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EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION OF AN ABSTRACT SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH STATE DEPENDENT NONLOCAL CONDITION

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of mild solutions to a semilinear Cauchy problem for an abstract fractional differential equation with state dependent nonlocal condition. Continuous dependence of solutions on initial conditions and local ϵ-approximate mild solution of the considered problem will be discussed.

1. Introduction

Many authors are interested in studying different classes of differential equations by using several forms of accompanying conditions. L. Byszewski [1] inaugurated the study of Cauchy problems for the abstract evolution differential equation $u'(t) + Au(t) = f(t, u(t))$, $t \in (t_0, t_0 + a]$, with the nonlocal condition $u(t_0) + g(t_1, t_2, \dots, t_p, u(\cdot)) = u_0$. K. Deng [2] indicated that the nonlocal condition can be applied in physics with more precise measurements, accurate results and better effect than the usual initial condition. Deng used the nonlocal form $g(u) = \sum_{k=1}^{p} c_k u(t_k)$, where $c_k, k = 1, 2, \dots, p$, are given constants. A. El-Sayed et al. [3] discussed the existence of solutions to the deviated-advanced nonlocal differential inclusion

$$\begin{aligned} x'(t) \in& F(t, x(t)) \text{ a.e. } t \in (0, 1),\\ \sum_{k=1}^{m} a_k x(\phi(\tau_k)) =& \alpha \sum_{j=1}^{n} b_j x(\psi(\eta_j)), \quad a_k, b_j > 0, \tau_k, \eta_j \in (0, 1), \end{aligned}$$

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where F is a set-valued function from $[0,1]\times\mathbb{R}$ into $P(\mathbb{R}^+)$ (the power set of $\mathbb{R}^+$), $\alpha > 0$ is a parameter and ϕ, ψ are, respectively, deviated and advanced given functions. In [3] some special forms of nonlocal conditions are displayed such as $\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = 0$, $\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \alpha x(\psi(\eta))$, $\tau_k, \eta \in (0,1)$, $\int_0^1 x(\phi(s))ds = 0$, $\int_0^1 x(\psi(s))ds = 0$ and $\int_0^1 x(\phi(s))ds = \alpha \int_0^1 x(\psi(s))ds$. E. Hernandez and D. O'Regan [8] investigated the existence and uniqueness of mild solutions for the class

$$u'(t) = Au(t) + F(t, u(\gamma(t))), \quad t \in [0,a],$$

with the state dependent nonlocal condition

$$u(0) = H(\sigma(u), u) \in X,$$

where A generates an analytic semigroup of linear operators on a Banach space X and $F(\cdot),\ \gamma(\cdot),\ H(\cdot)$ and $\sigma(\cdot)$ are suitable continuous functions. The state dependent nonlocal condition generalizes many types of nonlocal conditions. For instance, the conditions $u(0) = u_0,\ u(0) = \sum_{i=1}^{p} c_i u(t_i)$, with $0 \le t_1 < \cdots < t_p \le a$ and $u(0) = g(u)$, where $g \in C(C(J,X),X)$ can be considered as state dependent nonlocal conditions. For more details about the state dependent nonlocal conditions see [7]. For the history, applications and significant results on fractional derivatives and integrals, we refer the reader to [10, 12, 14–16, 19].

The aim of our manuscript is to discuss the existence and uniqueness of mild solutions to the state dependent nonlocal problem

$$\begin{aligned} {}^cD^\alpha u(t) =& Au(t) + F\left(t, u(t), u\left(\gamma(t)\right)\right), \quad t \in [0,b], && (1.1)\\ u(0) =& H\left(\sigma(u), u\right) \in X. && (1.2) \end{aligned}$$

${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$. The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$ of operators on X and $F(\cdot),\ \gamma(\cdot),\ H(\cdot)$ and $\sigma(\cdot)$ are appropriate continuous functions satisfying some hypotheses. We illustrate our results by giving an illustrative example. Further, we discuss the continuous dependence of solutions on initial conditions and local ϵ-approximate mild solution of problem (1.1). The results obtained are based upon the method of semigroups, the contraction mapping principle and the Krasnoselskii's fixed point theorem.

The rest of this paper is organized as follows. In Section 2, we display some notations, main definitions and theorems which are used through out the paper. The main results will be given in Section 3 where we investigate the existence and uniqueness of mild solutions to problem (1.1)–(1.2). In Section 4, we discuss the continuous dependence of solutions on initial conditions and study local ϵ-approximate mild solution of problem (1.1).

2. Preliminaries

Here, we introduce some notations, main definitions and theorems which are crucial in what follows.

Let $J=[0,b]$, where $b>0$, $(X,\|\cdot\|_X)$ be a Banach space, $B(X)$ be the space of all bounded linear operators from X into X, $C(J,X)$ be the set of all continuous functions $u:J\to X$ with the norm $\|u\|_C=\sup\{\|u(t)\|:\ u\in C(J,X), t\in J\}$, $C^n(J,X)$ be the set of all n-differentiable functions, with $u^{(n)}\in C(J,X)$, $AC(J,X)$ be the set of all absolutely continuous functions from J into X.

Let ϕ_η, $\eta>0$, be the function $\phi_\eta(t)=t^{\eta-1}/\Gamma(\eta)$ for $t>0$ and $\phi_\eta(t)=0$ for $t\le 0$. For $\eta=0$, $\phi_0(t)$ is the Dirac delta function.

Let $A:D(A)\subset X\to X$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$ of uniformly bounded linear operators on X.

Let $\rho(A)$ be the resolvent set of A, i.e., the set of all complex numbers λ for which $\lambda I-A$ is invertible. The family $\{(\lambda I-A)^{-1}\}_{\lambda\in\rho(A)}$ of bounded linear operators is called the resolvent of A.

A function $\gamma(t):J\to J$ is said to be a deviated function if $\gamma(t)\le t$ for all $t\in J$. As an example of a deviated function, we have $\gamma(t)=\beta t$, $\beta\in(0,1)$.

Farctional integral according to Riemann-Liouville approach and Caputo fractional derivative are given in what follows [10, 14].

Definition 2.1. The fractional integral of order $\alpha>0$ with the lower limit 0 of the function $u:[0,\infty)\to X$ is defined by

$$I^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds=(\phi_\alpha * u)(t),\quad t\ge 0,$$

provided that the right-hand side is point-wise defined. The symbol $*$ stands for the convolution operation and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Caputo derivative of order $\alpha\in(0,1)$ with the lower limit 0 for a function $u\in AC(J,X)$ is defined by

$${}^cD^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Du(s)ds=I^{1-\alpha}Du(t),\quad D=\frac{d}{dt}.$$

We recall some definitions and properties about C_0-semigroups [6, 13].

Definition 2.3. A family $\{T(t):0\le t<\infty\}$ of linear operators from X to X is called a C_0-semigroup if:

1. $\|T(t)\|\le\infty$, i.e., $\sup\{\|T(t)u\|:u\in X,\|u\|\le 1\}<\infty$ for each $t\ge 0$;
2. $T(t+s)u=T(t)T(s)u$ for all $u\in X$ and all $t,s\ge 0$;
3. $T(0)u=u$ for all $u\in X$;
4. $t\mapsto T(t)u$ is continuous for $t\ge 0$ for each $u\in X$.

Definition 2.4. For the C_0-semigroup $\{T(t)\}_{t\ge 0}$, the following holds.

1. There exist constants $N\ge 1$ and $\omega\ge 0$ such that $\|T(t)\|\le Ne^{\omega t}$ for $0\le t<\infty$.
2. $\{T(t)\}_{t\ge 0}$ is called a C_0-contraction semigroup if $\|T(t)\|\le 1$ for each $t\ge 0$.
3. $\{T(t)\}_{t\ge 0}$ is called a uniformly continuous semigroup if $t\mapsto T(t)$ is continuous in the uniform operator topology.

4. The linear operator $A : D(A) \subset X \to X$ is called the (infinitesimal) generator of $\{T(t)\}_{t\geq 0}$ where the domain $D(A)$ of A is the set of all functions $u \in X$ for which the limit $\lim\limits_{t\to 0^+}(T(t)u-u)/t$ exists in X. The previous limit gives Au in X. The domain $D(A)$ is dense in X and A is closed.
5. $\{T(t)\}_{t\geq 0}$ is a compact semigroup, if and only if $\{T(t)\}_{t\geq 0}$ is continuous in the uniform operator topology and $(\lambda I - A)^{-1}$ is compact for $\lambda \in \rho(A)$.
6. If $\{T(t)\}_{t\geq 0}$ is compact, then $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda s} T(s)ds$, $\mathrm{Re}\lambda > \omega$.

A useful Krasnoselskii's fixed point theorem [4] and Gronwall's inequality [18] are given in what follows.

Theorem 2.1. *Let X be a Banach space, Y be a bounded, closed and convex subset of X and K, Q be operators of Y into X such that $Ku + Qv \in Y$ for every pair $u, v \in Y$. If Q is a contraction and K is completely continuous, then the equation $Ku + Qu = u$ has a solution in Y.*

Theorem 2.2. *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on J, $g(t)$ is a nonnegative, nondecreasing continuous function defined on J, $g(t) \leq c$ (constant), and $u(t)$ is nonnegative and locally integrable on J with*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right] ds, \quad t \in J. \tag{2.1}$$

3. Existence and Uniqueness of Solutions

In this part, we investigate the existence and uniqueness of continuous mild solutions to the nonlocal problem (1.1)–(1.2).

Consider the one-sided stable probability density [11, 19]

$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-1-\alpha n}\frac{\Gamma(1+\alpha n)}{n!}\sin(\alpha n\pi), \quad \alpha \in (0,1), \theta \in (0,\infty), \tag{3.1}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \tag{3.2}$$

and consider the probability density function

$$h_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha}), \quad \theta \in (0,\infty), \tag{3.3}$$

which satisfies

$$h_\alpha(\theta) \geq 0, \quad \int_0^\infty h_\alpha(\theta)d\theta = 1 \text{ and } \int_0^\infty \theta^\nu h_\alpha(\theta) = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \in [0,1]. \tag{3.4}$$

We have relied on the following lemma to define a mild solution for problem (1.1)–(1.2).

Lemma 3.1. *The solution of the nonlocal problem* (1.1)–(1.2) *can be expressed by the integral equation*

$$\begin{aligned} u(t) = &\int_0^\infty h_\alpha(\theta)T(t^\alpha\theta)H\left(\sigma(u),u\right)d\theta \\ &+\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}h_\alpha(\theta)T((t-s)^\alpha\theta)F(s,u(s),u(\gamma(s)))d\theta ds. \end{aligned} \tag{3.5}$$

Proof. Let $u(t)$ be a solution of problem (1.1). Operating I^α on both sides of (1.1), we obtain

$$u(t) = u(0) + \phi_\alpha(t) * Au(t) + \phi_\alpha(t) * F(t,u(t),u(\gamma(t))). \tag{3.6}$$

Let $U(\lambda) = \int_0^\infty e^{-\lambda s}u(s)ds$ and $P(\lambda) = \int_0^\infty e^{-\lambda s}F(s,u(s),u(\gamma(s)))ds,\ \lambda > 0$. Taking Laplace transform for (3.6), we get

$$\begin{aligned} U(\lambda) &= \frac{1}{\lambda}u(0) + \frac{1}{\lambda^\alpha}AU(\lambda) + \frac{1}{\lambda^\alpha}P(\lambda) \\ &= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}u(0) + (\lambda^\alpha I - A)^{-1}P(\lambda) \\ &= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)u(0)ds + \left(\int_0^\infty e^{-\lambda^\alpha s}T(s)ds\right)P(\lambda), \end{aligned} \tag{3.7}$$

where I is the identity operator defined on X. Using (3.2), direct calculation gives that

$$\begin{aligned} \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)u(0)ds &= \int_0^\infty \alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)u(0)dt \\ &= \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}\left(e^{-(\lambda t)^\alpha}\right)T(t^\alpha)u(0)dt \\ &= \int_0^\infty\int_0^\infty \theta\psi_\alpha(\theta)e^{-\lambda t\theta}T(t^\alpha)u(0)d\theta dt \\ &= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u(0)d\theta\right]dt \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\left(\int_0^\infty e^{-\lambda^\alpha s}T(s)ds\right)P(\lambda) \qquad (3.9)\\ &= \left(\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)dt\right)P(\lambda) \\ &= \left(\int_0^\infty\int_0^\infty \alpha\psi_\alpha(\theta)e^{-\lambda t\theta}T\left(t^\alpha\right)t^{\alpha-1}d\theta dt\right)P(\lambda) \\ &= \left(\int_0^\infty e^{-\lambda t}\left(\alpha\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}d\theta\right)dt\right)P(\lambda) \\ &= \int_0^\infty e^{-\lambda t}\left[\left(\alpha\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}d\theta\right) * F(t,u(t),u(\gamma(t)))\right]dt \end{aligned}$$

$$=\int_0^\infty e^{-\lambda t}\left[\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds\right]dt.$$

Substituting (1.2), (3.8) and (3.9) into (3.7), we get

$$\begin{aligned}
&U(\lambda)\\
=&\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)H\left(\sigma(u),u\right)d\theta\right]dt\\
&+\int_0^\infty e^{-\lambda t}\left[\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds\right]dt.
\end{aligned}$$

Inverting the Laplace transform, we obtain

$$\begin{aligned}
u(t)=&\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)H\left(\sigma(u),u\right)d\theta\\
&+\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds.
\end{aligned}$$

Using (3.3), we get (3.5). This completes the proof. □

Define the operators $\{S_\alpha(t)\}_{t\geq 0}$ and $\{R_\alpha(t)\}_{t\geq 0}$ for any $u\in X$, by

$$S_\alpha(t)u=\int_0^\infty h_\alpha(\theta)T(t^\alpha\theta)ud\theta \text{ and } R_\alpha(t)u=\alpha\int_0^\infty \theta h_\alpha(\theta)T(t^\alpha\theta)ud\theta. \tag{3.10}$$

Now, the mild solution of the nonlocal problem (1.1)–(1.2) can be defined by following.

Definition 3.1. A function $u(t)\in C(J,X)$ is called a mild solution of the nonlocal problem (1.1)–(1.2) if $u(0)=H\left(\sigma(u),u\right)$ and

$$u(t)=S_\alpha(t)H\left(\sigma(u),u\right)+\int_0^t (t-s)^{\alpha-1}R_\alpha(t-s)F(s,u(s),u(\gamma(s)))ds. \tag{3.11}$$

The following lemma gives some basic properties of S_α and R_α which are useful in the sequel [9, 17].

Lemma 3.2. *The operators $S_\alpha(t)$, $t\geq 0$, and $R_\alpha(t)$, $t\geq 0$, have the following properties.*

1. *For any fixed $t\geq 0$, the operators $S_\alpha(t)$ and $R_\alpha(t)$ are linear and bounded operators, which means that for any $u\in X$*

$$\|S_\alpha(t)u\|\leq M\|u\| \quad \text{and} \quad \|R_\alpha(t)u\|\leq \frac{\alpha M}{\Gamma(1+\alpha)}\|u\|, \quad \text{for all } t\in J, \tag{3.12}$$

where $M:=\sup_{t\in[0,\infty)}\|T(t)\|_{B(X)}<\infty$.

2. *For every $u\in X$, $t\mapsto S_\alpha(t)u$ and $t\mapsto R_\alpha(t)u$ are continuous functions from $[0,\infty)$ into X.*

3. *The operators $S_\alpha(t)$, $t\geq 0$, and $R_\alpha(t)$, $t\geq 0$, are strongly continuous in $[0,\infty)$, which means that for all $u\in X$ and $0\leq t_1<t_2\leq b$, we have*

$$\|S_\alpha(t_2)u-S_\alpha(t_1)u\|\to 0 \quad \text{and} \quad \|R_\alpha(t_2)u-R_\alpha(t_1)u\|\to 0 \text{ as } t_2\to t_1.$$

4. *If $T(t)$ is a compact operator for every $t > 0$, then the operators $S_\alpha(t)$ and $R_\alpha(t)$ are also compact for every $t > 0$.*

In order to discuss the existence and uniqueness of mild solutions to the nonlocal problem (1.1)–(1.2), consider the following assumptions:

(H_1) $T(t)$ is a compact operator for each $t > 0$;
(H_2) $\gamma : J \to J$ is a deviated continuous function, i.e., $\gamma(t) \leq t$, $t \in J$;
(H_3) $\sigma : C(J, X) \to J$ is a Lipschitz function with Lipschitz constant L_σ;
(H_4) $F : J \times X^2 \to X$ is continuous and there exist constants $p, q > 0$ such that

$$\|F(t, u_1, v_1) - F(t, u_2, v_2)\| \leq p\|u_1 - u_2\| + q\|v_1 - v_2\|, \quad \text{with } f = \max_{t\in J} \|F(t, 0, 0)\|;$$

(H_5) $H : J \times C(J, X) \to X$ is continuous and there exists $\lambda > 0$ such that

$$\|H(u_1, u_2) - H(v_1, v_2)\| \leq \|u_1 - v_1\| + \lambda\|u_2 - v_2\|$$

and $H(\cdot)$ is bounded, with $h = \sup_{u\in C(J,X)} \|H(\sigma(u), u)\|$.

Noting that for all $u, v \in C(J, X)$:

(a) from (H_3) and (H_5),

$$\|H(\sigma(u), u) - H(\sigma(v), v)\| \leq (\lambda + L_\sigma)\|u - v\|; \tag{3.13}$$

(b) from (H_4),

$$\begin{aligned} \|F(t, u, v)\| &\leq \|F(t, u, v) - F(t, 0, 0)\| + \|F(t, 0, 0)\| \\ &\leq p\|u\| + q\|v\| + f. \end{aligned} \tag{3.14}$$

For the existence of mild solutions to problem (1.1)–(1.2), we give the following theorem.

Theorem 3.1. *Let the assumptions (H_1)–(H_5) be satisfied. Then, the nonlocal problem (1.1)–(1.2) has at least one mild solution $u \in C(J, X)$ if*

$$\max\left\{(\lambda + L_\sigma)M, \frac{M(p + q)b^\alpha}{\Gamma(1 + \alpha)}\right\} < 1.$$

Proof. Let $N(r)$ be the nonempty, closed and convex subset of $C(J, X)$ such that

$$N(r) = \left\{u \in C(J, X) : \|u\| \leq r,\, r = \frac{M[b^\alpha f + h\Gamma(1 + \alpha)]}{\Gamma(1 + \alpha) - M(p + q)b^\alpha}\right\}.$$

Let $W : C(J, X) \to C(J, X)$ be the operator given by $Wu(t) = Ku(t) + Qu(t)$, where

$$Ku(t) = \int_0^t (t - s)^{\alpha-1} R_\alpha(t - s)F(s, u(s), u(\gamma(s)))ds \tag{3.15}$$

and

$$Qu(t) = S_\alpha(t)H(\sigma(u), u). \tag{3.16}$$

The proof will be given in four steps.

Step 1. $Ku + Qv \in N(r)$ whenever $u, v \in N(r)$.

Using (3.15) and (3.16) with applying (3.12), we have

$$\begin{aligned}
&\|Ku(t)+Qv(t)\|\\
\le&\int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s,u(s),u(\gamma(s)))\|ds+\|S_\alpha(t)H\left(\sigma(v),v\right)\|\\
\le&\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u(s),u(\gamma(s)))\|ds+M\|H\left(\sigma(v),v\right)\|.
\end{aligned}$$

Using (3.14) and (H_5), we obtain

$$\|Ku(t)+Qv(t)\|\le\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u(s)\|+q\|u(\gamma(s))\|+f\right]ds+hM.$$

For $u\in N(r)$, we get

$$\|Ku+Qv\|\le M\left(\frac{b^\alpha[f+(p+q)r]}{\Gamma(\alpha+1)}+h\right)=r. \tag{3.17}$$

Thus, $Ku+Qv\in N(r)$ whenever $u,v\in N(r)$.

Step 2. K is continuous.

Let $\{u_n\}_{n=1}^\infty$ be a sequence in $C(J,X)$ such that u_n tends to $u\in C(J,X)$ as n tends to ∞ for all $t\in J$.

Using (3.15) and (3.12) we have

$$\begin{aligned}
&\|Ku_n(t)-Ku(t)\|\\
\le&\int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)\left[F(s,u_n(s),u_n(\gamma(s)))-F(s,u(s),u(\gamma(s)))\right]\|ds\\
\le&\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u_n(s),u_n(\gamma(s)))-F(s,u(s),u(\gamma(s)))\|ds.
\end{aligned}$$

Applying (H_4), we obtain

$$\begin{aligned}
&\|Ku_n(t)-Ku(t)\|\\
\le&\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u_n(s)-u(s)\|+q\|u_n(\gamma(s))-u(\gamma(s))\|\right]ds.
\end{aligned}$$

Then

$$\|Ku_n-Ku\|\le\frac{Mb^\alpha(p+q)}{\Gamma(\alpha+1)}\|u_n-u\|,$$

which tends to zero as n tends to ∞. Thus, K is a continuous operator.

Step 3. K is compact.

From (3.12), (3.14) and (3.15), we have

$$\begin{aligned}
\|Ku(t)\|&\le\int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s,u(s),u(\gamma(s)))\|ds\\
&\le\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u(s),u(\gamma(s)))\|ds
\end{aligned}$$

$$\leq \frac{\alpha M}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \left[p\|u(s)\| + q\|u(\gamma(s))\| + f\right] ds.$$

For $u, v \in N(r)$, we get

$$\|Ku\| \leq \frac{Mb^\alpha[f + (p+q)r]}{\Gamma(1+\alpha)}.$$

So, the class of functions $\{Ku(t)\}$ is uniformly bounded in $N(r)$.

Let $0 \leq t_1 \leq t_2 \leq b$. From (3.15), we have

$$\begin{aligned}
&\|Ku(t_2) - Ku(t_1)\| \\
&\leq \left\| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] R_\alpha(t_2-s) F(s, u(s), u(\gamma(s))) ds \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} R_\alpha(t_2-s) F(s, u(s), u(\gamma(s))) ds \right\| \\
&\quad + \left\| \int_0^{t_1} (t_1-s)^{\alpha-1} \left[R_\alpha(t_2-s) - R_\alpha(t_1-s)\right] F(s, u(s), u(\gamma(s))) ds \right\|.
\end{aligned}$$

Applying (3.12) and (3.14), we obtain

$$\begin{aligned}
&\|Ku(t_2) - Ku(t_1)\| \\
&\leq \frac{\alpha M[f + (p+q)r]}{\Gamma(1+\alpha)} \left[\int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right] \\
&\quad + \int_0^{t_1} (t_1-s)^{\alpha-1} \| \left(R_\alpha(t_2-s) - R_\alpha(t_1-s)\right) F(s, u(s), u(\gamma(s))) \| ds.
\end{aligned}$$

Since $\lim_{t_2 \to t_1} \|R_\alpha(t_2-s) - R_\alpha(t_1-s)\| = 0$ uniformly for $0 \leq s \leq t_1 \leq t_2 \leq b$, it is easy to see that $\|Ku(t_2) - Ku(t_1)\| \to 0$ as $t_2 \to t_1$. Thus, $\{Ku(t)\}$ is equicontinuous. By Arzela-Ascoli theorem, $\{Ku(t)\}$ is relatively compact and K is a compact operator.

Step 4. Q is a contraction.

Let $u, v \in N(r)$. From (3.16), we have

$$\|Qu(t) - Qv(t)\| \leq \|S_\alpha(t) \left(H(\sigma(u), u) - H(\sigma(v), v)\right)\|,$$

then by applying (3.12) and (3.13), we get $\|Qu - Qv\| \leq (\lambda + L_\sigma)M\|u - v\|$. Since $(\lambda + L_\sigma)M < 1$, Q is a contraction operator [5].

As a consequence of Krasonselskii's fixed point theorem, the operator W has at least one fixed point. Therefore, the nonlocal problem (1.1)–(1.2) has at least one mild solution $u \in N(r)$ which completes the proof. □

For the uniqueness of mild solutions to problem (1.1)–(1.2), we give the following theorem.

Theorem 3.2. *Let the assumptions* (H_1)-(H_5) *be satisfied. Then, the nonlocal problem* (1.1)–(1.2) *has a unique mild solution* $u \in C(J, X)$ *if*

$$M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right) < 1.$$

Proof. Consider the operator $W : C(J, X) \to C(J, X)$ such that

$$Wu(t) = S_\alpha(t)H(\sigma(u), u) + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s)F(s, u(s), u(\gamma(s)))ds. \tag{3.18}$$

The proof will be given in two steps.

Step 1. W maps $N(r)$ into itself. From (3.12) and (3.18), we have

$$\begin{aligned}\|Wu(t)\| &\le \|S_\alpha(t)H(\sigma(u), u)\| + \int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s, u(s), u(\gamma(s)))\|ds\\ &\le M\|H(\sigma(u), u)\| + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s, u(s), u(\gamma(s)))\|ds.\end{aligned}$$

Let $u \in N(r)$, with applying (3.14), we get

$$\|Wu\| \le M\left(\frac{b^\alpha[f + (p+q)r]}{\Gamma(1+\alpha)} + h\right) = r.$$

Therefore, $WN(r) \subseteq N(r)$.

Step 2. W is a contraction. Let $u, v \in N(r)$. Using (3.12), (3.13) and (3.18), we obtain

$$\begin{aligned}&\|Wu(t) - Wv(t)\|\\ \le&\|S_\alpha(t)\left[H(\sigma(u), u) - H(\sigma(v), v)\right]\|\\ &+ \int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)\left[F(s, u(s)u(\gamma(s))) - F(s, v(s), v(\gamma(s)))\right]\|ds\\ \le&M\|H(\sigma(u), u) - H(\sigma(v), v)\|\\ &+ \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s, u(s)u(\gamma(s))) - F(s, v(s), v(\gamma(s)))\|ds\\ \le&M(\lambda + L_\sigma)\|u(t) - v(t)\|+\\ &+ \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u(s) - v(s)\| + q\|u(\gamma(s)) - v(\gamma(s))\|\right]ds.\end{aligned}$$

Then

$$\|Wu - Wv\| \le M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right)\|u - v\|.$$

Since $M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right) < 1$, W is a contraction operator and it has a unique fixed point $u \in N(r)$ which is the unique mild solution of the nonlocal problem (1.1)–(1.2). Therefore, we get the required. □

We finalize this section by the following example to illustrate our results.

Example 3.1. Let $X = L_2([0, \pi], \mathbb{R})$, the space of all functions for which the 2^{nd} power of the absolute value is Lebesgue integrable. Consider a fractional partial differential equations of the form

$$\begin{cases} {}_tD^{0.4}x(t,z) + {}_zD^2x(t,z) = \dfrac{1}{19+e^t}\left(\dfrac{\|x(t)\|}{1+\|x(t)\|} + \dfrac{\|x(0.7t)\|}{1+\|x(0.7t)\|}\right), \quad t\in[0,1], \\ x(0,z) = 0.3x\left(\sigma(x),z\right) \in L_2\left([0,\pi],\mathbb{R}\right), \quad z\in[0,\pi], \end{cases}$$

where ${}_tD^{0.4}$ denotes Caputo fractional partial derivatives with $\alpha = 0.4$.

Let A be an operator defined by $Ax = -x''$ with the domain

$$D(A) = \left\{x(\cdot)\in L_2\left([0,\pi],\mathbb{R}\right) : x' \text{ is absolutely continuous}, x(0) = x(\pi) = 0\right\},$$

then A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ which is compact [21], that is (H_1) holds. Operator A has the natural eigenvalues $m, -m^2$, with normalized eigenvectors $x_m(t) = (2/\pi)^{0.5}\sin(mt)$. For each $x \in L_2\left([0,\pi],\mathbb{R}\right)$, $T(t)v = \sum_{m=1}^{\infty} e^{-m^2t}\langle v, x_m\rangle x_m$. In particular, $T(\cdot)$ is a uniformly stable semigroup and $\|T(t)\|_{L_2[0,\pi]} \leq e^{-t}$. Our problem can be reformed as the nonlocal problem (1.1)–(1.2).

Let $\sigma \in C_{Lip}\left(C\left([0,1], L_2([0,\pi],\mathbb{R})\right),[0,1]\right)$, with $L_\sigma = 0.2$. Defining $H(\cdot)$ by $H(t,x) = 0.3x$, then $\|H(t,x) - H(t,y)\| \leq 0.3\|x-y\|$. Clearly, $\lambda = 0.3$. Let

$$F(t,x(t),y(t)) = \frac{1}{19+e^t}\left(\frac{\|x(t)\|}{1+\|x(t)\|} + \frac{\|y(t)\|}{1+\|y(t)\|}\right),$$

then $f = 0$ and

$$\begin{aligned} &\|F(t,x_1(t),y_1(t)) - F(t,x_2(t),y_2(t))\| \\ \leq &\frac{1}{20}\left(\left\|\frac{\|x_1(t)\|}{1+\|x_1(t)\|} - \frac{\|x_2(t)\|}{1+\|x_2(t)\|}\right\| + \left\|\frac{\|y_1(t)\|}{1+\|y_1(t)\|} - \frac{\|y_2(t)\|}{1+\|y_2(t)\|}\right\|\right) \\ \leq &\frac{1}{20}\left(\|x_1(t)\| - \|x_2(t)\| + \|y_1(t)\| - \|y_2(t)\|\right) \\ \leq &\frac{1}{20}\left(\|x_1(t) - x_2(t)\| + \|y_1(t) - y_2(t)\|\right). \end{aligned}$$

So, we have $p = q = 0.05$. Therefore, all conditions of Theorem 3.2 are satisfied and the considered problem has a unique continuous mild solution.

4. Continuous Dependence and ϵ-Approximate Mild Solution

In this section, we discuss the continuous dependence of solutions on initial conditions and the local ϵ-approximate mild solution of problem (1.1).

Theorem 4.1. *Let the assumptions $(H_1),(H_2)$ and (H_4) be satisfied and $u_1(t)$ and $u_2(t)$ be the solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Then*

$$\|u_1 - u_2\| \leq M\|u_1^0 - u_2^0\|\left(1 + \sum_{n=1}^{\infty}\frac{[M(p+q)]^n}{\Gamma(1+n\alpha)}b^{n\alpha}\right). \tag{4.1}$$

Proof. Let $u_1(t)$ and $u_2(t)$ be the solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Hence,

$$^cD^\alpha u_1(t) = Au_1(t) + F\left(t, u_1(t), u_1\left(\gamma(t)\right)\right), \quad u_1(0) = u_1^0,\ t \in J,$$

and

$$^cD^\alpha u_2(t) = Au_2(t) + F\left(t, u_2(t), u_2\left(\gamma(t)\right)\right), \quad u_2(0) = u_2^0,\ t \in J.$$

This implies

$$u_1(t) = S_\alpha(t)u_1^0 + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s)))ds$$

and

$$u_2(t) = S_\alpha(t)u_2^0 + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s)))ds.$$

So, we have

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \\ \leq &\|S_\alpha(t)(u_1^0 - u_2^0)\| \\ &+ \int_0^t (t-s)^{\alpha-1} \|R_\alpha(t-s)[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))]\|ds. \end{aligned}$$

Using $(H_4), (H_2)$ and (3.12), we get

$$\|u_1(t) - u_2(t)\| \leq M\|u_1^0 - u_2^0\| + \frac{\alpha M(p+q)}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_1(s) - u_2(s)\|ds.$$

Applying Theorem 2.2, we obtain

$$\|u_1(t) - u_2(t)\| \leq M\|u_1^0 - u_2^0\| + \int_0^t \sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} M\|u_1^0 - u_2^0\|ds.$$

Therefore, it is easy to get the required inequality. $\square$

One can note the following.

(a) Since

$$\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+\alpha+1)} \leq \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+2)} = \frac{1}{\alpha n+1},$$

then

$$\lim_{n\to\infty} \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+\alpha+1)} = 0.$$

Therefore, by the ratio test,

$$\sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(1+n\alpha)} b^{n\alpha}$$

is a convergent series.

(b) Inequality (4.1) shows continuous dependence of solutions of the problem (1.1) on initial conditions as well as it gives the uniqueness which follows by putting $u_1^0 = u_2^0$.

Definition 4.1. A solution of the integral inequality

$$\left\| u(t) - S_\alpha(t)u(0) - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u(s), u(\gamma(s)))ds \right\| \leq \epsilon$$

is called a local ϵ-approximate mild solution of problem (1.1).

Theorem 4.2. *Let the assumptions* $(H_1), (H_1)$ *and* (H_4) *be satisfied. Suppose that* $u_1(t)$ *and* $u_2(t)$ *are* ϵ*-approximate mild solutions of problem* (1.1) *corresponding to* $u_1(0) = u_1^0$ *and* $u_2(0) = u_2^0$*, respectively. Then*

$$\|u_1 - u_2\| \leq \left(\epsilon_1 + \epsilon_2 + M\|u_1^0 - u_2^0\|\right)\left(1 + \sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(1+n\alpha)} b^{n\alpha}\right). \tag{4.2}$$

Proof. Let $u_1(t)$ and $u_2(t)$ be ϵ-approximate mild solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Hence

$$\left\| u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s)))ds \right\| \leq \epsilon_1$$

and

$$\left\| u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s)))ds \right\| \leq \epsilon_2.$$

We know that

$$\|z\| - \|y\| \leq \|z - y\| \leq \|z\| + \|y\|, \quad \text{for all } z, y \in X,$$

so let $z = u_1(t) - u_2(t)$ and

$$\begin{aligned} y =& S_\alpha(t)(u_1^0 - u_2^0) \\ &+ \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) \left[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))\right] ds. \end{aligned}$$

Hence,

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \\ &- \Big\| S_\alpha(t)(u_1^0 - u_2^0) \\ &+ \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s)[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))]ds \Big\| \\ \leq& \Big\| \Big[u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s)))ds\Big] \\ &- \Big[u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s)))ds\Big] \Big\| \\ \leq& \left\| u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s)))ds \right\| \\ &+ \left\| u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s)))ds \right\| \\ \leq& \epsilon_1 + \epsilon_2. \end{aligned}$$

Then

$$\begin{aligned}
&\|u_1(t) - u_2(t)\| \\
\leq &\epsilon_1 + \epsilon_2 + \|S_\alpha(t)(u_1^0 - u_2^0)\| \\
&+ \int_0^t (t-s)^{\alpha-1} \|R_\alpha(t-s)[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))]\| \, ds.
\end{aligned}$$

Using (3.12), (H_4) and (H_2), we obtain

$$\begin{aligned}
&\|u_1(t) - u_2(t)\| \\
\leq &\epsilon_1 + \epsilon_2 + M\|u_1^0 - u_2^0\| + \frac{\alpha M(p+q)}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_1(s) - u_2(s)\| ds.
\end{aligned}$$

Applying Theorem 2.2, we have

$$\begin{aligned}
\|u_1(t) - u_2(t)\| \leq &\epsilon_1 + \epsilon_2 + M\|u_1^0 - u_2^0\| \\
&+ \int_0^t \sum_{n=1}^{\infty} \frac{\left[\frac{\alpha M(p+q)\Gamma(\alpha)}{\Gamma(1+\alpha)}\right]^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \left(\epsilon_1 + \epsilon_2 + M\|u_1^0 - u_2^0\|\right) ds.
\end{aligned}$$

Therefore, we get the required. □

Remark 4.1. From Definition 4.1, if $\epsilon = 0$, then $u(t)$ is a solution of the integral equation

$$u(t) = S_\alpha(t)u(0) + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u(s), u(\gamma(s))) ds,$$

which is a mild solution of problem (1.1).

Remark 4.2. From (4.2), $\epsilon_1 = \epsilon_2 = 0$ implies $u_1(t)$ and $u_2(t)$ are the mild solutions of (1.1) corresponding to the initial conditions $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Further, (4.2) reduced to (4.1), which gives continuous dependence of mild solutions of (1.1) corresponding to initial conditions.

Remark 4.3. (4.2) proves the uniqueness of mild solutions of (1.1) if $\epsilon_1 = \epsilon_2 = 0$ and $u_1^0 = u_2^0$.

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