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GROWTH OF SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

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Abstract. In this paper, we investigate the growth of solutions of the differential equation

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0,$$

where $A(z)$, $B(z)$ are analytic functions in the closed complex plane except at z_0 and a, b are complex constants such that $ab \neq 0$ and $a = cb$, $c > 1$. Another case has been studied for higher order linear differential equations with analytic coefficients having the same order near a finite singular point.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [11, 15, 20]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [2, 12–14, 16]. Recently in [6, 10], Fettouch and Hamouda investigated the growth of solutions of certain linear differential equations near a finite singular point. In this paper, we continue this investigation near a finite singular point to study other types of linear differential equations.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function near z_0

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$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t} dt - n(\infty,f)\log r, \tag{1.1}$$

where $n(t,f)$ counts the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ each pole according to its multiplicity and the proximity function by

$$m_{z_0}(r,f) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^+ \left| f\left(z_0 - re^{i\varphi}\right) \right| d\varphi. \tag{1.2}$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f). \tag{1.3}$$

In addition, the order of meromorphic function $f(z)$ near z_0 is defined by

$$\sigma_T(f,z_0) = \limsup_{r\to 0} \frac{\log^+ T_{z_0}(r,f)}{-\log r}. \tag{1.4}$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$\sigma_M(f,z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ M_{z_0}(r,f)}{-\log r}, \tag{1.5}$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$.

If $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $0 < \sigma_T(f,z_0) = \sigma < \infty$, then we can define the type of f as the following:

$$\tau_T(f,z_0) = \limsup_{r\to 0} r^{\sigma} T_{z_0}(r,f).$$

If $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $0 < \sigma_M(f,z_0) = \sigma < \infty$, we have also another definition of the type of f as the following:

$$\tau_M(f,z_0) = \limsup_{r\to 0} r^{\sigma} \log^+ M_{z_0}(r,f).$$

In the usual manner, we define the hyper order near z_0 as follows:

$$\sigma_{2,T}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ T_{z_0}(r,f)}{-\log r}, \tag{1.6}$$

$$\sigma_{2,M}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ \log^+ M_{z_0}(r,f)}{-\log r}. \tag{1.7}$$

Remark 1.1. It is shown in [6] that if f is a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ and $g(w) = f\left(z_0 - \frac{1}{w}\right)$ then $g(w)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R,g) = T_{z_0}\left(\frac{1}{R}, f\right),$$

and so $\sigma(f,z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then $g(w)$ is entire and thus $\sigma_T(f,z_0) = \sigma_M(f,z_0)$ and $\sigma_{2,T}(f,z_0) = \sigma_{2,M}(f,z_0)$.

So, we can use the notation $\sigma(f, z_0)$ without any ambiguity. But concerning the type, as in the complex plane, $\tau_T(f, z_0)$ does not equal to $\tau_M(f, z_0)$. For example, for the function $f(z) = \exp\left\{\frac{1}{z_0 - z}\right\}$, we have $M_{z_0}(r, f) = \exp\left\{\frac{1}{r}\right\}$, then $\sigma_M(f, z_0) = 1$ and $\tau_M(f, z_0) = 1$. On the other side, we have

$$T_{z_0}(r, f) = m_{z_0}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+\left|f\left(z_0 - re^{i\varphi}\right)\right| d\varphi = \frac{1}{\pi r},$$

so $\sigma_T(f, z_0) = 1$ and $\tau_T(f, z_0) = \frac{1}{\pi}$.

Definition 1.1. The linear measure of a set $E \subset (0, \infty)$ is defined as $\int_0^\infty \chi_E(t)\, dt$ and the logarithmic measure of E is defined by $\int_0^\infty \frac{\chi_E(t)}{t} dt$, where $\chi_E(t)$ is the characteristic function of the set E.

The linear differential equation

$$f'' + A(z)\, e^{az} f' + B(z)\, e^{bz} f = 0, \tag{1.8}$$

where $A(z)$ and $B(z)$ are entire functions, is investigated by many authors; see for example [1, 3, 4, 7]. In [3], Chen proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$ or $c > 1$, then every solution $f(z) \not\equiv 0$ of (1.8) is of infinite order. In 2012, Hamouda proved results similar to (1.8) in the unit disc concerning the differential equation

$$f'' + A(z)\, e^{\frac{a}{(z_0 - z)^\mu}} f' + B(z)\, e^{\frac{b}{(z_0 - z)^\mu}} f = 0, \tag{1.9}$$

where $\mu > 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$, see [8]. Recently, Fettouch and Hamouda proved the following two results.

Theorem 1.1 ([6])**.** *Let z_0, a, b be complex constants such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$ and n be a positive integer. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z) \exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0$$

satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = n$.

Theorem 1.2 ([6])**.** *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.10}$$

satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In this paper, we will investigate the case $c > 1$ to complete the remaining case in Theorem 1.1, in the following two results.

Theorem 1.3. *Let $n \in \mathbb{N} \setminus \{0\}$, $A(z) \not\equiv 0$, $B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that* $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. *Let a, b be complex constants such that $ab \neq 0$ and $a = cb$, $c > 1$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \tag{1.11}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfies $\sigma(f, z_0) = \infty$.

Theorem 1.4. *Let $n \in \mathbb{N} \setminus \{0\}$, $A(z) \not\equiv 0$, $B(z) \not\equiv 0$ be polynomials. Let a, b be complex constants such that $ab \neq 0$ and $a = cb$, $c > 1$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A\left(\frac{1}{z_0 - z}\right)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B\left(\frac{1}{z_0 - z}\right)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \tag{1.12}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = n$.

In the following result, we will improve Theorem 1.2 by studying the case when $\max\{\sigma(A_j, z_0) : j \neq 0\} \leq \sigma(A_0, z_0)$.

Theorem 1.5. *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying the following conditions*

i) $0 < \sigma(A_j, z_0) \leq \sigma(A_0, z_0) < \infty$, $j = 1, \ldots, k-1$;

ii) $\max\{\tau_M(A_j, z_0) : \sigma(A_j, z_0) = \sigma(A_0, z_0)\} < \tau_M(A_0, z_0)$.

Then every solution $f(z) \not\equiv 0$ of (1.10) *that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.*

Remark 1.2. If we replace τ_M by τ_T in the condition *ii*) in Theorem 1.5 we get the same result.

We can find the analogs of Theorem 1.5 in the complex plane and in the unit disc in ([18, Theorem 1], [9, Theorem 3]).

We signal here that when the coefficients $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$, it may happen that the solution f of (1.10) is not analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$. For example, $f(z) = z$ is a solution of the differential equation

$$f'' - \exp\left\{\frac{1}{z}\right\} f' + \frac{1}{z}\exp\left\{\frac{1}{z}\right\} f = 0, \tag{1.13}$$

where the coefficients of (1.13) are analytic in $\overline{\mathbb{C}} \setminus \{0\}$, but the solution $f(z) = z$ is not analytic in $\overline{\mathbb{C}} \setminus \{0\}$. That's why we wrote in our results (every solution $f(z) \not\equiv 0$ of (1.10), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}, \ldots$) So, it is a priori assumed that f is analytic in Theorem 1.1 and Theorem 1.2. It is similar to the case when the coefficients are meromorphic in $\mathbb{C}$, it is well known that the solutions of (1.10) may be non meromorphic in $\mathbb{C}$.

2. Preliminary Lemmas

To prove these results we need the following lemmas.

Lemma 2.1 ([6]). *Let $A(z) \not\equiv 0$ be analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$, with $\sigma(A, z_0) < n$, n is a positive integer. Set $g(z) = A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\}$, where $a = \alpha + i\beta \neq 0$ is complex number, $z_0 - z = re^{i\varphi}$, $\delta_a(\varphi) = \alpha \cos(n\varphi) + \beta \sin(n\varphi)$ and $H = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$ (obviously, H is of linear measure zero). Then for any given $\varepsilon > 0$ and for any $\varphi \in [0, 2\pi) \setminus H$, there exists $r_0 > 0$ such that for $0 < r < r_0$, we have*

(i) if $\delta_a(\varphi) > 0$, then

$$\exp\left\{(1-\varepsilon)\,\delta_a(\varphi)\,\frac{1}{r^n}\right\} \le |g(z)| \le \exp\left\{(1+\varepsilon)\,\delta_a(\varphi)\,\frac{1}{r^n}\right\}; \tag{2.1}$$

(ii) if $\delta_a(\varphi) < 0$, then

$$\exp\left\{(1+\varepsilon)\,\delta_a(\varphi)\,\frac{1}{r^n}\right\} \le |g(z)| \le \exp\left\{(1-\varepsilon)\,\delta_a(\varphi)\,\frac{1}{r^n}\right\}. \tag{2.2}$$

Lemma 2.2 ([6]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Let $\alpha > 0$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$. Then*

i) there exists a set $E_1 \subset (0, 1)$ that has finite logarithmic measure and a constant $A > 0$ that depends on α and j such that for all $r = |z - z_0|$ satisfying $r \in (0, 1) \setminus E_1$ we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le A\left[\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f)\right]^j; \tag{2.3}$$

ii) there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero and a constant $A > 0$ that depends on α and j such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that (2.3) holds for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ and $r = |z - z_0| < r_0$.

Lemma 2.3 ([10]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma(f, z_0) < \infty$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E_1 \subset (0, 1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0, 1) \setminus E_1$ we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le \frac{1}{r^{k(\sigma+1)+\varepsilon}}, \qquad k \in \mathbb{N}. \tag{2.4}$$

Lemma 2.4. *Let $f(z)$ be a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Then $\sigma(f', z_0) = \sigma(f, z_0)$.*

Proof. By Remark 1.1, $g(w) = f\left(z_0 - \frac{1}{w}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma(g) = \sigma(f, z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g') = \sigma(g)$ (see [17, 19]). We have $f'(z) = \frac{1}{w^2} g'(w)$. Set $h(w) = \frac{1}{w^2} g'(w)$. Obviously, we have $\sigma(h) = \sigma(g')$. In the other hand, by Remark 1.1, we have $\sigma(h) = \sigma(f', z_0)$. So, we conclude that $\sigma(f', z_0) = \sigma(f, z_0)$. □

Lemma 2.5. *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ and suppose that $\left|f^{(k)}(z)\right|$ is unbounded on some ray $\arg(z_0 - z) = \theta$. Then there exists an infinite sequence of points $z_m = z_0 - r_m e^{i\theta}$, $m = 1, 2, \ldots$, where $r_m \to 0$, such that $f^{(k)}(z_m) \to \infty$ and*

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leq M, \quad M > 0,\ j = 0, 1, \ldots, k-1. \tag{2.5}$$

Proof. Let $M\left(r, \theta, f^{(k)}\right)$ denotes the maximum modulus of $f^{(k)}$ on the line segment $\left[z_0 - r_1 e^{i\theta}, z_0 - re^{i\theta}\right]$. Clearly, we may construct a sequence of points $z_m = z_0 - r_m e^{i\theta}$, $m \geq 1$, $r_m \to 0$, such that $M\left(r, \theta, f^{(k)}\right) = f^{(k)}(z_m) \to \infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1, z_m]$ we have

$$\begin{aligned} f^{(j)}(z_m) =& f^{(j)}(z_1) + f^{(j+1)}(z_1)(z_m - z_1) \\ &+ \cdots + \frac{1}{(k-j-1)} f^{(k-1)}(z_1)(z_m - z_1)^{k-j-1} + \int_{z_1}^{z_m} \cdots \int_{z_1}^{y} f^{(k)}(x)\, dx dy \cdots dt, \end{aligned}$$

and by an elementary triangle inequality estimate we obtain

$$\begin{aligned} \left|f^{(j)}(z_m)\right| \leq & \left|f^{(j)}(z_1)\right| + \left|f^{(j+1)}(z_1)\right| |z_m - z_1| \\ &+ \cdots + \frac{1}{(k-j-1)} \left|f^{(k-1)}(z_1)\right| |z_m - z_1|^{k-j-1} \\ &+ \frac{1}{(k-j)} \left|f^{(k)}(z_m)\right| |z_m - z_1|^{k-j}. \end{aligned} \tag{2.6}$$

From (2.6) and by taking into account that when $m \to \infty$, $f^{(k)}(z_m) \to \infty$, $z_m \to z_0$, we obtain

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leq M, \quad M > 0.$$

□

Lemma 2.6. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma(f, z_0) = \sigma > 0$ and finite type $\tau_M(f, z_0) = \tau > 0$. Then for any given $0 < \beta < \tau$ there exists a set $F \subset (0, 1)$ of infinite logarithmic measure such that for all $r \in F$ we have*

$$\log M_{z_0}(r, f) > \frac{\beta}{r^\sigma}.$$

Proof. By the definition of $\tau_M(f, z_0)$, there exists a decreasing sequence $\{r_m\} \to 0$ satisfying $\frac{m}{m+1} r_m > r_{m+1}$ and

$$\lim_{m \to \infty} r_m^\sigma \log M_{z_0}(r_m, f) = \tau.$$

Then there exists m_0 such that for all $m > m_0$ and for a given $\varepsilon > 0$ we have

$$\log M_{z_0}(r_m, f) > \frac{\tau - \varepsilon}{r_m^\sigma}. \tag{2.7}$$

There exists m_1 such that for all $m > m_1$ and for a given $0 < \varepsilon < \tau - \beta$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{\tau - \varepsilon}. \tag{2.8}$$

By (2.7) and (2.8), for all $m > m_2 = \max\{m_0, m_1\}$ and for any $r \in \left[\frac{m}{m+1} r_m, r_m\right]$, we have

$$\log M_{z_0}(r, f) > \log M_{z_0}(r_m, f) > \frac{\tau - \varepsilon}{r_m^{\sigma}} > \frac{\tau - \varepsilon}{r^{\sigma}}\left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{r^{\sigma}}.$$

Set $F = \bigcup\limits_{m=m_2}^{\infty} \left[\frac{m}{m+1} r_m, r_m\right]$. Then we have

$$\sum_{m=m_2}^{\infty} \int\limits_{\frac{m}{m+1} r_m}^{r_m} \frac{dt}{t} = \sum_{m>m_2} \log \frac{m+1}{m} = \infty. \qquad \square$$

Lemma 2.7. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with hyper-order $\sigma_2(f, z_0) = \sigma$ and let $V_{z_0}(r)$ be the central index of f (see* [10]*). Then*

$$\limsup_{r \to 0} \frac{\log^+ \log^+ V_{z_0}(r)}{-\log r} = \sigma. \tag{2.9}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f, z_0) = \sigma$ and if $V(R)$ denotes the central index of g, then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [5, Lemma 2], we have

$$\limsup_{R \to +\infty} \frac{\log^+ \log^+ V(R)}{\log R} = \sigma. \tag{2.10}$$

Substituting R by $\frac{1}{r}$ in (2.10), we get (2.9). $\square$

Lemma 2.8. *Let $A_j(z)$, $j = 0, \ldots, k-1$, be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that $\sigma(A_j, z_0) \le \alpha < \infty$. If f is a solution of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.11}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then $\sigma_2(f, z_0) \le \alpha$.

Proof. For any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $0 < r = |z_0 - z| < r_0$, we have

$$|A_j(z)| \le \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.12}$$

By the Wiman-Valiron near a finite singular point (see [10]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^{j}, \quad j = 0, \ldots, k-1, \tag{2.13}$$

where $V_{z_0}(r)$ is the central index of f and $|f(z_r)| = M(r,f) = \max_{|z_0-z|=r} |f(z)|$. From (2.11), we can write

$$-\frac{f^{(k)}}{f} = A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z). \tag{2.14}$$

Substituting (2.12) and (2.13) into (2.14), we obtain

$$(1+o(1))\frac{(V_{z_0}(r))^k}{r^k} \le k\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}\frac{(V_{z_0}(r))^{k-1}}{r^{k-1}}(1+o(1)),$$

and so

$$V_{z_0}(r) \le kr\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}(1+o(1)). \tag{2.15}$$

By (2.15), we get

$$\sigma_2(f,z_0) \le \alpha.$$

□

It is easy to prove the following lemma.

Lemma 2.9. *Let $P(z) = a_n z^n + \cdots + a_0$, with $a_n \ne 0$ be a polynomial and $A(z) = P\left(\frac{1}{z_0-z}\right)$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $0 < r = |z_0 - z| \le r_0$, the inequalities*

$$(1-\varepsilon)\frac{|a_n|}{r^n} \le |P(z)| \le (1+\varepsilon)\frac{|a_n|}{r^n}$$

hold.

Lemma 2.10. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with the hyper-order $\sigma_2(f,z_0) = \alpha$, and let $V_{z_0}(r)$ be the central index of f. Let $E \subset (0,1]$ be a set of finite logarithmic measure. Then, there exists a sequence of points $\left\{z_m = z_0 - r_m e^{i\theta_m}\right\}$, $m \ge 1$, such that $|f(z_m)| = M_{z_0}(r_m,f)$, $\lim_{m\to\infty}\theta_m = \theta^* \in [0,2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have*

$$\limsup_{r\to 0}\frac{\log^+ V_{z_0}(r)}{-\log r} = \infty, \tag{2.16}$$

$$\exp\left\{\frac{1}{r^{\alpha-\varepsilon}}\right\} \le V_{z_0}(r) \le \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.17}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f,z_0) = \alpha$ and if $V(R)$ denotes the central index of g then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [3, Remark 1] we have

$$\limsup_{R\to\infty}\frac{\log^+ V_{z_0}(R)}{\log R} = \infty, \tag{2.18}$$

$$\exp\left\{R^{\alpha-\varepsilon}\right\} \le V(R) \le \exp\left\{R^{\alpha+\varepsilon}\right\}. \tag{2.19}$$

Substituting R by $\frac{1}{r}$ in (2.18) and (2.19), we get (2.16) and (2.17). □

3. Proof of Theorems

Proof of Theorem 1.3. We assume that $\sigma(f, z_0) = \sigma < \infty$, and we prove that is failing. By Lemma 2.3, for any given $\varepsilon > 0$ there exists a set $E \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \backslash E$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \backslash E$ and $r = |z - z_0| < r_0$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{r^{\sigma+1+\varepsilon}}. \tag{3.1}$$

Set $a = \alpha + i\beta$, $z_0 - z = re^{i\theta}$, $\delta = \delta_a(\theta) = \alpha\cos(n\theta) + \beta\sin(n\theta)$,

$$H = \{\theta \in [0, 2\pi) : \delta_a(\theta) = 0\}, \tag{3.2}$$

(obviously, H is of linear measure zero). By Lemma 2.1, *for any given* $0 < \varepsilon < 1$ and *for any* $\theta \in [0, 2\pi) \backslash E \cup H$, *there exists* $r_0 > 0$ *such that for* $0 < r < r_0$, (2.1) and (2.2) hold.

Now we take $\theta \in [0, 2\pi) \backslash E \cup H$ (obviously, $E \cup H$ is of linear measure zero). Then we have two cases: $\delta_a(\theta) < 0$ or $\delta_a(\theta) > 0$.

Case (i). $\delta_a = \delta < 0$. By $a = cb$, $c > 1$, $\delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta$. By (1.11), we get

$$1 \leq \left|A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\}\right| \cdot \left|\frac{f'}{f''}\right| + \left|B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\}\right| \cdot \left|\frac{f}{f''}\right|. \tag{3.3}$$

If $|f''(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.5 there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$, $m \geq 1$, where $r_m \to 0$ such that $f''(z_m) \to \infty$ and

$$\left|\frac{f(z_m)}{f''(z_m)}\right| \leq M_1, \quad \left|\frac{f'(z_m)}{f''(z_m)}\right| \leq M_2. \tag{3.4}$$

Using Lemma 2.1 and (3.4) into (3.3), we get as $m \to \infty$

$$1 \leq M_1 \exp\left\{(1 - \varepsilon)\frac{\delta}{r_m^n}\right\} + M_2 \exp\left\{(1 - \varepsilon)\frac{1}{c}\frac{\delta}{r_m^n}\right\} \to 0,$$

a contradiction. Hence,

$$|f''(z)| \leq C_1, \tag{3.5}$$

holds on $\arg(z_0 - z) = \theta$, where C_1 is a constant. By integration along the line segment $\left[z_0 - r_1 e^{i\theta}, z_0 - re^{i\theta}\right]$, from (3.5) and the equality

$$f'(z) = f'(z_1) + \int_{z_1}^{z} f''(t)\,dt,$$

we obtain

$$|f'(z)| \leq C_2 + C_1|z - z_1| \leq C_3, \tag{3.6}$$

as $z \to z_0$. Analogously, by (3.6), we can obtain

$$|f(z)| \le C_4, \tag{3.7}$$

holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Case (ii). $\delta > 0$. We have $\delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta > 0$. By (1.11), we have

$$\left|A(z)\exp\left\{\frac{a}{(z_0 - z_k)^n}\right\}\right| \le \left|\frac{f''(z)}{f'(z)}\right| + \left|B(z)\exp\left\{\frac{b}{(z_0 - z_k)^n}\right\}\right| \cdot \left|\frac{f(z)}{f'(z)}\right|. \tag{3.8}$$

If $|f'(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.5, there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$, $m \ge 1$, where $r_m \to 0$ such that $f'(z_m) \to \infty$ and

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \le M_3. \tag{3.9}$$

Substituting (3.1) and (3.9) into (3.8) and by Lemma 2.1, we obtain

$$\begin{aligned}\exp\left\{(1-\varepsilon)\frac{\delta}{r_m^n}\right\} &\le \frac{1}{r_m^{\sigma+1+\varepsilon}} + M_3 \exp\left\{(1+\varepsilon)\frac{1}{c}\cdot\frac{\delta}{r_m^n}\right\} \\ &\le \frac{M_3}{r_m^{\sigma+1+\varepsilon}} \exp\left\{(1+\varepsilon)\frac{1}{c}\cdot\frac{\delta}{r_m^n}\right\},\end{aligned}$$

which implies that

$$1 \le \frac{M_3}{r_m^{\sigma+1+\varepsilon}} \exp\left\{\left[(1+\varepsilon)\frac{1}{c} - (1-\varepsilon)\right]\frac{\delta}{r_m^n}\right\}. \tag{3.10}$$

By taking $0 < \varepsilon < \frac{c-1}{1+c}$, a contradiction follows in (3.10) as $m \to \infty$. So, $|f'(z)| \le C_5$. As above, we obtain that $|f(z)| \le C_6$, holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Now, we proved that $|f(z)| \le C$ on any ray $\arg(z_0 - z) = \theta \in [0, 2\pi) \setminus E \cup H$. Set $g(w) = f(z)$ such that $w = \frac{1}{z_0 - z}$. $g(w)$ is entire function in $\mathbb{C}$ and $|g(w)| \le C'$ $(C' > 0)$ on any ray $\arg(w) = -\theta$ such that $\theta \in [0, 2\pi) \setminus E \cup H$. By Phragmen-Lindelof theorem in sectors, we get that $|g(w)| \le C'$ in $\mathbb{C}$ and By Liouville theorem we conclude that $g(w)$ is a constant. So, $f(z)$ is constant. We know that the only constant solution of (1.11) is $f \equiv 0$. Hence, every solution $f(z) \not\equiv 0$ of (1.11) is of infinite order. □

Proof of Theorem 1.4. Assume that $f \not\equiv 0$ is an analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (1.12). By Theorem 1.3 and Lemma 2.8, we have $\sigma(f, z_0) = \infty$ and $\sigma_2(f, z_0) = \alpha \le n$. We assume that $\sigma_2(f, z_0) = \alpha < n$, and we prove that is failing. Since the Wiman-Valiron near a finite singular point (see [10]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j, \quad j = 1, 2, \tag{3.11}$$

where $|f(z_r)| = M_{z_0}(r,f) = \max_{|z_0-z|=r}|f(z)|$. By Lemma 2.10, there is a sequence $\left\{z_m = z_0 - r_m e^{i\theta_m}\right\}$, $m \geq 1$, such that $|f(z_m)| = M_{z_0}(r_m, f)$, $\lim_{m\to\infty}\theta_m = \theta^* \in [0,2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have

$$\limsup_{m\to\infty}\frac{\log V_{z_0}(r_m)}{-\log r_m} = \infty,$$

$$\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\} \leq V_{z_0}(r_m) \leq \exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}. \tag{3.12}$$

Set $a = \alpha + i\beta$, $z_0 - z = re^{i\theta_0}$, $\delta = \delta_a(\theta^*) = \alpha\cos(n\theta^*) + \beta\sin(n\theta^*)$. Since $a = cb$, $c > 1$, we have $\delta_b(\theta^*) = \frac{1}{c}\delta_a(\theta^*) = \frac{1}{c}\delta$. There is three cases: (i) $\delta < 0$; (ii) $\delta > 0$; (iii) $\delta = 0$.

Case (i). $\delta < 0$. By $\lim_{m\to\infty}\theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m < 0$, $\delta_a(\theta_m) = c\delta_m < 0$. From (1.12), we can write

$$\left|\exp\left\{\frac{-b}{(z_0 - z_m)^n}\right\}\right| \cdot \left|\frac{f''}{f}\right| \leq \left|A\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{a-b}{(z_0 - z_m)^n}\right\}\right| \cdot \left|\frac{f'}{f}\right| + \left|B\left(\frac{1}{z_0 - z_m}\right)\right|. \tag{3.13}$$

Substituting (3.11)–(3.12) into (3.13) and by Lemma 2.1 and Lemma 2.9, for any given ε $(0 < \varepsilon < n - \alpha)$ as m is sufficiently large, we have

$$\begin{aligned}
&\exp\left\{(1-\varepsilon)\frac{-\delta_m}{r_m^n}\right\}\exp\left\{\frac{2}{r_m^{\alpha-\varepsilon}}\right\}\frac{1}{r_m^2}(1+o(1)) \\
\leq &\left|\exp\left\{\frac{-b}{(z_0 - z_m)^n}\right\}\right| \cdot \left|\frac{f''}{f}\right| \\
\leq &\exp\left\{(1-\varepsilon)(c-1)\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1+o(1)}{r_m} + \frac{1}{r_m^{d+1}} \\
\leq &\exp\left\{(1-\varepsilon)(c-1)\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1}{r_m^{d+2}},
\end{aligned}$$

where $d = \deg B$, which implies

$$\exp\left\{\frac{2}{r_m^{\alpha-\varepsilon}}\right\}(1+o(1)) \leq \exp\left\{(1-\varepsilon)c\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1}{r_m^d}. \tag{3.14}$$

By taking $0 < \varepsilon < \max\{1, n-\alpha\}$, the right side of inequality (3.14) tends to zero as $m \to \infty$. This is a contradiction.

Case (ii). $\delta > 0$. By $\lim_{m\to\infty}\theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m > 0$, $\delta_a(\theta_m) = c\delta_m > 0$. By (1.12), we can write

$$\left|A\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{a}{(z_0 - z_m)^n}\right\}\right| \cdot \left|\frac{f'}{f}\right| \leq \left|\frac{f''}{f}\right| + \left|B\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{b}{(z_0 - z_m)^n}\right\}\right|. \tag{3.15}$$

Substituting (3.11)–(3.12) into (3.15) and by Lemma 2.1, as m is sufficiently large,we have

$$\begin{aligned}
&\exp\left\{(1-\varepsilon)\frac{c\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}\frac{(1+o(1))}{r_m}\\
\leq&\left|A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f'}{f}\right|\\
\leq&\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\frac{(1+o(1))}{r_m^2}+\exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\}\\
\leq&\frac{1}{r_m^2}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\},
\end{aligned}$$

which implies the following inequality

$$\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}(1+o(1))\leq\frac{1}{r_m}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{[(1+\varepsilon)-(1-\varepsilon)c]\frac{\delta_m}{r_m^n}\right\}. \tag{3.16}$$

By taking $0<\varepsilon<\max\left\{\frac{c-1}{c+1},n-\alpha\right\}$, the right side of inequality (3.16) tends to zero as $m\to\infty$ and so a contradiction follows.

Case (iii) $\delta=0$. Since $\arg(z_0-z)=\theta^*$ is an asymptotic line of $\frac{a}{(z_0-z_m)^n}$, there is $m_0>0$ such that as $m>m_0$ we have

$$e^{-1}\leq\left|\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\leq e, \tag{3.17}$$

$$e^{\frac{-1}{c}}\leq\left|\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}\right|\leq e^{\frac{1}{c}}. \tag{3.18}$$

By (1.12), (3.11) and (3.17)–(3.18), we obtain

$$\begin{aligned}
-\left(\frac{V_{z_0}(r_m)}{z_0-z_m}\right)^2(1+o(1))=&A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\left(\frac{V_{z_0}(r)}{z_0-z_m}\right)(1+o(1))\\
&+B\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}.
\end{aligned} \tag{3.19}$$

By (3.17)–(3.19) and Lemma 2.1, for m large enough, we have

$$\left(\frac{V_{z_0}(r_m)}{r_m}\right)^2(1+o(1))\leq\frac{1}{r_m^{d+1}}\left(\frac{V_{z_0}(r_m)}{r_m}\right)(1+o(1)),$$

and so

$$V_{z_0}(r_m)\leq\frac{1}{r_m^d}(1+o(1)), \tag{3.20}$$

where $d=\max\{\deg A,\deg B\}$. (3.20) contradicts (3.12). Thus, $\sigma_2(f,z_0)\geq n$ and by Lemma 2.8, we obtain $\sigma_2(f,z_0)=n$. □

Proof of Theorem 1.5. Assume that $f \not\equiv 0$ is an analytic solution of (1.10) in $\overline{\mathbb{C}} \setminus \{z_0\}$. From (1.10), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \cdot \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \cdot \left|\frac{f'}{f}\right|. \tag{3.21}$$

By Lemma 2.2, for any given $\alpha > 0$ there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $\lambda > 0$ that depends only on α such that for all $r = |z - z_0|$ satisfying $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j}, \quad j = 1, \ldots, k. \tag{3.22}$$

There exist β_1, β_2 such that $\max\{\tau_M(A_j, z_0) : \sigma(A_j, z_0) = \sigma(A_0, z_0)\} < \beta_1 < \beta_2 < \tau_M(A_0, z_0)$. There exists a set $E_2 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0|$ satisfying $r \notin E_2$, we have

$$|A_j(z)| \leq \exp\left\{\frac{\beta_1}{r^\sigma}\right\}, \quad j = 1, \ldots, k. \tag{3.23}$$

By Lemma 2.6, there exists a set $F \subset (0,1)$ of infinite logarirhmic measure such that for all $r \in F$ we have

$$M_{z_0}(r, A_0) > \exp\left\{\frac{\beta_2}{r^\sigma}\right\}. \tag{3.24}$$

From (3.21)–(3.24), for all z satisfying $r = |z - z_0| \in F \backslash E_1 \cup E_2$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we obtain

$$\exp\left\{\frac{\beta_2}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k} \exp\left\{\frac{\beta_1}{r^\sigma}\right\},$$

and thus

$$\exp\left\{\frac{\beta_2 - \beta_1}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k}. \tag{3.25}$$

From (3.25), it is easy to obtain that $\sigma_2(f, z_0) \geq \sigma$ and combining this with Lemma 2.8, we get the equality $\sigma_2(f, z_0) = \sigma = \sigma(A_0, z_0)$. □

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