_i) for $i = 1, 2, \ldots, n$;

(b) at each point x_i, for $i = 1, 2, \ldots, n-1$, the one-sided limits:

$$\lim_{x \to x_i^-} f(x) \quad \text{and} \quad \lim_{x \to x_i^+} f(x)$$

exist and are finite.

Theorem 2.1. *The space $C([a,b])$ of continuous, real-valued (or complex-valued) functions on $[a,b]$ with the sup-norm is a complete normed space, hence a Banach space. More generally, the space $C(K)$ of continuous functions on a compact metric space K equipped with the sup-norm is a Banach space.*

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J=[1,T]$ into $\mathbb{R}$ endowed with the norm $\|u\|_\infty=\sup_{t\in J}|u(t)|$.

Theorem 2.2. *The set of piecewise continuous functions*

$$PC(J,\mathbb{R})=\Big\{u:J\to\mathbb{R}:\ u|_{(t_k,t_{k+1})}\in C\left((t_k,t_{k+1}),\mathbb{R}\right),k=0,1,\dots,m,\\ u\left(t_k^-\right)=u\left(t_k\right)\ \text{and}\ u\left(t_k^+\right)\ \text{exists}\ \Big\}$$

equipped with the norm $\|u\|_{PC}=\sup_{t\in J}|u(t)|$, is a Banach space.

Definition 2.2. Let X and Y be Banach spaces (both over $\mathbb{R}$ or $\mathbb{C}$), and let $T:X\to Y$ be an operator. The operator T is said to be:

(a) **Bounded** if it maps bounded subsets of X into bounded subsets of Y;
(b) **Compact** if it maps bounded subsets of X into subsets of Y whose closure is compact; that is, the image of any bounded set under T has compact closure in Y;
(c) **Completely continuous** if it is both compact and continuous.

Theorem 2.3 (PC-type Ascoli-Arzelá theorem)**.** *Let E be a Banach space, and let $W\subset PC(J,E)$ be such that:*

- *W is a uniformly bounded subset of $PC(J,E)$;*
- *W is equicontinuous on each interval (t_k,t_{k+1}), for $k=0,1,\dots,m$, where $t_0=1$ and $t_{m+1}=T$;*
- *for all $t\in J\setminus\{t_1,\dots,t_m\}$, the sets*

$$W(t):=\{u(t)\mid u\in W\},\\ W(t_k^+):=\{u(t_k^+)\mid u\in W\},\\ W(t_k^-):=\{u(t_k^-)\mid u\in W\}$$

are relatively compact subsets of E.

Then, W is a relatively compact subset of $PC(J,E)$.

Theorem 2.4 (Banach Fixed Point Theorem)**.** *Let $T:X\to X$, where X is a complete metric space. If T is a strict contraction, then the set of fixed points of T, denoted by* $\mathrm{Fix}(T)$*, consists of exactly one element $x\in X$, that is, $T(x)=x$.*

Theorem 2.5 (Schaefer's Fixed Point Theorem)**.** *Let E be a Banach space, and let $U\subset E$ be a convex set such that $0\in U$. Let $T:U\to U$ be a completely continuous operator. If*

$$\Omega=\{u\in U:u=\lambda Tu,\ \lambda\in(0,1)\}$$

is bounded, then T has at least one fixed point in E.

Theorem 2.6 (Krasnoselskii's Fixed Point Theorem)**.** *Let S be a nonempty, closed, convex subset of a Banach space E. Let P and Q be two operators satisfying the following conditions:*

(a) *$Px + Qy \in S$ whenever $x, y \in S$;*
(b) *P is a contraction mapping;*
(c) *Q is compact and continuous.*

Then, there exists $z \in S$ such that $z = Pz + Qz$, that is, the operator $P + Q$ has a fixed point in S.

Let f be a continuous function on the interval (a, b), where $0 \leq a < b \leq +\infty$. Consider an interval in $\mathbb{R}^+$, and let $\mu, \alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$.

Definition 2.3. The Hadamard fractional integral of order α of the function f is defined by:

$$ {}^{H}I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau, $$

where $a < t < b$.

Definition 2.4. The Caputo-Hadamard fractional derivative of order $\alpha \in (n-1, n)$, where $n \in \mathbb{N}$, is defined by:

$$ {}^{CH}{}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n f(\tau)\, d\tau, $$

where a is the lower limit of integration, $t > a$, and $\delta^n = \left(t\frac{d}{dt}\right)^n$.

Theorem 2.7. *Let $\mathrm{Re}(\alpha) > 0$, $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$, and let $f \in C^n([a, b])$, with $0 < a < b < +\infty$. Then,*

$$ {}^{H}I^{\alpha}_{a} \left({}^{CH}{}_{a}D^{\alpha}_{t} f(t)\right) = f(t) - \sum_{j=0}^{n-1} \frac{\delta^j f(a)}{j!} \left(\log \frac{t}{a}\right)^j, $$

where $\delta^j = \left(t\frac{d}{dt}\right)^j$.

3. Local Backward Impulsive Fractional Differential Equations

In this section, we focus on the main results concerning the existence of solutions to the problem (1.1). We discuss the conditions under which this problem admits either a unique solution or at least one solution.

To study the problem (1.1), we work under the following assumptions.

(A1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A2) There exists a positive constant λ such that, for any $t_1, t_2 \in J$ and $u, v \in PC(J, \mathbb{R})$,

$$ \int_{t_1}^{t_2} |f(s, u(s)) - f(s, v(s))|\, ds \leq \lambda |u(t) - v(t)|, \quad t \in [t_1, t_2]. $$

(A3) There exists a positive constant θ such that $|f(t,u(t))|<\theta$ for all $t\in J$ and $u\in PC(J,\mathbb{R})$.

(A4) There exists a constant $\mu>0$ such that

$$|I_k(u(t))-I_k(v(t))|\leq \mu|u(t)-v(t)|,$$

for all $u,v\in PC(J,\mathbb{R}),\ t\in J$ and $k=1,\dots,m$.

(A5) The functions $I_k:\mathbb{R}\to\mathbb{R}$ are continuous, and there exists a positive constant γ such that

$$|I_k(u(t))|<\gamma,$$

for all $u\in PC(J,\mathbb{R}),\ t\in J$ and $k=1,\dots,m$.

A function $u\in PC(J,\mathbb{R})$ is called a solution to (1.1) if its derivative exists on $J'=J\setminus\{t_k:k=1,2,\dots,m\}$, and u satisfies the equation

$${}^{CH}{}_tD_b^{\alpha}u(t)=f(t,u(t)),\quad t\in J',$$

together with the conditions

$$\begin{aligned}&\Delta u\Big|_{t=t_k}=I_k\left(u(t_k^-)\right),\quad k=1,\dots,m,\\ &u(T)=u_T.\end{aligned}$$

Lemma 3.1. *A function u is a solution of the integral equation*

$$u(t)=\begin{cases}u_T-\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\ +\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds, & \text{if } t\in(t_m,T),\\ u_T-\sum\limits_{p=0}^{k-1}I_{m-p}\left(u\left(t_{m-p}^-\right)\right)\\ -\frac{1}{\Gamma(\alpha)}\sum\limits_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\ +\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds, & \text{if } t\in(t_{m-k},t_{m-k+1})\end{cases}\tag{3.1}$$

if and only if u is a solution of the backward impulsive differential equation

$$\begin{cases}{}^{CH}{}_tD_b^{\alpha}u(t)=h(t), & t\in J=[1,T],\ t\neq t_k,\\ \Delta u\Big|_{t=t_k}=I_k\left(u\left(t_k^-\right)\right), & k=1,\dots,m,\\ u(T)=u_T.\end{cases}\tag{3.2}$$

Proof. Assume that u satisfies (3.2). For $t\in(t_m,t_{m+1})$, where $t_{m+1}=T$, we have

$$u(t)=u_T-\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds.$$

For $t\in(t_{m-1},t_m)$, we get

$$u(t)=u\left(t_m^-\right)-\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\left(\ln\frac{t_m}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
=&-\left(-u\left(t_m^-\right)\right)-\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\left(\ln\frac{t_m}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
=&-\Delta u(t_m)+u\left(t_m^+\right)-\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\left(\ln\frac{t_m}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
=&-I_m\left(u\left(t_m^-\right)\right)+u_T-\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\left(\ln\frac{t_m}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds.
\end{aligned}
$$

For $t\in(t_{m-2},t_{m-1})$, one has

$$
\begin{aligned}
u(t)=&u\left(t_{m-1}^-\right)-\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t_{m-1}}\left(\ln\frac{t_{m-1}}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
=&-\left(-u\left(t_{m-1}^-\right)\right)-\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t_{m-1}}\left(\ln\frac{t_{m-1}}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
=&-\triangle\left(u\left(t_{m-1}\right)\right)+u\left(t_{m-1}^+\right)\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t_{m-1}}\left(\ln\frac{t_{m-1}}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-2}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
=&-I_m\left(u\left(t_m^-\right)\right)-I_{m-1}\left(u\left(t_{m-1}^-\right)\right)+u_T\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\left(\ln\frac{t_m}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds
\end{aligned}
$$

$$- \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} \left(\ln \frac{t_{m-1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds$$

Then, by induction, we obtain: for $t \in (t_{m-k}, t_{m-k+1})$, where $k = 0, \ldots, m$,

$$\begin{aligned} u(t) = u_T &- \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right) \\ &- \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} \frac{h(s)}{s}\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s}\, ds, \end{aligned}$$

for $k = 1, 2, \ldots, m$.

Conversely, assume that u satisfies the impulsive integral equation (3.2). If $t \in (t_m, T)$, then $u(T) = u_T$. If $t \in (t_{m-k}, t_{m-k+1})$, where $k = 0, \ldots, m$, then by differentiation, we get

$$^{CH}{}_{t}D_{b}^{\alpha} u(t) = h(t).$$

Moreover, we can easily show that

$$\Delta u\Big|_{t=t_k} = I_k\left(u\left(t_k^{-}\right)\right), \quad k = 1, \ldots, m. \qquad \square$$

Let us begin by discussing the conditions under which problem (1.1) admits a unique solution. The following result is based on the Banach fixed point theorem.

Theorem 3.1. *Assume that the function f satisfies conditions* **(A1)**, **(A2)** *and* **(A4)**, *and that*

$$m\mu + \frac{\lambda(m+1)}{\Gamma(\alpha+1)} (\ln T)^{\alpha} < 1. \tag{3.3}$$

Then, the problem (1.1) *has a unique solution in $PC(J, \mathbb{R})$.*

Proof. We transform problem (1.1) into a fixed point problem. Consider the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$\begin{aligned} F(u)(t) = u_T &- \sum_{t<t_k<T} I_k\left(u\left(t_k^{-}\right)\right) \\ &- \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s}\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s}\, ds. \end{aligned}$$

Clearly, a fixed point of the operator F is a solution of problem (1.1). We apply the Banach contraction principle to prove that F has a fixed point. To this end, we show that F is a contraction.

Let $u, v \in PC(J, \mathbb{R})$. Then, for each $t \in J$, we have

$$\begin{aligned}
|F(u)(t) - F(v)(t)| \leq & \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u(s)) - f(s,v(s))|}{s} ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s)) - f(s,v(s))|}{s} ds \\
\leq & \mu \sum_{t<t_k<T} \left|u\left(t_{m-p}^-\right) - v\left(t_{m-p}^-\right)\right| \\
& + \frac{\lambda}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|u(s) - v(s)|}{s} ds \\
& + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|u(s) - v(s)|}{s} ds \\
\leq & m\mu \|u - v\| + \frac{m\lambda (\ln T)^{\alpha}}{\Gamma(\alpha)} \|u - v\| + \frac{\lambda (\ln T)^{\alpha}}{\Gamma(\alpha+1)} \|u(t) - v(t)\| \\
\leq & m\mu \|u - v\| + \frac{m\lambda (\ln T)^{\alpha}}{\Gamma(\alpha+1)} \|u - v\| + \frac{\lambda (\ln T)^{\alpha}}{\Gamma(\alpha+1)} \|u - v\| \\
= & \left[m\mu + \frac{\lambda(m+1)}{\Gamma(\alpha+1)} (\ln T)^{\alpha}\right] \|u - v\|
\end{aligned}$$

Hence, by (3.3), F is a contraction. Therefore, by the Banach contraction principle, we conclude that F has a unique fixed point which is a solution of problem (1.1). □

Remark 3.1. It is worth situating condition (**A2**) relative to the closest related works in the literature, both in terms of the type of hypothesis it replaces and the structural setting in which it is used.

The most direct point of comparison is [12] (Mahmoudi and Kessi), which treats the same backward impulsive differential equations on Banach spaces. Their existence and uniqueness result relies on the pointwise Lipschitz hypothesis

$$|f(t,x) - f(t,y)| \leq \lambda |x - y|, \quad t \in [0,T],\ x, y \in \mathbb{R},$$

and yields the contraction condition $m\mu + \lambda T < 1$. Our condition (A2) replaces this with an integral-averaged Lipschitz bound,

$$\int_{t_1}^{t_2} |f(s,u(s)) - f(s,v(s))|\, ds \leq \lambda |u(t) - v(t)|, \quad t \in [t_1, t_2],$$

which is strictly weaker: it is satisfied by functions such as $f(x) = \sqrt{x}$ on $[0,1]$, which fail the pointwise Lipschitz condition used in [12]. Formally, our contraction constant $m\mu + \frac{\lambda(m+1)}{\Gamma(\alpha+1)}(\ln T)^{\alpha}$ reduces to theirs, $m\mu + \lambda T$, in the limit $\alpha \to 1$ (with the Caputo–Hadamard derivative degenerating to the classical derivative and $\ln T \to T$),

confirming that Theorem 3.1 is a generalization of [12, Theorem 6] rather than a reformulation under a different fractional operator.

A second, structurally different point of comparison is Hao and Liu [9]. There the authors study a semilinear impulsive evolution equation in a general Banach space, in which the linear part is generated by a C_0-semigroup $(T(t))_{t\geq 0}$, and existence of a mild solution is obtained through measure-of-noncompactness arguments combined with a Krasnoselskii- or Sadovskii-type fixed point theorem. The resulting hypotheses are compactness- and equicontinuity-type conditions on the nonlinearity and on the semigroup, made necessary by the presence of an unbounded generator A. By contrast, problem (1.1) does not involve an infinitesimal generator: it is a backward (terminal-value) problem governed by the Caputo-Hadamard fractional derivative in the scalar/finite-dimensional setting of $PC(J,\mathbb{R})$. This structural difference allows us to dispense with any compactness or semigroup-regularity assumption altogether, and to replace it with the single explicit, easily verifiable algebraic inequality (**A2**) on the Lipschitz constants μ and λ.

Taken together, these two comparisons show that the reduction in restrictiveness works along two distinct axes: relative to [12], condition (A2) weakens a hypothesis of the same type (pointwise vs. integral-averaged Lipschitz control) within the same finite-dimensional, non-semigroup setting; relative to [9], it is not a term-by-term weakening of an identical hypothesis at all, but a change in the type of condition required: an analytic (compactness/semigroup) requirement is replaced by a purely algebraic one, which is generally simpler to verify in applications.

The following result provides sufficient conditions for the existence of at least one solution to problem (1.1). It is based on Schaefer's fixed point theorem.

Theorem 3.2. *Suppose that conditions* **(A1)**, **(A2)**, **(A3)**, **(A4)** *and* **(A5)** *are satisfied. Then, the problem* (1.1) *has at least one solution in* $PC(J,\mathbb{R})$.

Proof. The proof of this result is divided into several steps.

Step 1: The operator F is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ on J. For all $t \in [1,T]$, we have

$$\begin{aligned} F_n(t) \leq & \sum_{t<t_k<T} \left| I_k\left(u_n\left(t_k^-\right)\right) - I_k\left(u\left(t_k^-\right)\right)\right| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u_n(s)) - f(s,u(s))|}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u_n(s)) - f(s,u(s))|}{s} ds, \end{aligned}$$

where $F_n(t) = |F(u_n)(t) - F(u)(t)|$. Since f and I_k, for $k = 1,\ldots,m$, are continuous functions, it follows that

$$\|Fu_n - Fu\|_\infty \to 0 \quad \text{as } n \to +\infty,$$

which implies that F is continuous.

Step 2: The operator F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$. For all $u \in B_r$, we have

$$\begin{aligned}
|F(u)(t)| &= \Bigg| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \\
&\quad - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s}\, ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s}\, ds \Bigg| \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s}\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s}\, ds \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| \\
&\quad + \frac{\|f\|}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \frac{\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1}}{s}\, ds + \frac{\|f\|}{\Gamma(\alpha)} \int_{t_k}^{t} \frac{\left(\ln \frac{t}{s}\right)^{\alpha-1}}{s}\, ds \\
&\leq |u_T| + m\gamma + \frac{m\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \\
&= |u_T| + m\gamma + \frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \\
&= \rho.
\end{aligned}$$

Hence, the operator F maps the bounded set B_r into a bounded set B_ρ.

Step 3: The operator F maps bounded sets into equicontinuous subsets of $PC(J,\mathbb{R})$. Let $\tau_1, \tau_2 \in [1,T]$ with $\tau_1 < \tau_2$, and let $u \in B_r$. Then,

$$|F(u)(\tau_2) - F(u)(\tau_1)| \leq \sum_{\tau_1<t_k<\tau_2} \left| I_k\left(u\left(t_k^-\right)\right)\right| + \int_{\tau_1}^{\tau_2} |f\,(s,u(s))|\, \frac{ds}{s},$$

which tends to zero as $\tau_1 \to \tau_2$. By the previous steps, together with the Arzelá-Ascoli theorem, it follows that F is equicontinuous on each interval $[t_k, t_{k+1}]$.

As a consequence of Steps 1–3 and the PC-type Arzelá-Ascoli theorem, we conclude that the operator $F : B_r \to B_\rho$ is continuous and completely continuous.

Step 4: We show that the set

$$\Omega = \{u \in PC(J,\mathbb{R}) : u = \lambda F(u),\ 0 < \lambda < 1\}$$

is bounded. Let $u \in \Omega$, then $u = \lambda F(u)$ for some $0 < \lambda < 1$. Then, for all $t \in [0, T]$, we have

$$\begin{aligned}
|u(t)| =& \lambda \Bigg| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \\
&- \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s}\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s}\, ds \Bigg| \\
\leq& |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s}\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s}\, ds \\
\leq& |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| + \frac{\|f\|}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{1}{s}\, ds \\
&+ \frac{\|f\|}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}\, ds \\
\leq& |u_T| + m\gamma + \frac{m\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \\
=& |u_T| + m\gamma + \frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

This shows that the set Ω is uniformly bounded. By Schaefer's fixed point theorem, the operator F has a fixed point, which is a solution to the problem (1.1). □

4. Nonlocal Backward Impulsive Fractional Differential Equations

Now, we extend the results of the previous section to the nonlocal impulsive differential equation (1.2). We introduce the following assumptions.

(A6) There exists a positive constant C such that $|g(u) - g(v)| \leq C\|u - v\|$, for all $u, v \in PC(J, \mathbb{R})$.

(A7) There exists a positive constant κ such that $|g(u)| \leq \kappa$ for all $u \in PC(J, \mathbb{R})$.

The equation (1.2) is equivalent to the following integral equation:

$$\begin{aligned}
F(u)(t) = u_T + g(u) &- \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \\
&- \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s}\, ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}\,ds.$$

Theorem 4.1. *Assume that the function f satisfies the conditions* **(A2)** *and* **(A4)**, *and that*

$$C+m\mu+\frac{\lambda(m+1)(\ln T)^{\alpha}}{\Gamma(\alpha+1)}<1. \tag{4.1}$$

Then, the problem (1.2) *has a unique solution in $PC(J,\mathbb{R})$.*

Proof. We transform the problem (1.2) into a fixed point problem. Consider the operator $F: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ defined by

$$\begin{aligned}F(u)(t)=&u_T+g(u)-\sum_{t<t_k<T}I_k\left(u\left(t_k^-\right)\right)\\&-\frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}\,ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}\,ds.\end{aligned}$$

It is clear that the fixed points of the operator F are solutions to the problem (1.2). We use the Banach contraction principle to prove that F has a unique fixed point. Specifically, we will show that F is a contraction mapping.

Let $u,v\in PC(J,\mathbb{R})$. Then, for each $t\in J$, we have

$$\begin{aligned}|F(u)(t)-F(v)(t)|\le&|g(u)-g(v)|+\sum_{t<t_k<T}\left|I_k\left(u\left(t_k^-\right)\right)-I_k\left(v\left(t_k^-\right)\right)\right|\\&+\frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\frac{ds}{s}\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\frac{ds}{s}\\\le&C\|u-v\|+\mu\sum_{t<t_k<T}\|u-v\|\\&+\frac{\lambda}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{\alpha-1}\|u-v\|\frac{ds}{s}\\&+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\|u-v\|\frac{ds}{s}\\\le&C\|u-v\|+m\mu\|u-v\|+\frac{\lambda m(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|\\&+\frac{\lambda(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|\\=&\left(C+m\mu+\frac{\lambda m(\ln T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\right)\|u-v\|\end{aligned}$$

$$= \left(C + m\mu + \frac{\lambda(m+1)(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \right) \|u - v\|.$$

Hence, by the condition (4.1), the operator F is a contraction. Therefore, by the Banach contraction principle, F has a unique fixed point, which corresponds to the unique solution of the problem (1.2). □

Theorem 4.2. *If assumptions* **(A1)**, **(A3)**, **(A5)**, **(A6)** *and* **(A7)** *are satisfied, and if* $C < 1$ *and*

$$\frac{|u_T|}{1 - \left(C + m\gamma + \frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \right)} \leq r, \tag{4.2}$$

then the problem (1.2) *has at least one solution in* $PC(J, \mathbb{R})$.

Proof. Define the operators P and Q on the compact set $B_r = \{u \in X : \|u\| \leq r\} \subset X$ by $Pu(t) = u_T + g(u(t))$ and

$$\begin{aligned} Qu(t) = &- \sum_{t<t_k<T} I_k \left(u\left(t_k^-\right) \right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{f(s, u(s))}{s}\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{f(s, u(s))}{s}\, ds. \end{aligned}$$

Firstly, we show that P maps B_r into itself, i.e., $PB_r \subset B_r$. For all $u \in B_r$, we have $|Pu(t)| = |u_T + g(u(t))| \leq |u_T| + |g(u(t))| \leq r$. Hence, the operator P maps B_r into itself.

Next, we prove that P is a contraction. Let $u, v \in PC(J, \mathbb{R})$. Then,

$$|Pu(t) - Pv(t)| = |g(u(t)) - g(v(t))| \leq C|u(t) - v(t)|,$$

which shows that P satisfies the contraction property.

Now, consider the operator Q. For any $v \in B_r$, we estimate

$$\begin{aligned} |Qv(t)| &\leq \sum_{t<t_k<T} \left| I_k \left(v\left(t_k^-\right) \right) \right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{|f(s, u(s))|}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{|f(s, u(s))|}{s}\, ds \\ &\leq \sum_{t<t_k<T} \left| I_k \left(v\left(t_k^-\right) \right) \right| + \frac{\|f\|}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{1}{s}\, ds \\ &\quad + \frac{\|f\|}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{1}{s}\, ds \\ &\leq m\gamma + \frac{m\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \\ &= \left(m\gamma + \frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \right) \|v\|. \end{aligned}$$

Therefore, for $u, v \in B_r$, we obtain

$$\begin{aligned}
|Pu(t) + Qv(t)| &\leq |Pu(t)| + |Qv(t)| \\
&\leq |u_T| + C\|u\| + \left(m\gamma + \frac{(m+1)\theta(\ln T)^\alpha}{\Gamma(\alpha+1)}\right)\|v\| \\
&\leq |u_T| + Cr + \left(m\gamma + \frac{(m+1)\theta(\ln T)^\alpha}{\Gamma(\alpha+1)}\right) r \\
&= |u_T| + \left(C + m\gamma + \frac{(m+1)\theta(\ln T)^\alpha}{\Gamma(\alpha+1)}\right) r \\
&\leq r.
\end{aligned}$$

Thus, if $u, v \in B_r$, then $Pu + Qv \in B_r$.

Clearly, in view of assumption **(A1)**, the operator Q is continuous. Furthermore, by inequality (4.2), it is uniformly bounded on B_r. The equicontinuity of $Qv(t)$ follows from Theorem 3.2. Hence, by the Arzelá-Ascoli theorem, $Q(B_r)$ is relatively compact, which implies that Q is compact.

Therefore, applying Krasnoselskii's fixed point theorem, there exists a solution to equation (1.2). □

5. Ulam Stability

In this section, we study the Ulam stability of the solution to the problem (1.1).

Let $\varepsilon > 0$, $\psi \geq 0$, and let $\varphi \in PC(J, \mathbb{R}^+)$ be nondecreasing. We consider the following inequalities:

$$\begin{cases} |D^\alpha u(t) - f(t, u(t))| \leq \varepsilon, & t \in J',\ t \neq t_k, \\ \left|\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)\right| \leq \varepsilon, & k = 1, \ldots, m, \end{cases} \tag{5.1}$$

$$\begin{cases} |D^\alpha u(t) - f(t, u(t))| \leq \varphi(t), & t \in J',\ t \neq t_k, \\ \left|\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)\right| \leq \psi, & k = 1, \ldots, m, \end{cases} \tag{5.2}$$

and

$$\begin{cases} |D^\alpha u(t) - f(t, u(t))| \leq \varepsilon\varphi(t), & t \in J',\ t \neq t_k, \\ \left|\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)\right| \leq \varepsilon\psi, & k = 1, \ldots, m. \end{cases} \tag{5.3}$$

Definition 5.1. Equation (1.1) is Ulam-Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (5.1), there exists a solution $x \in PC^1(J, \mathbb{R})$ of the equation (1.1) with

$$|y(t) - x(t)| \leq c_{f,m}\varepsilon, \quad t \in J.$$

Definition 5.2. Equation (1.1) is generalized Ulam-Hyers stable if there exists a function $\theta_{f,m} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\theta_{f,m}(0) = 0$, such that for each solution $y \in PC^1(J, \mathbb{R})$

of the inequality (5.1), there exists a solution $x \in PC^1(J,\mathbb{R})$ of the equation (1.1) with

$$|y(t)-x(t)| \leq \theta_{f,m}(\varepsilon), \quad t \in J.$$

Definition 5.3. Equation (1.1) is Ulam-Hyers-Rassias stable with respect to (φ,ψ) if there exists a constant $c_{f,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J,\mathbb{R})$ of the inequality (5.3), there exists a solution $x \in PC^1(J,\mathbb{R})$ of the equation (1.1) with

$$|y(t)-x(t)| \leq c_{f,m,\varphi}\varepsilon(\varphi(t)+\psi), \quad t \in J.$$

Definition 5.4. Equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (φ,ψ) if there exists a constant $c_{f,m,\varphi} > 0$ such that for each solution $y \in PC^1(J,\mathbb{R})$ of inequality (5.2), there exists a solution $x \in PC^1(J,\mathbb{R})$ of equation (1.1) with

$$|y(t)-x(t)| \leq c_{f,m,\varphi}(\varphi(t)+\psi), \quad t \in J.$$

Remark 5.1. A function $y \in PC^1(J,\mathbb{R})$ is a solution of the inequality (5.1) if and only if there exist a function $g \in PC(J,\mathbb{R})$ and a sequence $\{g_k\}$, $k = 1,2,\ldots,m$, such that:

(i) $|g(t)| \leq \varepsilon$, for all $t \in J$, and $|g_k| \leq \varepsilon$, for all $k = 1,2,\ldots,m$,
(ii) $D^\alpha y(t) = f(t,y(t)) + g(t)$, for all $t \in J'$,
(iii) $\Delta y(t_k) = I_k\left(y\left(t_k^-\right)\right) + g_k$, for all $k = 1,2,\ldots,m$.

Similar remarks hold for the inequalities (5.2) and (5.3).

Thus, Ulam stabilities of impulsive differential equations correspond to certain types of data dependence of their solutions.

Theorem 5.1. *If $y \in PC^1(J,\mathbb{R})$ is a solution of inequality* (5.1)*, then y is a solution of the following inequality:*

$$\begin{aligned}&\left|y(t)-u_T+\sum_{t<t_k<T} I_k\left(y\left(t_k^-\right)\right)+\frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{\alpha-1} f(s,y(s))\frac{ds}{s}\right.\\ &\left.-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\frac{ds}{s}\right|\\ \leq&\frac{m\left(\alpha\Gamma(\alpha)+(\ln T)^\alpha\right)+\left(\ln\frac{t}{T}\right)^\alpha}{\alpha\Gamma(\alpha)}\varepsilon, \quad t \in J.\end{aligned}$$

Proof. Indeed, by remark (5.1), we have that

$$\begin{cases} D^\alpha y(t) = f(t,y(t)) + g(t), & t \in J' \\ \Delta y\left(t_k\right) = I_k\left(y\left(t_k^-\right)\right) + g_k, & k = 1,2,\ldots,m. \end{cases}$$

Then, for $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0,\ldots,m$.

$$\left|y(t)-u_T+\sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right)+\sum_{p=0}^{k-1} g_i\right.$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\Big|\\
\leq&\Big|y(t)-u_T+\sum_{p=0}^{k-1}I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\Big|\\
&+\sum_{p=0}^{k-1}|g_i|+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}.
\end{aligned}
$$

From this it follows

$$
\begin{aligned}
&\Big|y(t)-u_T+\sum_{p=0}^{k-1}I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,y(s))\frac{ds}{s}\Big|\\
\leq&\sum_{i=1}^{m}\varepsilon+\frac{\varepsilon}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left(\ln\frac{t_{i+1}}{s}\right)^{\alpha-1}\frac{ds}{s}+\frac{\varepsilon}{\Gamma(\alpha)}\int_{T}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
\leq&\sum_{i=1}^{m}\varepsilon-\frac{\varepsilon}{\alpha\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left(\ln\frac{t_{i+1}}{s}\right)^{\alpha-1}\frac{ds}{s}+\frac{\varepsilon}{\alpha\Gamma(\alpha)}\int_{T}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
\leq&m\varepsilon+\frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(\sum_{i=1}^{m}\left(\ln\frac{t_{i+1}}{t_i}\right)^{\alpha}+\left(\ln\frac{t}{T}\right)^{\alpha}\right)\\
\leq&m\varepsilon+\frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(m(\ln T)^{\alpha}+\left(\ln\frac{t}{T}\right)^{\alpha}\right)\\
\leq&\frac{\left(\alpha m\Gamma(\alpha)+m(\ln T)^{\alpha}+\left(\ln\frac{t}{T}\right)^{\alpha}\right)}{\alpha\Gamma(\alpha)}\varepsilon.
\end{aligned}
$$

□

Theorem 5.2. *If the assumptions* **(A1)**, **(A2)** *and* **(A4)** *hold, then the equation* (1.1) *is generalized Ulam-Hyers stable.*

Proof. Let $u \in PC^1(J,\mathbb{R})$ be a solution of the inequality (5.2). Denote by x the unique solution of the backward impulsive problem:

$$\begin{cases} D^\alpha u(t) = f(t,u(t)), \quad t \in J = [1,T],\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad k = 1,\dots,m, \\ u(T) = u_T. \end{cases}$$

Then, we have

$$\begin{aligned} u(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s}, \end{aligned}$$

where $t \in (t_{m-k}, t_{m-k+1})$, for $k = 0,\dots,m$. For each such t, we estimate

$$\begin{aligned} &\Bigg| u(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ &- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \Bigg| \\ \leq & \sum_{i=1}^{m} |g_i| + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \left(\ln \frac{t_{i+1}}{s}\right)^{\alpha-1} \varepsilon \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \varepsilon \frac{ds}{s} \\ \leq & m\varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \sum_{i=0}^{m} \left(\ln \frac{t_{i+1}}{t_i}\right)^{\alpha} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(\ln \frac{t}{t_{m-k}}\right)^{\alpha} \\ \leq & m\varepsilon + \frac{\varepsilon}{\Gamma(\alpha+1)} m (\ln T)^\alpha + \frac{\varepsilon}{\Gamma(\alpha+1)} (\ln T)^\alpha \\ = & \frac{\varepsilon}{\Gamma(\alpha+1)} \left[m\Gamma(\alpha+1) + (m+1)(\ln T)^\alpha \right]. \end{aligned}$$

Hence, for each $t \in (t_{m-k}, t_{m-k+1})$, $k = 0,\dots,m$, we obtain:

$$\begin{aligned} |u(t) - x(t)| \leq \Bigg| & u(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ & - \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \Bigg| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{p=0}^{k-1}\left|I_{m-p}(u(t_{m-p}^{-}))-I_{m-p}(x(t_{m-p}^{-}))\right| \\
&+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,x(s))|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,x(s))|\frac{ds}{s}.
\end{aligned}
$$

Using assumptions **(A1)**, **(A2)**, and **(A4)**, we continue estimating: since (A4) provides a single constant μ common to every I_k, each of the $k \le m$ terms

$$\left|I_{m-p}(u(t_{m-p}^{-}))-I_{m-p}(x(t_{m-p}^{-}))\right|, \quad p=0,\dots,k-1,$$

is bounded by $\mu\|u-x\|_{PC}$, so their sum is bounded by $k\mu\|u-x\|_{PC} \le m\mu\|u-x\|_{PC}$. Hence,

$$
\begin{aligned}
|u(t)-x(t)| &\le \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\Gamma(\alpha+1)+(m+1)(\ln T)^{\alpha}\right)+m\mu\,\|u-x\|_{PC} \\
&\quad+\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}\left(\ln\frac{t_{m-p+1}}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\quad+\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\le \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\Gamma(\alpha+1)+(m+1)(\ln T)^{\alpha}\right) \\
&\quad+\|u-x\|_{PC}\left(m\mu+\frac{\lambda m(\ln T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\right).
\end{aligned}
$$

Thus,

$$\|u-x\|_{PC} \le \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\Gamma(\alpha+1)+(m+1)(\ln T)^{\alpha}\right)+\|u-x\|_{PC}D,$$

where

$$D=m\mu+\frac{\lambda m(\ln T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda(\ln T)^{\alpha}}{\Gamma(\alpha+1)}=m\mu+\frac{\lambda(m+1)(\ln T)^{\alpha}}{\Gamma(\alpha+1)}.$$

Assuming $D<1$, we get

$$\|u-x\|_{PC} \le \frac{\varepsilon}{S}\cdot\left(m\Gamma(\alpha+1)+(m+1)(\ln T)^{\alpha}\right),$$

where $S=\Gamma(\alpha+1)\,m\mu+\lambda m(\ln T)^{\alpha}+\lambda(\ln T)^{\alpha}$. Therefore, the equation (1.1) is generalized Ulam-Hyers stable. □

Remark 5.2. The constant

$$D=m\mu+\frac{\lambda(m+1)(\ln T)^{\alpha}}{\Gamma(\alpha+1)}$$

may be read as an overall perturbation-amplification factor for the backward impulsive fractional system: it bounds, in a worst-case sense, how far a discrepancy between an

approximate and an exact solution can grow as it is propagated backward through the impulses and the continuous fractional dynamics. The condition $D < 1$ then plays the role of a small-gain condition, requiring the combined feedback of the system to stay below unity for the perturbation estimate to close.

The term $m\mu$ collects the impulsive contribution: by **(A4)**, μ bounds how strongly every map I_k can amplify a state discrepancy at its jump $t = t_k$, and since at most m impulses are encountered as the perturbation is propagated backward, their combined worst-case amplification is bounded by $m\mu$. The term $\lambda(m+1)(\ln T)^\alpha/\Gamma(\alpha+1)$ is the continuous contribution: λ, from **(A2)**, measures the sensitivity of f along the flow between impulses, while $(\ln T)^\alpha/\Gamma(\alpha+1)$ encodes the memory-weighted, logarithmic time scale characteristic of the Caputo-Hadamard kernel $\ln(t/s)$, in place of the linear elapsed time T that would appear in the classical (integer-order) setting. Notably, this is exactly the same contraction constant that appears in condition (3.3) of Theorem 3.1, so D can equivalently be read as the contraction constant of the underlying fixed-point map, now controlling the propagation of Ulam-type perturbations rather than the distance between two exact solutions.

Thus, $D < 1$ says that the total worst-case amplification - summed over the discrete jumps and the continuous fractional evolution between them - stays below the level at which the perturbation estimate would reproduce itself, allowing the discrepancy to be absorbed into the stability inequality; this is precisely the mechanism used in the proof of Theorem 5.2. If $D \geq 1$, this particular argument no longer closes, though this does not by itself imply instability of the underlying system.

6. Application

Example 6.1. Consider the backward impulsive fractional differential equation:

$$\begin{cases} {}^{CH}{}_tD_e^\alpha u(t) = 0, \quad t \in J' = [1,e] \setminus \left\{e^{1/3}\right\}, \\ \Delta u|_{t=e^{1/3}} = \dfrac{1}{2} \cdot \dfrac{\left|u\left(e^{1/3}\right)\right|}{1+\left|u\left(e^{1/3}\right)\right|}, \\ u(e) = 1, \end{cases} \tag{6.1}$$

where $0 < \alpha < 1$, and the inequalities

$$\begin{cases} \left|{}^{CH}{}_tD_e^\alpha y(t)\right| \leq \varepsilon, \quad t \in J' = [1,e] \setminus \left\{e^{1/3}\right\}, \\ \left|\Delta y|_{t=e^{1/3}} - \dfrac{1}{2} \cdot \dfrac{\left|y\left(e^{1/3}\right)\right|}{1+\left|y\left(e^{1/3}\right)\right|}\right| \leq \varepsilon, \end{cases} \tag{6.2}$$

hold for some $\varepsilon > 0$.

Here $f \equiv 0$, so **(A1)** is immediate and **(A2)** holds trivially with $\lambda = 0$. The map $I_1(x) = \frac{1}{2}|x|/(1+|x|)$ satisfies

$$|I_1(x) - I_1(y)| \leq \tfrac{1}{2}|x-y|, \quad x, y \in \mathbb{R},$$

so **(A4)** holds with $\mu = \frac{1}{2}$. With $m = 1$ impulse and $\lambda = 0$, the quantity D of Theorem 5.2 is $D = m\mu = 1 \cdot \frac{1}{2} = \frac{1}{2} < 1$, so Theorem 5.2 applies and equation (6.1) is generalized Ulam-Hyers stable. We now make the resulting bound explicit.

Since $f \equiv 0$, u is constant on each of $(e^{1/3}, e)$ and $(1, e^{1/3})$. The terminal condition gives $u(t) = 1$ on $(e^{1/3}, e)$. Writing $a := u(e^{1/3})$, the jump condition gives

$$1 - a = I_1(a) = \frac{1}{2} \cdot \frac{a}{1+a}, \quad a \geq 0,$$

whose unique solution is $a = \frac{\sqrt{17}-1}{4} \approx 0.781$. Thus,

$$u(t) = \begin{cases} 1, & t \in \left(e^{1/3}, e\right], \\ \dfrac{\sqrt{17}-1}{4}, & t \in \left[1, e^{1/3}\right). \end{cases}$$

Let $y \in PC^1([1,e], \mathbb{R})$ to solve (6.2), by Remark 5.1, there exist $g \in PC([1,e], \mathbb{R})$, $g_1 \in \mathbb{R}$ with $|g| \leq \varepsilon$, $|g_1| \leq \varepsilon$, $D^\alpha y = g$ on J', and $\Delta y|_{t=e^{1/3}} = I_1(y(e^{1/3-})) + g_1$. Write $b := y(e^{1/3-})$, $c := y(e^{1/3+})$. Taking $t \to e^{1/3+}$ in the integral representation on $(e^{1/3}, e)$,

$$|c - 1| \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_{e^{1/3}}^{e} \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{ds}{s} = \frac{\varepsilon}{\Gamma(\alpha+1)} \left(\frac{2}{3}\right)^{\alpha} =: \delta.$$

From $b + I_1(b) = c - g_1$ and $a + I_1(a) = 1$, subtracting and using that I_1 is $\frac{1}{2}$-Lipschitz, $|b - a| \leq |c - 1| + |g_1| + \frac{1}{2}|b - a|$ which implies $|b - a| \leq 2\delta + 2\varepsilon = 2\varepsilon + \frac{2\varepsilon}{\Gamma(\alpha+1)} \left(\frac{2}{3}\right)^{\alpha}$. This is the step that could not be closed when $\mu = 1$; with $\mu = \frac{1}{2}$ it closes explicitly. For $t \in (e^{1/3}, e)$,

$$|y(t) - u(t)| \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \left[\left(\frac{2}{3}\right)^{\alpha} + \left(\ln t - \frac{1}{3}\right)^{\alpha}\right] \leq \frac{2\varepsilon}{\Gamma(\alpha+1)} \left(\frac{2}{3}\right)^{\alpha}.$$

For $t \in (1, e^{1/3})$, using $u(t) = a$ and the bound on $|b - a|$ above,

$$\begin{aligned} |y(t) - u(t)| &\leq |b - a| + \frac{\varepsilon}{\Gamma(\alpha+1)} \left[\left(\frac{1}{3}\right)^{\alpha} + (\ln t)^{\alpha}\right] \\ &\leq 2\varepsilon + \frac{2\varepsilon}{\Gamma(\alpha+1)} \left[\left(\frac{2}{3}\right)^{\alpha} + \left(\frac{1}{3}\right)^{\alpha}\right]. \end{aligned}$$

Combining both branches, for all $t \in [1, e]$,

$$|y(t) - u(t)| \;\leq\; 2\varepsilon + \frac{2\varepsilon}{\Gamma(\alpha+1)} \left[\left(\frac{2}{3}\right)^{\alpha} + \left(\frac{1}{3}\right)^{\alpha}\right] \;=\; c_{f,1}\, \varepsilon,$$

with $c_{f,1} := 2 + \dfrac{2}{\Gamma(\alpha+1)} \left[\left(\frac{2}{3}\right)^{\alpha} + \left(\frac{1}{3}\right)^{\alpha}\right]$. Since this bound is linear in ε (not merely of the form $\theta(\varepsilon)$), equation (6.1) is in fact Ulam-Hyers stable in the sense of Definition 5.1, a stronger conclusion than generalized Ulam-Hyers stability.

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