Kragujevac Journal of Mathematics
Volume 49(6) (2025), Pages 831–846.

ON THE q-BESSEL TRANSFORM OF LIPSCHITZ AND DINI-LIPSCHITZ FUNCTIONS ON WEIGHTED SPACE $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$

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Abstract. E. C. Titchmarsh proved some theorems (Theorems 84 and 85) on the classical Fourier transform of functions satisfying conditions related to the Cauchy-Lipschitz conditions in the one-dimensional case. In this paper, we obtain a generalization of those theorems for the q-Bessel transform of a set of functions satisfying the q-Bessel-Lipschitz condition of certain order in suitable weighted spaces $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, where $1 < p \leq 2$. In addition, we introduce the q-Bessel-Dini-Lipschitz condition and we obtain analogous of Titchmarsh's theorems in this occurrence.

1. Introduction

By definition, a function $f = f(t)$ on $\mathbb{R}$ belong to the Lipschitz class $\mathrm{Lip}(\alpha, p; \mathbb{R})$, $0 < \alpha \leq 1$, $p \in [1, +\infty)$, if $f \in L^p(\mathbb{R})$ and

$$\left(\int_{\mathbb{R}} |f(t+h) - f(t)|^p dt\right)^{1/p} = \mathcal{O}(h^\alpha), \tag{1.1}$$

as $h \to 0$. It was first considered by Lipschitz in 1864 while studying the convergence of the Fourier series of a periodic function f. He proved that the inequality (1.1) is sufficient to have that the Fourier series of f converges everywhere to the value of f. A strengthening criterion was introduced by Dini in 1872 whose conclusion states that the convergence is in addition uniform.

A first classical result of Titchmarsh [19, Theorem 84] says that if $0 < \alpha \leq 1$, $1 < p \leq 2$, and

$$f \in \mathrm{Lip}(\alpha, p; \mathbb{R}),$$

Key words and phrases. q-Bessel operator, q-Bessel transform, generalized q-Bessel translation, Lipschitz class, Dini-Lipschitz class, Titchmarsh's theorems.

2020 *Mathematics Subject Classification.* Primary: 33D15. Secondary: 42B10, 42A38.

DOI

Received: May 31, 2022.

Accepted: February 28, 2023.

then the classical Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{p}{p+\alpha p-1}<\beta\leq\overline{p}=\frac{p}{p-1}.$$

On the other hand, Younis in [21] studied the same phenomena for the wider Dini-Lipschitz class as well as for some other allied classes of functions. More precisely, he proved that if $f\in L^p(\mathbb{R})$ with $1<p\leq 2$, is such that

$$\left(\int_{\mathbb{R}}|f(t+h)-f(t)|^p dt\right)^{1/p}=\mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right),$$

as $h\to 0$, where $0<\alpha\leq 1$, then its Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{p}{p+\alpha p-1}<\beta\leq\overline{p}=\frac{p}{p-1}\quad\text{and}\quad\frac{1}{\beta}<\gamma.$$

A second result of Titchmarch [19, Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform. He proved that if $\alpha\in(0,1)$, then the following statement

$$f\in \operatorname{Lip}(\alpha,2;\mathbb{R}),$$

is equivalent to the statement

$$\int_{|\lambda|\geq N}|\widehat{f}(\lambda)|^2 d\lambda=\mathcal{O}(N^{-2\alpha})\quad\text{as}\quad N\to+\infty.$$

An extension of these theorems to functions of several variables on $\mathbb{R}^n$ and on the torus group $\mathbb{T}^n$ was studied by Younis [20,21], and has also been generalized to general compact Lie groups [22]. Recently, it has also been extended to the case of compact groups [3]. Later, analogous results were given, where considering generalized Fourier transforms: Bessel, Dunkl, q-Dunkl, Jacobi, ... One can cite [2,4–7,11,14].

One may naturally ask what are the analogous results for the q-Bessel transform of Titchmarsh theorems? As far as we know, this question has not been answered yet. In this paper, we try to explore the validity of those theorems in case of the q-Bessel transform in the space $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q,t^{2\nu+1}d_qt)$, where $1<p\leq 2$. For this generalization, we use a generalized q-Bessel translation operator.

This paper is arranged as follows. In Section 2, we state some basic notions and results from q-harmonic analysis related to the q-Bessel transform $\mathcal{F}_{q,\nu}$ that will be needed throughout this paper. Section 3 is devoted to proving Titchmarsh's theorem [19, Theorem 84] for the q-Bessel transform for functions satisfying the q-Bessel-Lipschitz condition in the weighted space $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, where $1<p\leq 2$, and we extend this theorem to functions satisfying the q-Bessel-Dini-Lipschitz condition. In the last section, we obtain a generalization of Titchmarsh's theorem [19, Theorem 85] on the image under the q-Bessel transform of a class functions satisfying a generalized Lipschitz and Dini-Lipschitz condition in the Sobolev space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$.

2. Harmonic Analysis Associated with the q-Bessel Operator

Throughout this paper we consider $0 < q < 1$ and $\nu > -1/2$. We refer to [15] and [17] for the definitions, notations and properties of the q-shifted factorials, the q-hypergeometric functions, the Jackson's q-derivative and the Jackson's q-integrals. The references [8–10, 13, 18] are devoted to the q-Bessel Fourier analysis.

We introduce the following set

$$\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}.$$

We notice that in q-calculus all functions are assumed to have $\mathbb{R}_q^+$ as a domain of definition.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{l=0}^{n-1}(1-aq^l), \quad n = 1, 2, \ldots, \tag{2.1}$$

$$(a;q)_\infty = \lim_{n\to+\infty}(a;q)_n = \prod_{l=0}^{+\infty}(1-aq^l).$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C},$$

and

$$[n]_q! = [1]_q \times [2]_q \times \cdots \times [n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad n = 0, 1, 2, \ldots$$

The q-gamma function is given by (see [1])

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1,$$
$$\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x), \quad \mathrm{Re}(x) > 0.$$

The q-derivative $\mathcal{D}_q f$ of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0.$$

$\mathcal{D}_q f(0) = f'(0)$ provided $f'(0)$ exists.

The q-Jackson integrals are defined by (see [16])

$$\int_0^a f(t)d_qt = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_a^b f(t)d_qt = (1-q)\sum_{n=0}^{+\infty} q^n \left[bf(bq^n) - af(aq^n)\right],$$

$$\int_0^{+\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(t)d_qt = (1-q)a\sum_{n=-\infty}^{-1} q^n f(aq^n).$$

Note that

$$\mathcal{D}_q\left(\int_x^a f(t)d_qt\right) = -f(x) \tag{2.2}$$

and

$$\int_a^b \mathcal{D}_q f(t)d_qt = f(b)-f(a).$$

The q-integration by parts formula is given by

$$\int_a^b g(t)\mathcal{D}_q f(t)d_qt = [f(b)g(b)-f(a)g(a)] - \int_a^b f(qt)\mathcal{D}_q g(t)d_qt. \tag{2.3}$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b f\left(\frac{\lambda}{s}\right)\lambda^{2\nu+1}d_q\lambda = s^{2\nu+2}\int_{\frac{a}{s}}^{\frac{b}{s}} f(t)t^{2\nu+1}d_qt, \quad \text{for all } s\in\mathbb{R}_q^+.$$

Let $\nu \geq -1/2$. The normalized third Jackson q-Bessel function of order ν is defined by (see [1])

$$j_\nu(x,q^2) = \sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma_{q^2}(\nu+1)q^{n(n+1)}}{\Gamma_{q^2}(\nu+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{2.4}$$

For $\lambda\in\mathbb{C}$, the function $x\mapsto j_\nu(\lambda x,q^2)$ is a solution of the following q-differential equation

$$\begin{cases} \Delta_{q,\nu}f(x) = -\lambda^2 f(x), \\ f(0)=1, \end{cases}$$

where $\Delta_{q,\nu}$ is the q-Bessel operator defined by

$$\Delta_{q,\nu}f(x) = \frac{1}{x^2}\left[f(q^{-1}x)-(1+q^{2\nu})f(x)+q^{2\nu}f(qx)\right], \quad x\in\mathbb{R}_q^+.$$

Lemma 2.1. *i) The following inequalities are valid for a q-Bessel function:*

$$j_\nu(t,q^2) = \mathcal{O}(1), \quad \textit{if } t\geq 0 \textit{ and } t\in\mathbb{R}_q^+, \tag{2.5}$$

$$1-j_\nu(t,q^2) = \mathcal{O}(1), \quad \textit{if } t\geq 1 \textit{ and } t\in\mathbb{R}_q^+,$$

$$1-j_\nu(t,q^2) = \mathcal{O}(t^2), \quad \textit{if } t\leq 1 \textit{ and } t\in\mathbb{R}_q^+. \tag{2.6}$$

ii) The inequality

$$|1-j_\nu(t,q^2)| \geq c \tag{2.7}$$

is true with $t\geq 1$, $t\in\mathbb{R}_q^+$, where $c>0$ is a certain constant.

Proof. See [12, Lemma 3.1]. □

Moreover, by the relation (2.4), a simple calculation yields

$$\lim_{t\to 0}\frac{1-j_\nu(t,q^2)}{t^2}=\frac{1}{[\nu+1]_{q^2}}\left(\frac{q}{q+1}\right)^2\neq 0, \tag{2.8}$$

hence, there exist $c'>0$ and $\eta>0$ satisfying

$$|t|<\eta \;\Rightarrow\; |1-j_\nu(t,q^2)|\geq c't^2. \tag{2.9}$$

For $1\leq p<+\infty$, we denote by $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ the set of all real functions on $\mathbb{R}^+_q$ for which

$$\|f\|_{q,p,\nu}=\left(\int_0^{+\infty}|f(t)|^p t^{2\nu+1}d_qt\right)^{1/p}<+\infty,$$

and $\mathcal{C}_{q,0}(\mathbb{R}^+_q)$, for the space of functions defined on $\mathbb{R}^+_q$ tending to 0 as $t\to+\infty$ and continuous at 0. The space $\mathcal{C}_{q,0}(\mathbb{R}^+_q)$, when equipped with the topology of uniform convergence, is a complete normed linear space with norm

$$\|f\|_{q,\infty}=\sup_{t\in\mathbb{R}^+_q}|f(t)|.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ associated with the q-Bessel operator $\Delta_{q,\nu}$ is defined for every function f in $\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$ by

$$\mathcal{F}_{q,\nu}(f)(\lambda)=C_{q,\nu}\int_0^{+\infty}f(x)j_\nu(\lambda x,q^2)x^{2\nu+1}d_qx,\quad \text{for all } \lambda\in\mathbb{R}^+_q,$$

where

$$C_{q,\nu}=\frac{(1+q)^{-\nu}}{\Gamma_{q^2}(\nu+1)}.$$

Moreover, it was shown in [8, 13], that q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ satisfies the following properties.

(i) *Riemann-Lebesgue lemma.* Let $f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$, then $\mathcal{F}_{q,\nu}(f)\in\mathcal{C}_{q,0}(\mathbb{R}^+_q)$ and we have

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty}\leq\mathcal{B}_{q,\nu}\|f\|_{q,1,\nu},$$

where

$$\mathcal{B}_{q,\nu}=\frac{1}{1-q}\cdot\frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

(ii) *q-Inversion formula.* If $f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$ such that $\mathcal{F}_{q,\nu}f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$, then for all $x\in\mathbb{R}^+$, we have

$$f(x)=C_{q,\nu}\int_0^{+\infty}\mathcal{F}_{q,\nu}(f)(\lambda)j_\nu(\lambda x,q^2)\lambda^{2\nu+1}d_q\lambda,\quad \text{for all } \lambda\in\mathbb{R}^+_q.$$

(iii) *q-Plancherel formula.* The q-Bessel transform $\mathcal{F}_{q,\nu}$ can be uniquely extended to an isometric isomorphism on $\mathcal{L}^2_{q,\nu}(\mathbb{R}^+_q)$ with

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}=\|f\|_{q,2,\nu}. \tag{2.10}$$

(iv) *q-Hausdorff-Young inequality.* Let $1 < p \leq 2$. If $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, then $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}^{\overline{p}}_{q,\nu}(\mathbb{R}^+_q)$ and

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\overline{p},\nu} \leq \mathcal{B}_{q,\nu}^{\frac{2}{p}-1}\|f\|_{q,p,\nu}, \tag{2.11}$$

where the numbers p and $\overline{p}$ above are conjugate exponents:

$$\frac{1}{p} + \frac{1}{\overline{p}} = 1.$$

The q-generalized translation operator associated with the q-Bessel transform $T^\nu_{q,h}$, $h \in \mathbb{R}^+_q$ was introduced in [13] and rectified in [8], where it is defined by the use of Jackson's q-integral and the q-shifted factorial as

$$T^\nu_{q,h}f(x) = \int_0^{+\infty} f(t)\mathcal{K}_{q,\nu}(h,x,t)t^{2\nu+1}d_qt,$$

where

$$\mathcal{K}_{q,\nu}(h,x,y) = C^2_{q,\nu}\int_0^{+\infty} j_\nu(ht,q^2)j_\nu(xt,q^2)j_\nu(yt,q^2)t^{2\nu+1}d_qt.$$

In particular the product formula

$$T^\nu_{q,h}j_\nu(x,q^2) = j_\nu(h,q^2)j_\nu(x,q^2)$$

holds.

Lemma 2.2. *For $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $p \geq 1$, we have $T^\nu_{q,h}f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ and*

$$\|T^\nu_{q,h}f\|_{q,p,\nu} \leq \|f\|_{q,p,\nu}.$$

Let $\mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, be the Sobolev space constructed by the q-Bessel operator $\Delta_{q,\nu}$, that is,

$$\mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q) := \{f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q) : \Delta^j_{q,\nu}f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q), j = 1,2,\ldots,m\},$$

where

$$\Delta^0_{q,\nu}f = f, \quad \Delta^j_{q,\nu}f = \Delta_{q,\nu}(\Delta^{j-1}_{q,\nu}f), \quad j = 1,2,\ldots,m.$$

Lemma 2.3. *For $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, we have*

$$\mathcal{F}_{q,\nu}(T^\nu_{q,h}f)(\lambda) = j_\nu(\lambda h,q^2)\mathcal{F}_{q,\nu}(f)(\lambda), \tag{2.12}$$

and if $f \in \mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q)$, we get

$$\mathcal{F}_{q,\nu}(\Delta^m_{q,\nu}f)(\lambda) = (-1)^m\lambda^{2m}\mathcal{F}_{q,\nu}(f)(\lambda).$$

Proof. See [9]. □

For every $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, we define the differences $\mathcal{Z}^m_h f$ of order m, $m = 1,2,\ldots$, with step $h > 0$, $h \in \mathbb{R}^+_q$ by:

$$\mathcal{Z}^1_h f(x) = \mathcal{Z}_h f(x) := T^\nu_{q,h}f(x) - f(x),$$

$$\mathcal{Z}^m_h f(x) = \mathcal{Z}_h(\mathcal{Z}^{m-1}_h f(x)), \quad \text{for } m \geq 2.$$

Also, we can write that

$$\mathcal{Z}_h^m f(x) = (T_{q,h}^{\nu} - \mathcal{I})^m f(x),$$

where $\mathcal{I}$ is the identity operator in $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$.

3. Lipschitz and Dini-Lipschitz Condition in the q-Bessel Setting on the Space $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$

In this section, we prove Titchmarsh's theorem [19, Theorem 84] for the q-Bessel transform for functions satisfying the q-Bessel-Lipschitz condition and Younis's theorem [21, Theorem 3.3] on the image under the q-Bessel transform of a class of functions satisfying the q-Bessel-Dini-Lipschitz condition in the space $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+, t^{2\nu+1}d_qt)$, where $1 < p \leq 2$. We begin with auxiliary results interesting in themselves.

Lemma 3.1. *Let $f \in \mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ and $h > 0$ with $h \in \mathbb{R}_q^+$, then*

$$\int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{m\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h^m f\|_{q,p,\nu}^{\overline{p}}, \tag{3.1}$$

where K is a positive constant and $m = 0, 1, 2, \ldots$

Proof. By formula (2.12), we obtain

$$\begin{aligned} \mathcal{F}_{q,\nu}(\mathcal{Z}_h f)(\lambda) &= \mathcal{F}_{q,\nu}(T_{q,h}^{\nu} f)(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda) \\ &= j_\nu(\lambda h, q^2)\mathcal{F}_{q,\nu}(f)(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda) \\ &= (j_\nu(\lambda h, q^2) - 1)\mathcal{F}_{q,\nu}(f)(\lambda). \end{aligned}$$

Using the proof of recurrence for m, we have

$$\mathcal{F}_{q,\nu}(\mathcal{Z}_h^m f)(\lambda) = (j_\nu(\lambda h, q^2) - 1)^m \mathcal{F}_{q,\nu}(f)(\lambda), \quad \text{for all } h \in \mathbb{R}_q^+.$$

Now by q-Hausdorff-Young inequality (2.11), we have the result. ☐

Remark 3.1. If $f \in \mathcal{W}_{q,p,\nu}^m(\mathbb{R}_q^+)$, from (3.1) we get

$$\int_0^{+\infty} \lambda^{2k\overline{p}} |1 - j_\nu(\lambda h, q^2)|^{m\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h^m(\Delta_{q,\nu}^k f)\|_{q,p,\nu}^{\overline{p}}, \tag{3.2}$$

where $k = 0, 1, \ldots, m$.

Definition 3.1. Let $0 < \alpha < 1$. A function $f \in \mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ is said to be in the q-Bessel-Lipschitz class, denoted by q-Lip$(\alpha; p, \nu)$, if

$$\|\mathcal{Z}_h f\|_{q,p,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

Lipschitz classes have been constantly employed in Fourier analysis, although they appear in the realm of trigonometric series, more than they occur in Fourier transforms. Now, we are going to give some results associated with Lipschitz functions in the space $L_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ for q-Bessel transform. We here prove the following theorem.

Theorem 3.1. *Let f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}_q^+)$, $1 < p \leq 2$ and let f also belong to q-$\mathcal{L}ip(\alpha; p, \nu)$. Then, $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^\beta_{q,\nu}(\mathbb{R}_q^+)$, where*

$$\frac{2p\nu + 2p}{2p + 2\nu(p-1) + \alpha p - 2} < \beta \leq \overline{p} = \frac{p}{p-1}. \tag{3.3}$$

Proof. By using the Hausdorff-Young formula (2.11), we note that the theorem is proved in the case where $\beta = \overline{p}$.

Indeed, we have $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}^{\overline{p}}_{q,\nu}(\mathbb{R}_q^+)$ and for $\nu > -1/2$, $0 < \alpha < 1$, we get

$$\begin{aligned} 1 + \frac{\alpha\overline{p}}{2\nu+2} > 1 \ &\Rightarrow \frac{p-1}{p} \cdot \frac{2\nu + 2 + \alpha\overline{p}}{2\nu+2} > \frac{p-1}{p} \\ &\Rightarrow \frac{(2\nu+2)(p-1) + \alpha p}{p(2\nu+2)} > \frac{1}{\overline{p}} \\ &\Rightarrow \frac{p(2\nu+2)}{(2\nu+2)(p-1)+\alpha p} < \beta = \overline{p} = \frac{p}{p-1}. \end{aligned}$$

Then, we get

$$\frac{2p\nu + 2p}{2p + 2\nu(p-1) + \alpha p - 2} < \beta = \overline{p} = \frac{p}{p-1}.$$

Assume now that $\beta < \overline{p}$. If f belong to q-$\mathcal{L}ip(\alpha; p, \nu)$, then we have

$$\|\mathcal{Z}_h f\|_{q,p,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

It follows from the formula (3.1) that

$$\int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h f\|^{\overline{p}}_{q,p,\nu} \leq K' h^{\alpha\overline{p}},$$

where K' is a positive constant, being the last inequality valid for sufficiently small values of h.

If $0 \leq \lambda \leq \frac{\eta}{h}$, then $0 \leq \lambda h \leq \eta$ and inequality (2.9) imply that

$$|1 - j_\nu(\lambda h, q^2)| \geq c' \lambda^2 h^2.$$

From this, we get

$$\int_0^{\eta/h} h^{2\overline{p}} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{\alpha\overline{p}}).$$

Then

$$\int_0^{\eta/h} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{(\alpha-2)\overline{p}}) \quad \text{as} \quad h \to 0.$$

Thus,

$$\int_0^{\xi} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(\xi^{(2-\alpha)\overline{p}}) \quad \text{as} \quad \xi \to +\infty.$$

We consider the function ψ defined by

$$\psi(\xi) = \int_1^{\xi} |\lambda^2 \mathcal{F}_{q,\nu}(f)(\lambda)|^\beta \lambda^{(2\nu+1)\frac{\beta}{\overline{p}}} d_q\lambda.$$

Then, by Hölder inequality we obtain

$$\begin{aligned}\psi(\xi) &\leq \left(\int_1^{\xi} |\lambda^2 \mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda\right)^{\beta/\overline{p}} \left(\int_1^{\xi} d_q\lambda\right)^{(\overline{p}-\beta)/\overline{p}} \\ &= \mathcal{O}\left(\xi^{(2-\alpha)\overline{p}\,\frac{\beta}{\overline{p}}} \xi^{\frac{\overline{p}-\beta}{\overline{p}}}\right) \\ &= \mathcal{O}\left(\xi^{2\beta-\alpha\beta+1-\frac{\beta}{\overline{p}}}\right).\end{aligned}$$

Furthermore, by (2.2), we can see that

$$\mathcal{D}_q\psi(\lambda) = \left(|\lambda^2 \mathcal{F}_{q,\nu}(f)(\lambda)|^{\beta}\right) \lambda^{(2\nu+1)\frac{\beta}{\overline{p}}}. \tag{3.4}$$

According to the q-integration by parts formula (2.3), we get

$$\begin{aligned}&\int_1^{\xi} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\beta} \lambda^{2\nu+1} d_q\lambda = \int_1^{\xi} \lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}} \lambda^{2\nu+1} \mathcal{D}_q\psi(\lambda) d_q\lambda \\ =&\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+1}\psi(\xi) - \int_1^{\xi} \psi(q\lambda)\mathcal{D}_q(\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+1}) d_q\lambda \\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right) - [-2\beta-(2\nu+1)\beta/\overline{p}+2\nu+1]_q \\ &\times \int_1^{\xi} \psi(q\lambda)\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu} d_q\lambda \\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right) + \mathcal{O}\left(\int_1^{\xi} \lambda^{1-\alpha\beta+\beta(\frac{p+1}{p})}\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu} d_q\lambda\right) \\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right)\end{aligned}$$

and this is bounded as $\xi \to +\infty$ if

$$-2\beta - (2\nu+1)\frac{\beta}{\overline{p}} + 2\nu + 2 - \alpha\beta + \beta\left(\frac{p+1}{p}\right) < 0,$$

that is

$$\beta > \frac{2p\nu + 2p}{2p + 2\nu(p-1) + \alpha p - 2}.$$

We do the same proof for the integral over $(-\xi, -1)$, this proves Theorem 3.1. □

We now generalize Theorem 3.1 as follows.

Corollary 3.1. *Let f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}_q^+)$, $1 < p \leq 2$ and if*

$$\|\mathcal{Z}_h^m f\|_{q,p,\nu} = \mathcal{O}(h^{\alpha}), \quad 0 < \alpha < m \quad \text{as} \quad h \to 0,$$

then $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^{\beta}_{q,\nu}(\mathbb{R}_q^+)$, where (3.3) holds.

Remark 3.2. The condition (3.3) can be written in a simple and easier to understand way when it is replaced by

$$\left(\frac{p-1}{p}+\frac{\alpha}{2(\nu+1)}\right)^{-1}<\beta\leq\overline{p}=\frac{p}{p-1}.$$

This also shows directly that the width of the interval for β shrinks as ν increases.

In 1986, Younis studied the same phenomena "Younis's thoerem [21, Theorem 84]" for the wider Dini-Lipschitz class, he replaced $O(h^\alpha)$ by Younes Dini-Lipschitz condition $O\left(h^\alpha(\log(\frac{1}{|h|}))^{-\gamma}\right)$, $\gamma\geq 0$ as $h\to 0$. We now show that Theorem 3.1 could be extended. We begin to define the q-Bessel-Dini Lipschitz class.

Definition 3.2. Let $0<\alpha<1$ and $\gamma\geq 0$, we define the q-Bessel-Dini-Lipschitz class and we denote q-$\mathfrak{D}\,\mathrm{Lip}((\alpha,\gamma);p,\nu)$, the set of functions f belonging to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ satisfying

$$\|\mathcal{Z}_h f\|_{q,p,\nu}=O\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right)\quad\text{as}\quad h\to 0.$$

Theorem 3.2. *If* $\alpha>2$, $\gamma\geq 0$ *and* f *belong to* q-$\mathfrak{D}\,\mathrm{Lip}((\alpha,\gamma);p,\nu)$, *then* f *is null almost everywhere on* $\mathbb{R}^+_q$.

Proof. Assume that $f\in q$-$\mathfrak{D}\,\mathrm{Lip}((\alpha,\gamma);p,\nu)$. Then we have

$$\|\mathcal{Z}_h f\|_{q,p,\nu}\leq C\frac{h^\alpha}{(\log\frac{1}{h})^\gamma},\quad \gamma\geq 0.$$

where C is a positive constant, being the last inequality valid for sufficiently small values of h.

From the relation (3.1), we get

$$\int_0^{+\infty}|1-j_\nu(\lambda h,q^2)|^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda\leq KC^{\overline{p}}\frac{h^{\alpha\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}.$$

Then

$$\frac{1}{h^{2\overline{p}}}\int_0^{+\infty}|1-j_\nu(\lambda h,q^2)|^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda\leq KC^{\overline{p}}\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}.$$

Since $\alpha>2$, we have

$$\lim_{h\to 0}\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}=0.$$

Thus,

$$\lim_{h\to 0}\int_0^{+\infty}\lambda^{2\overline{p}}\left(\frac{|1-j_\alpha(\lambda h,q^2)|}{\lambda^2h^2}\right)^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda=0.$$

Hence, from relation (2.8), one gets

$$\|\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)\|^{\overline{p}}_{q,\overline{p},\nu}=\int_0^{+\infty}\lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda=0.$$

Hence $\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda) = 0$ for all $\lambda \in \mathbb{R}_q^+$. The injectivity of the q-Bessel transform yields to the wanted result. □

Remark 3.3. The same conclusion holds if we consider a function f such that

$$\|\mathcal{Z}_h^m f\|_{q,p,\nu} = \mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right) \quad \text{as} \quad h \to 0,$$

provided that $\alpha > 2m$, $\gamma \geq 0$ and $1 < p \leq 2$.

Since the same technics previously are available, then we remove details in the proofs of the theorems below.

Theorem 3.3. *Let f belong to q-$\mathcal{D}\,\mathrm{Lip}((\alpha,\gamma);p,\nu)$, $1 < p \leq 2$. Then $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}_{q,\nu}^\beta(\mathbb{R}_q^+)$, where* (3.3) *holds.*

Proof. By analogy with the proof of Theorem 3.1, we can establish the following result:

$$\int_0^{\eta/h} \lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}\right),$$

hence

$$\int_0^{\xi} \lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{\xi^{(2-\alpha)\overline{p}}}{(\log\xi)^{\gamma\overline{p}}}\right).$$

Let

$$\psi(\xi) = \int_1^{\xi} |\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\lambda^{(2\nu+1)\frac{\beta}{\overline{p}}}d_q\lambda.$$

Then, if $\beta < \overline{p}$, by Hölder inequality we obtain

$$\psi(\xi) = \mathcal{O}\left(\frac{\xi^{2\beta-\alpha\beta+1-\frac{\beta}{\overline{p}}}}{(\log\xi)^{\gamma\beta}}\right) \quad \text{as} \quad \xi \to +\infty.$$

By using (3.4), an q-integration by parts yields

$$\int_1^{\xi} |\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}}{(\log\xi)^{\gamma\beta}}\right)$$

and for the right hand of this estimate to be bounded as $\xi \to +\infty$ one must have

$$-2\beta - (2\nu+1)\frac{\beta}{\overline{p}} + 2\nu + 2 - \alpha\beta + \beta\left(\frac{p+1}{p}\right) < 0,$$

therefore

$$\frac{2p\nu+2p}{2p+2\nu(p-1)+\delta p-2} < \beta \leq \overline{p} = \frac{p}{p-1},$$

and we do the same proof for the integral over $(-\xi,-1)$, this ends the proof of this theorem. □

We shall also generalize Theorem 3.3 to the following corollary.

Corollary 3.2. *Let $\gamma \geq 0$ and f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$ such that*

$$\|\mathcal{Z}^m_h f\|_{q,p,\nu} = \mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right), \quad 0 < \alpha < m \quad as \quad h \to 0.$$

Then, $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^\beta_{q,\nu}(\mathbb{R}^+_q)$, where (3.3) *holds.*

4. An Equivalence Theorem for the q-Bessel-Lipschitz Class Functions

In this section, we consider $p = 2$. We try to put the previous theorem, Theorem 3.1, into form in which it is reversible. More precisely, we will give a generalization of Titchmarsh's theorem [19, Theorem 85] on the image under the q-Bessel transform of a class functions satisfying a generalized Lipschitz condition in the space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$. The q-Hausdorff-Young inequality (2.11) will likewise be replaced by the q-Plancherel formula (2.10).

We need first to define the q-Bessel-Lipschitz class in the space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$.

Definition 4.1. Let $0 < \alpha < m$, $m \in \mathbb{N}$. A function $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$ is said to be in the q-Bessel-Lipschitz class, denoted by q-$\mathcal{L}ip(\alpha; 2, m, \nu)$, if

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|_{q,2,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

Lemma 4.1. *Let $f \in \mathcal{L}^2_{q,\nu}(\mathbb{R}^+_q)$ and $h > 0$ with $h \in \mathbb{R}^+_q$, then*

$$\|\mathcal{Z}^m_h f\|^2_{q,2,\nu} = \int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda, \tag{4.1}$$

where $m = 0, 1, 2, \ldots$

Proof. The result follows easily by using the q-Plancherel formula (2.10), (2.12) and an induction on m. □

Remark 4.1. If $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$, from (4.1) we get

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|^2_{q,2,\nu} = \int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda, \tag{4.2}$$

where $k = 0, 1, \ldots, m$.

Theorem 4.1. *Let $0 < \alpha < m$ and assume that $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$. Then the following statements are equivalent:*

(1) $f \in q$-Lip$(\alpha; 2, m, \nu)$.
(2) $\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha})$ *as $N \to +\infty$.*

Proof. (1) ⇒ (2) Assume that $f \in q$-$\mathcal{L}ip(\alpha; 2, m, \nu)$. Then we have:

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|_{q,2,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

From (4.2), we have

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|^2_{q,2,\nu} = \int_0^{\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda.$$

If $\frac{1}{h} \leq \lambda \leq \frac{2}{h}$, then $\lambda h \geq 1$ and inequality (2.7) implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_\nu(\lambda h, q^2)|^{2m}.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda &\leq \frac{1}{c^{2m}} \int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq \frac{1}{c^{2m}} \int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &= \frac{1}{c^{2m}} \|\mathcal{Z}_h^m(\Delta_{q,\nu}^k f)\|_{q,2,\nu}^2 \\ &= \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h \to 0.\end{aligned}$$

So, we obtain

$$\int_N^{2N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty.$$

Thus there exists $C > 0$ such that

$$\int_N^{2N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \leq C N^{-2\alpha}.$$

Furthermore, we have

$$\begin{aligned}\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda &= \sum_{l=0}^{+\infty} \int_{2^l N}^{2^{l+1} N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq C \sum_{l=0}^{+\infty} (2^l N)^{-2\alpha} \\ &= C_\alpha N^{-2\alpha},\end{aligned}$$

where $C_\alpha = C(1 - 2^{-2\alpha})^{-1}$. This proves that

$$\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty.$$

(2) $\Rightarrow$ (1) Suppose now that

$$\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty,$$

we have to show that

$$\int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h \to 0.$$

We write

$$\int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 = \int_0^{1/h} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda$$

and

$$\mathcal{J}_2=\int_{1/h}^{+\infty}\lambda^{4k}|1-j_\nu(\lambda h,q^2)|^{2m}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda.$$

Let us estimate the summands $\mathcal{J}_1$ and $\mathcal{J}_2$ from above. From inequality (2.5) of Lemma 2.1, there exists a constant c_1 such that

$$|j_\nu(\lambda h,q^2)|\leq c_1.$$

Hence, from this we conclude that

$$\begin{aligned}\mathcal{J}_2&=\int_{1/h}^{+\infty}\lambda^{4k}|1-j_\nu(\lambda h,q^2)|^{2m}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\\&\leq(1+c_1)^{2m}\int_{1/h}^{+\infty}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\\&=\mathcal{O}(h^{2\alpha})\quad\text{as}\quad h\to0.\end{aligned}\tag{4.3}$$

Now, let us estimate $\mathcal{J}_1$. From inequality (2.6) of Lemma 2.1, there exists a constant c_2 such that

$$|1-j_\nu(\lambda h,q^2)|\leq c_2\lambda^2h^2.$$

Then

$$\begin{aligned}\mathcal{J}_1&=\int_0^{1/h}\lambda^{4k}|1-j_\nu(\lambda h,q^2)|^{2m}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\\&\leq c_2^{2m}\int_0^{1/h}h^{4m}\lambda^{4m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\\&\leq c_2^{2m}h^{2m}\int_0^{1/h}\lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda.\end{aligned}\tag{4.4}$$

Now, we need to introduce the function φ defined by

$$\varphi(\lambda)=\int_\lambda^{+\infty}t^{4k}|\mathcal{F}_{q,\nu}(f)(t)|^2t^{2\nu+1}d_qt.$$

Therefore, it follows from (2.2) that

$$\mathcal{D}_q\varphi(\lambda)=-\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}.$$

Now, we apply the q-integration by parts formula (2.3). We obtain

$$\begin{aligned}\int_0^{1/h}\lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda&=\int_0^{1/h}-\lambda^{2m}\mathcal{D}_q\varphi(\lambda)d_q\lambda\\&=-\frac{1}{h^{2m}}\varphi\left(\frac{1}{h}\right)+\int_0^{1/h}\varphi(q\lambda)\mathcal{D}_q\lambda^{2m}d_q\lambda\\&\leq\int_0^{1/h}\varphi(q\lambda)\mathcal{D}_q\lambda^{2m}d_q\lambda\\&=[2m]_q\int_0^{1/h}\varphi(q\lambda)\lambda^{2m-1}d_q\lambda\\&=[2m]_q\int_0^{1/h}\mathcal{O}((q\lambda)^{-2\alpha})\lambda^{2m-1}d_q\lambda\end{aligned}$$

$$=\mathcal{O}\left(\int_0^{1/h} \lambda^{2m-2\alpha-1} d_q\lambda\right).$$

Since

$$\int_0^{1/h} \lambda^{2m-2\alpha-1} d_q\lambda = \left((1-q)\sum_{n=0}^{+\infty} q^{2n(m-\alpha)}\right) h^{2(\alpha-m)}.$$

Then, we conclude that

$$\int_0^{1/h} \lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{2(\alpha-m)}). \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\mathcal{J}_1 = \mathcal{O}\left(h^{2m}\int_0^{1/h} \lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1} d_q\lambda\right) = \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h\to 0.$$

Finally, from this and (4.3), we deduce that

$$\int_0^{+\infty} \lambda^{4k}|1-j_\nu(\lambda h, q^2)|^{2m}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h\to 0,$$

which completes the proof of this theorem. □

Corollary 4.1. *Let $f\in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}_q^+)$ and let $f\in q$-Lip$(\alpha; 2, m, \nu)$. Then*

$$\int_N^{+\infty} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-4k-2\alpha}) \quad as \quad N\to +\infty.$$

We do the same technique of the proof of Theorem 4.1. We get the following theorem.

Theorem 4.2. *Let $0<\alpha<m$, $\gamma\geq 0$ and assume that $f\in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}_q^+)$. Then*

$$\|\mathcal{Z}_h^m(\Delta^k_{q,\nu} f)\|_{q,2,\nu} = \mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right) \quad as \quad h\to 0$$

is equivalent to

$$\int_N^{+\infty} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1} d_q\lambda = \mathcal{O}\left(\frac{N^{-2\alpha-4k}}{(\log N)^{2\gamma}}\right) \quad as \quad N\to +\infty.$$

Acknowledgements

The authors are grateful to the anonymous referees for the useful comments and suggestions in improving the presentation of the paper.

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