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COUNTING FUZZY SUBGROUPS OF CERTAIN FINITE GROUPS BY AUTOMORPHISMS

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Abstract. One of the main interests in fuzzy group theory is the classification and enumeration of distinct fuzzy subgroups of finite groups. The main purpose of this paper is to count the distinct fuzzy subgroups of specific classes of finite groups with respect to an equivalence relation. Exactly, the number of fuzzy subgroups of semi-dihedral group SD_{8n} and two groups U_{6n} and $V_{8(2n-1)}$, where $n \in \mathbb{N}$, is determined by an equivalence relation based on automorphism group. According to the used equivalence relation, the number of fuzzy subgroups of a group G is equal to the number of non-isomorphic chains of subgroups of G that terminate in G. In this regard, first maximal subgroups and automorphism group of SD_{8n}, U_{6n} and $V_{8(2n-1)}$ are studied. Then, by inclusion-exclusion principle, recurrence relations are obtained which solving of them enables us to count the distinct fuzzy subgroups for three mentioned classes of non-abelian groups.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy subset as a function mapping a non-empty set to a closed unit interval [22]. The theory of fuzzy sets subsequently underwent extensive development and found diverse applications across various fields. In [14], Rosenfeld utilized this concept to introduce the theory of fuzzy groups. Since the notion of fuzzy group is a generalization of the notion of group, many fundamental properties of groups carry over to fuzzy groups. However, not all results from classical group theory have counterparts in the fuzzy context. In [8], the authors discussed some group theoretic facts that do not generally extend to fuzzy settings. Moreover, certain

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straightforward results of classical group theory when extended to fuzzy setting, often have complicated proofs.

The notion of level subsets, introduced in [22], is a useful tool for developing results in fuzzy set theory and their applications. Using this concept, Das defined the notion of level subgroup of a given fuzzy group and proved that the family of level subgroups forms a chain [4]. This result contributes to the development of the theory of fuzzy groups within the framework of classical group theory.

The classification of fuzzy subgroups of finite groups has emerged as a fundamental and remarkable problem in recent years. Since without any equivalence relation, the number of fuzzy subgroups of a finite group is infinite, researchers treated the classification of the fuzzy subgroups of groups with respect to various equivalence relations. In fact, this discussion originates from the paper [23], where Zhang and Zou investigated the number of equivalence classes of fuzzy subgroups of a group G. In [13], Murali and Makamba employed the equivalence relation $\sim_M$ on the set of all fuzzy subgroups of G. They explored the conditions under which the equivalence relation of fuzzy subgroups can be equivalently described by level subgroups. Afterwards, Volf develops the equivalence relation $\sim_M$ by introducing the natural relation $\sim$ on the set of all fuzzy subgroups of G as follows [21]: for two fuzzy subgroups μ and η of G, $\mu \sim \eta$ if and only if for all $x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\eta(x) > \eta(y)$. According to this equivalence relation, counting all distinct fuzzy subgroups of G with respect to $\sim$ is equivalent to finding the number of all chains of subgroups of G that terminate in G. The above equivalence relation has been used to some remarkable classes of finite groups, including cyclic groups, elementary abelian p-groups, dihedral groups, symmetric groups, hamiltonian groups, finite p-groups having a cyclic maximal subgroup, dicyclic groups and non-abelian groups of order p^3 and 2^4 in [1,2,5–7,16–19].

In [20], Tărnăuceanu has treated the problem of classifying the fuzzy subgroups of a finite group by a new equivalence relation $\approx$. In this equivalence relation, the corresponding equivalence classes of fuzzy subgroups of a group G are closely connected to the automorphism group and the chains of subgroups of G. The equivalence relation $\approx$ was successfully used to count the number of distinct fuzzy subgroups for various finite groups, including symmetric groups, dihedral groups, dicyclic groups and non-abelian groups of order p^3 [10–12, 20].

In the present paper, we will extend the above studies by determining the number of fuzzy subgroups with respect to $\approx$ for the finite semi-dihedral group SD_{8n}, and two groups U_{6n} and $V_{8(2n-1)}$, introduced in [9]. In this regard, a clear understanding of the subgroup structure and properties of these groups is required. Their fundamental structural properties are discussed in [3] and [9]. Moreover, the subgroup lattice of U_{6n} has been extensively investigated in [12]. Additionally, several numerical invariants related to the subgroup lattice have been studied for these groups. In particular, in [15], the authors compute various subgroup-theoretic parameters for several families of these groups. Their results highlight the importance of determining explicit counting formulas for subgroup-related structures.

This paper is organized as follows. Section 2 reviews some preliminary definitions and necessary results on fuzzy subgroups. It also introduces the technique for classifying the fuzzy subgroups of finite groups by the new equivalence relation $\approx$. Section 3 focuses on determining the number of distinct fuzzy subgroups of SD_{8n}, U_{6n} and $V_{8(2n-1)}$ with respect to $\approx$. In the final section, some conclusions and further research directions are indicated.

2. Preliminaries

In this section, we recall some preliminary definitions and necessary results on fuzzy subgroups, which we will need in the next section, for more details, see [14, 20]. Throughout this section, G denotes a finite group and most of our notations are standard and will usually not be repeated here.

Definition 2.1 ([14]). A fuzzy subset μ of G is called a fuzzy subgroup if for all $x, y \in G$,

$$\min\{\mu(x), \mu(y)\} \leq \mu(xy) \quad \text{and} \quad \mu(x) \leq \mu(x^{-1}).$$

The set $FL(G)$ consisting of all fuzzy subgroups of G forms a lattice with respect to fuzzy set inclusion.

For all $t \in [0, 1]$, the level subgroup corresponding to t is defined as $U(\mu, t) = \{x \in G \mid \mu(x) \geq t\}$ [4]. The non-empty level subgroups of a fuzzy subgroup of G are subgroups of G in the ordinary sense. Exactly, suppose that μ is a fuzzy subgroup of G such that $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$U(\mu, \alpha_1) \subset U(\mu, \alpha_2) \subset \cdots \subset U(\mu, \alpha_r) = G. \tag{2.1}$$

For classification of fuzzy subgroups of G, consider the equivalence relation $\approx$ on $FL(G)$ which is described with chains of subgroups of G as follows [20]: Let $\mu, \eta \in FL(G)$ and define $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, $\eta(G) = \{\beta_1, \beta_2, \ldots, \beta_m\}$ such that $\beta_1 > \beta_2 > \cdots > \beta_m$. Then, μ and η determine the following chains of type (2.1):

$$\mathcal{C}_\mu : U(\mu, \alpha_1) \subset \cdots \subset U(\mu, \alpha_n) = G \quad \text{and} \quad \mathcal{C}_\eta : U(\eta, \beta_1) \subset \cdots \subset U(\eta, \beta_m) = G.$$

The equivalence relation $\approx$ on $FL(G)$ is defined by $\mu \approx \eta$ if and only if there exists $f \in \mathrm{Aut}(G)$ such that $f(\mathcal{C}_\eta) = \mathcal{C}_\mu$. More precisely, $\mu \approx \eta$ if and only if $m = n$ and there exists $f \in \mathrm{Aut}(G)$ such that for all $1 \leq i \leq n$, $f(U(\mu, \alpha_i)) = U(\mu, \beta_i)$. In fact, in the equivalence relation $\approx$, the image of μ and η is not necessarily equal, but certainly there is a bijection between $\mathrm{Im}\,(\mu)$ and $\mathrm{Im}\,(\eta)$.

Now, we describe how to determine the number of fuzzy subgroups of G with respect to $\approx$. In other words, we aim to find the number of distinct equivalence classes of $FL(G)$ modulo $\approx$ which is denoted by $\mathcal{N}(G)$. Let $\bar{\mathcal{C}}$ represent the set consisting of all chains of subgroups of G that terminate in G. Since every level subgroup of G is a subgroup in the ordinary sense, the equivalence relation $\approx$ can be constructed in the following manner [20]: for two chains

$$\mathcal{C}_1 : H_1 \subset H_2 \subset \cdots \subset H_m = G \quad \text{and} \quad \mathcal{C}_2 : K_1 \subset K_2 \subset \cdots \subset K_n = G$$

of $\bar{\mathcal{C}}$, we put $\mathcal{C}_1 \approx \mathcal{C}_2$ if and only if $m = n$ and there exists $f \in \operatorname{Aut}(G)$ such that $f(H_i) = K_i$, for all $1 \leq i \leq n$. Then, $\mathcal{N}(G)$ is equal to the number of all non-isomorphic chains of subgroups of G that terminate in G. It is important to note that $\approx$ generalizes the natural equivalence relation $\sim$ defined in [21]. In fact, these two equivalence relations coincide for a group G if and only if G is cyclic [20]. Therefore, since the number of fuzzy subgroups of finite cyclic groups with respect $\sim$ is determined in [16], we have the following theorem.

Theorem 2.1. *If G is a finite cyclic group of order n and $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ is the decomposition of n as a product of prime factors, then the number of all distinct fuzzy subgroups of G with respect to $\approx$ is given by the equality:*

$$\mathcal{N}(G) = 2^{\sum\limits_{\alpha=1}^{s} m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \cdots \sum_{i_s=0}^{m_s} (-1/2)^{\sum\limits_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum\limits_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha},$$

and the above iterated sums are equal to 1 *for* $s = 1$.

In particular, the number of all distinct fuzzy subgroups of the finite cyclic group G of order p^n is 2^n and of order $p^n q^m$ (p, q primes) is given by the equality:

$$\mathcal{N}(G) = 2^{n+m} \sum_{r=0}^{m} (1/2^r) \binom{n}{r} \binom{m}{r}.$$

Next, we concentrate on determining the number of distinct fuzzy subgroups of the finite non-cyclic group G with respect to $\approx$, that is the number of distinct equivalence classes of $\bar{\mathcal{C}}$ modulo $\approx$. For this purpose, we will prove the following theorem which is similar to what has been presented in [17] for counting fuzzy subgroups under the natural equivalence relation $\sim$. Suppose that H is a subgroup of a non-cyclic group G. According to the notation used in this paper, $\mathcal{N}(H)$ denotes the number of distinct equivalence classes of H under $\approx$, or in other words, the number of non-isomorphic chains of subgroups of H that terminate in H with respect to the automorphisms of H. Subsequently, we denote by $\mathcal{N}_G(H)$, the number of non-isomorphic chains of subgroups of H that terminate in H under the automorphisms of G. It is obvious that $\mathcal{N}(H) \leq \mathcal{N}_G(H)$ and if every automorphism of H can be extended to an automorphism of G, then $\mathcal{N}_G(H) = \mathcal{N}(H)$.

Theorem 2.2. *Let $M_1, M_2, \ldots, M_k$ be non-isomorphic maximal subgroups of a non-cyclic group G. Then, the number of all distinct fuzzy subgroups with respect to $\approx$ of G is given by the following equality:*

$$\mathcal{N}(G) = 2 \left(\sum_{i=1}^{k} \mathcal{N}_G(M_i) - \sum_{1 \leq i_1 < i_2 \leq k} \mathcal{N}_G(M_{i_1} \cap M_{i_2}) + \cdots + (-1)^{k-1} \mathcal{N}_G \left(\bigcap_{i=1}^{k} M_i \right) \right).$$

Proof. For $1 \leq i \leq k$, suppose that $\mathcal{C}_i$ is the set of non-equivalent chains of subgroups of G which are contained in M_i, of type $H_1 \subset H_2 \subset \cdots H_r \neq G$. Hence, $\mathcal{N}(G) = 1 + \left|\bigcup_{i=1}^{k} \mathcal{C}_i\right|$. By applying inclusion-exclusion principle, we get

$$\mathcal{N}(G) = 1 + \sum_{i=1}^{k} |\mathcal{C}_i| - \sum_{1\leq i_1<i_2\leq k} \mathcal{N}\left|\mathcal{C}_{i_1} \cap \mathcal{C}_{i_2}\right| + \cdots + (-1)^{k-1}\left|\bigcap_{i=1}^{k} \mathcal{C}_i\right|. \tag{2.2}$$

Clearly, for every $1 \leq l \leq k$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq k$, the set $\bigcap_{j=1}^{l} \mathcal{C}_{i_j}$ consists of all chains which are contained in $Q_{i_l} = \bigcap_{j=1}^{l} M_{i_j}$. If $\mathcal{D}_{i_l}$ is the set of non-equivalent chains of subgroups of Q_{i_l} with respect to automorphisms of G of type $H_1 \subset H_2 \subset \cdots \subset H_r = Q_{i_l}$, then $|\mathcal{D}_{i_l}| = \mathcal{N}_G(Q_{i_l})$ and $\left|\bigcap_{j=1}^{l} \mathcal{C}_{i_j}\right| = 2\,|\mathcal{D}_{i_l}| - 1$. Therefore,

$$\left|\bigcap_{j=1}^{l} \mathcal{C}_{i_j}\right| = 2\mathcal{N}_G(Q_{i_l}) - 1 = 2\mathcal{N}_G\left(\bigcap_{j=1}^{l} M_{i_j}\right) - 1.$$

Then, by (2.2), $\mathcal{N}(G)$ is given by the following equality:

$$\begin{aligned}\mathcal{N}(G) =& 1 + \sum_{i=1}^{k}(2\mathcal{N}_G(M_i) - 1) - \sum_{1\leq i_1<i_2\leq k}(2\mathcal{N}_G(M_{i_1} \cap M_{i_2}) - 1)\\ &+ \cdots + (-1)^{k-1}\left(2\mathcal{N}_G\left(\bigcap_{i=1}^{k} M_i\right) - 1\right)\\ =& 2\left(\sum_{i=1}^{k}\mathcal{N}_G(M_i) - \sum_{1\leq i_1<i_2\leq k}\mathcal{N}_G(M_{i_1} \cap M_{i_2}) + \cdots + (-1)^{k-1}\mathcal{N}_G\left(\bigcap_{i=1}^{k} M_i\right)\right)\\ &+ 1 + \sum_{i=1}^{k}(-1) - \sum_{1\leq i_1<i_2\leq k}(-1) + \sum_{1\leq i_1<i_2<i_3\leq k}(-1) - \cdots + (-1)^{k-1}(-1).\end{aligned}$$

On the other hand,

$$\begin{aligned}&1 + \sum_{i=1}^{k}(-1) - \sum_{1\leq i_1<i_2\leq k}(-1) + \sum_{1\leq i_1<i_2<i_3\leq k}(-1) - \cdots + (-1)^{k-1}(-1)\\ =& 1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k\binom{k}{k} = \sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0.\end{aligned}$$

Therefore,

$$\mathcal{N}(G) = 2\left(\sum_{i=1}^{k}\mathcal{N}_G(M_i) - \sum_{1\leq i_1<i_2\leq k}\mathcal{N}_G(M_{i_1} \cap M_{i_2}) + \cdots + (-1)^{k-1}\mathcal{N}_G\left(\bigcap_{i=1}^{k} M_i\right)\right).$$

□

Therefore, one can calculate $\mathcal{N}(G)$ for any finite group G whose subgroup lattice, maximal subgroups, and automorphism group are known. The above theorem plays an essential role in this paper in solving the counting problem for the finite semi-dihedral group SD_{8n} and two groups U_{6n} and V_{8n} which have the following presentation:

$$\begin{aligned} SD_{8n} =&\langle x, y \mid x^{4n} = y^2 = e, yxy = x^{2n-1}\rangle, \\ U_{6n} =&\langle x, y \mid x^{2n} = y^3 = e, yxy = x\rangle, \\ V_{8n} =&\langle x, y \mid x^{2n} = y^4 = e, xyx = y^{-1}, xy^{-1}x = y\rangle. \end{aligned}$$

These groups are supersolvable, that is, they have an invariant normal series where all the factors are cyclic groups. The numbers of their maximal, normal maximal, self-normalizing maximal, maximal normal, and minimal normal subgroups are computed in [15]. In this paper, we aim to determine the number of fuzzy subgroups of these groups with respect to $\approx$. For this purpose, we first recall the definition of dihedral and dicyclic groups, along with their principal properties.

It is well-known that the finite dihedral group D_{2n}, $n \geq 2$, can be described by the presentation $D_{2n} = \langle x, y \mid x^n = y^2 = e, y^{-1}xy = x^{-1}\rangle$. Note that D_4 is isomorphic to the direct product of two cyclic groups of order 2.

By the structure of maximal subgroups characterized in [17], the subgroup lattice of D_{2n} exhibits the following property: for every divisor r of n, D_{2n} contains the subgroup ${H_0}^r = \langle x^{n/r}\rangle \cong \mathbb{Z}_r$ and n/r subgroups of form ${H_i}^r = \langle x^{n/r}, x^{i-1}y\rangle \cong D_{2r}$, where $1 \leq i \leq n/r$. The automorphism group of D_{2n} is the set $\mathrm{Aut}(D_{2n}) = \{f_{\alpha,\beta} \mid 0 \leq \alpha, \beta \leq n-1 \text{ and } (\alpha, \mathrm{n}) = 1\}$, where $f_{\alpha,\beta}(x) = x^\alpha$ and $f_{\alpha,\beta}(y) = x^\beta y$ [20]. By $\mathrm{Aut}(D_{2n})$, for every $1 \leq i, j \leq n/r$, two subgroups ${H_i}^r = \langle x^{n/r}, x^{i-1}y\rangle$ and ${H_j}^r = \langle x^{n/r}, x^{j-1}y\rangle$ are isomorphic by the automorphism $f_{1,j-i}$.

Utilizing the above statements, the number of fuzzy subgroups of dihedral groups with respect to $\approx$ is determined as follows.

Theorem 2.3 ([10])**.** *The number of all distinct fuzzy subgroups of the group D_{2n}, $n \geq 2$, under the equivalence relation $\approx$ is determined by the equality*

$$\mathcal{N}(D_{2n}) = 2\mathcal{N}(\mathbb{Z}_n) + 2 + \sum_{\substack{r|n \\ r\neq 1,n}} \mathcal{N}(D_{2r}).$$

Now, we recall the dicyclic group T_{4n}, also called the binary dihedral group with parameter $n \geq 3$, as any group having the presentation $T_{4n} = \langle x, y \mid x^{2n} = e, y^2 = x^n, y^{-1}xy = x^{-1}\rangle$. Note that T_4 and T_8 are considered as groups isomorphic to $\mathbb{Z}_4$ and the quaternion group Q_8, respectively. It is well known that the if the prime factorization of n is $p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}$, then the structure of the subgroup lattice of T_{4n} is as follows: for every divisor r of $2n$, T_{4n} possesses a subgroup isomorphic to $\mathbb{Z}_r$, namely ${H_0}^r = \langle x^{\frac{2n}{r}}\rangle$ and for every divisor s of n, T_{4n} possesses n/s subgroups isomorphic to T_{4s}, namely ${H_i}^s = \langle x^{n/s}, x^{i-1}y\rangle$, where $1 \leq i \leq n/s$.

The automorphism group of T_{4n} denoted by $\mathrm{Aut}(T_{4n})$ is the set $\{f_{\alpha,\beta} \mid 0 \leq \alpha, \beta \leq 2n-1 \text{ and } (\alpha, 2\mathrm{n}) = 1\}$, where $f_{\alpha,\beta}(x) = x^\alpha$ and $f_{\alpha,\beta}(y) = x^\beta y$. By $f_{1,j-i}$, we have ${H_i}^s \cong {H_j}^s$. The number of fuzzy subgroups of T_{4n} under $\approx$ is determined in the next theorem as follows.

TABLE 1. The number of distinct fuzzy subgroups of some groups under $\approx$

G	$\mathcal{N}(G)$	G	$\mathcal{N}(G)$	G	$\mathcal{N}(G)$
$\mathbb{Z}_{2^m}$	2^m	$\mathbb{Z}_{2p^m}$	$2^m(m+2)$	$\mathbb{Z}_{4p^m}$	$2^{m-1}(m^2+7m+8)$
$\mathbb{Z}_{8p}$	40	$\mathbb{Z}_{2pq}$	26	$\mathbb{Z}_{4pq}$	88
T_8	12	T_{16}	32	T_{32}	80
T_{64}	192	T_{4p}	16	T_{4p^2}	52
T_{4p^3}	152	T_{4p^4}	416	T_{8p}	64, $(p \neq 2)$
T_{4pq}	88	D_{2p}	6	D_{2p^2}	16
D_{2p^3}	40	D_{2p^4}	96	D_{2pq}	26
D_{4p^2}	88, $(p \neq 2)$	D_{4p^3}	264, $(p \neq 2)$	D_{4p^4}	736, $(p \neq 2)$
D_{8p}	88, $(p \neq 2)$	D_{4pq}	150, $(p \neq 2)$		

Theorem 2.4 ([11]). *The number of all distinct fuzzy subgroups of the group T_{4n}, $n \geq 2$, under the equivalence relation $\approx$ is determined by the equality*

$$\mathcal{N}(T_{4n}) = 2\mathcal{N}(\mathbb{Z}_{2n}) + 4 + \sum_{\substack{r|n \\ r\neq 1,n}} \mathcal{N}(T_{4r}).$$

For more details of Theorems 2.3 and 2.4, we refer the reader to [10] and [11], respectively. For the mentioned groups, according to Theorems 2.1, 2.3, and 2.4, the specific values of $\mathcal{N}$ in some required cases in the next section are listed in Table 1.

3. The Number of Fuzzy Subgroups of the Groups SD_{8n}, U_{6n}, and V_{8n}

In this section, we classify the fuzzy subgroups of the groups SD_{8n}, U_{6n} and V_{8n} with respect to $\approx$. In this regard, first we study maximal subgroups and automorphism group of them.

3.1. The number of fuzzy subgroups of the semi-dihedral group SD_{8n}.

Consider the semi-dihedral group SD_{8n}, having the following presentation:

$$SD_{8n} = \langle x, y \mid x^{4n} = y^2 = e, yxy = x^{2n-1} \rangle, \quad n \geq 2.$$

In fact, for $n = 1$ we have $SD_8 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. According to the presentation of SD_{8n}, we get $x^i y = yx^{i(2n-1)}$, $o(x^{2i}y) = 2$ and $o(x^{2i+1}y) = 4$. The group SD_{8n} is supersolvable and its maximal subgroups are of prime index. Let $n = 2^{m_0}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors. Then, SD_{8n} possesses maximal subgroups $M_1 = \langle x \rangle \cong \mathbb{Z}_{4n}$, $M_2 = \langle x^2, y \rangle \cong D_{4n}$, $M_3 = \langle x^2, xy \rangle \cong T_{4n}$ and p_i maximal subgroups of type $SD_{\frac{8n}{p_i}}$, namely, $M_{ij} = \langle x^{p_i}, x^{2(j-1)}y \rangle$, $j = 1, \ldots, p_i$ and $i = 1, \ldots, k$.

By the properties of automorphism group and the order of elements of SD_{8n}, we find that $\mathrm{Aut}(SD_{8n}) = \{f_{\alpha,\beta} \mid 0 \leq \beta \leq 2n-1, 0 \leq \alpha \leq 4n-1 \text{ and } (\alpha, 4\mathrm{n}) = 1\}$, where $f_{\alpha,\beta}(x) = x^\alpha$ and $f_{\alpha,\beta}(y) = x^{2\beta}y$. This implies that $|\mathrm{Aut}(SD_{8n})| = 2n\varphi(4n)$, where φ is Euler's Totient function. By $\mathrm{Aut}(SD_{8n})$, it is clear that for every $1 \leq i \leq k$, two subgroups $M_{it} = \langle x^{p_i}, x^{2(t-1)}y\rangle$ and $M_{is} = \langle x^{p_i}, x^{2(s-1)}y\rangle$ are isomorphic by

$$f_{1,s-t} = \begin{cases} x \to x, \\ y \to x^{2(s-t)}y. \end{cases}$$

Therefore, non-isomorphic maximal subgroups of SD_{8n} are

$$M_1 = \langle x\rangle = \{x^t \mid 1 \le t \le 4n\} \cong \mathbb{Z}_{4n},$$
$$M_2 = \langle x^2, y\rangle = \{x^{2t}, x^{2t}y \mid 1 \le t \le 2n\} \cong D_{4n},$$
$$M_3 = \langle x^2, xy\rangle = \{x^{2t}, x^{2t+1}y \mid 1 \le t \le 2n\} \cong T_{4n},$$
$$M_{i+3} = M_{i1} = \langle x^{p_i}, y\rangle = \left\{x^{p_it}, x^{p_it}y \mid 1 \le t \le \frac{4n}{p_i}\right\} \cong SD_{\frac{8n}{p_i}}, \quad \text{where } 1 \le i \le k.$$

Consequently, an arbitrary intersection of the non-isomorphic maximal subgroups of SD_{8n} is as follows, where $1 \le r \le k$, $i_r \in \{4, \dots, k+3\}$,

$$M_1 \cap M_2 = M_1 \cap M_3 = M_2 \cap M_3 = M_1 \cap M_2 \cap M_3 = \langle x^2\rangle \cong \mathbb{Z}_{2n},$$
$$M_1 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}, y\rangle \cong D_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}, x^{p_{i_1}p_{i_2}\cdots p_{i_r}}y\rangle \cong T_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_1 \cap M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}};$$
$$M_1 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_2 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_1 \cap M_2 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}, y\rangle \cong SD_{\frac{8n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}.$$

By Theorem 2.2, we get

$$\begin{aligned}
&\mathcal{N}(SD_{8n}) \\
=&2\Bigg(\mathcal{N}_{SD}(\mathbb{Z}_{4n}) + \mathcal{N}_{SD}(D_{4n}) + \mathcal{N}_{SD}(T_{4n}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2n}) \\
&- \sum_{i=1}^{k} \mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_i}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_i}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_i}}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_i}}) - \mathcal{N}_{SD}(SD_{\frac{8n}{p_i}}) \\
&+ \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_ip_j}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_ip_j}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_ip_j}}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_ip_j}}) - \mathcal{N}_{SD}(SD_{\frac{8n}{p_ip_j}}) \\
&- \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=j+1}^{k} \Big(\mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_ip_jp_l}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_ip_jp_l}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_ip_jp_l}}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}})
\end{aligned}$$

$$- \mathcal{N}_{SD}(SD_{\frac{8n}{p_ip_jp_l}})\Big)$$
$$+ \cdots + (-1)^k\Big(\mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_1\cdots p_k}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_1\cdots p_k}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_1\cdots p_k}}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_1\cdots p_k}})$$
$$- \mathcal{N}_{SD}(SD_{\frac{8n}{p_1\cdots p_k}})\Big)\Big).$$

By rewriting the above formula, we obtain the following theorem.

Theorem 3.1. *Let $n = 2^{m_0}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors and $n \geq 2$. Then, the number of all distinct fuzzy subgroups of the group SD_{8n} with respect to $\approx$ is given by the equality*

$$\begin{aligned}\mathcal{N}(SD_{8n}) =& 2\Bigg(\mathcal{N}_{SD}(\mathbb{Z}_{4n}) + \mathcal{N}_{SD}(D_{4n}) + \mathcal{N}_{SD}(T_{4n}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2n}) \\ &+ \sum_{r=1}^{k}\Bigg(\sum_{i_1=1}^{k-r+1}\sum_{i_2=i_1+1}^{k-r+2}\sum_{i_3=i_2+1}^{k-r+3}\cdots\sum_{i_r=i_{r-1}+1}^{k}(-1)^r\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}}\Bigg)\Bigg),\end{aligned}$$

where

$$\begin{aligned}\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} =& \mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) \\ &- 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}_{SD}(SD_{\frac{8n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}),\end{aligned}$$

and the above iterated sum is equal to 0 for $k = 0$.

In Theorem 3.1, $\mathcal{N}_{SD}(\mathbb{Z}_{4n}) = \mathcal{N}(\mathbb{Z}_{4n})$, $\mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) = \mathcal{N}(\mathbb{Z}_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}})$ and these two values can be calculated by Theorem 2.1. For calculation of other values of $\mathcal{N}_{SD}$, suppose that $H_r = \langle x^{2r}, y\rangle \cong D_{\frac{4n}{r}}$, $K_r = \langle x^{2r}, x^r y\rangle \cong T_{\frac{4n}{r}}$, and $L_r = \langle x^r, y\rangle \cong SD_{\frac{8n}{r}}$, where r is an odd divisor of n and $r \neq n$. If $f \in \mathrm{Aut}(H_r)$, $g \in \mathrm{Aut}(K_r)$ and $h \in \mathrm{Aut}(L_r)$, then they are as

$$f = \begin{cases} x^{2r} \to x^{2r\alpha}, & 1 \leq \alpha \leq \frac{2n}{r}, (\alpha, \frac{2n}{r}) = 1, \\ y \to x^{2r\beta}y, & 1 \leq \beta \leq \frac{2n}{r}, \end{cases}$$
$$g = \begin{cases} x^{2r} \to x^{2r\alpha}, & 1 \leq \alpha \leq \frac{2n}{r}, (\alpha, \frac{2n}{r}) = 1, \\ x^r y \to x^{2r\beta+r}y, & 1 \leq \beta \leq \frac{2n}{r}, \end{cases}$$
$$h = \begin{cases} x^{r} \to x^{r\alpha}, & 1 \leq \alpha \leq \frac{4n}{r}, (\alpha, \frac{4n}{r}) = 1, \\ y \to x^{2r\beta}y, & 1 \leq \beta \leq \frac{2n}{r}, \end{cases}$$

which can be extended to an automorphism of SD_{8n} as

$$f_E = \begin{cases} x \to x^{\alpha}, \\ y \to x^{2r\beta}y, \end{cases} \quad g_E = \begin{cases} x \to x^{\alpha}, \\ y \to x^{2r\beta-r\alpha+r}y, \end{cases} \quad h_E = \begin{cases} x \to x^{\alpha}, \\ y \to x^{2r\beta}y. \end{cases}$$

Therefore, $\mathcal{N}_{SD}(SD_{\frac{8n}{r}}) = \mathcal{N}(SD_{\frac{8n}{r}})$ and values of $\mathcal{N}_{SD}(D_{\frac{4n}{r}})$ and $\mathcal{N}_{SD}(T_{\frac{4n}{r}})$ in Theorem 3.1 are equal to $\mathcal{N}(D_{\frac{4n}{r}})$ and $\mathcal{N}(T_{\frac{4n}{r}})$, which can be calculated by Theorems 2.3 and 2.4, respectively. Note that $\mathcal{N}_{SD}(SD_8) = \mathcal{N}_{SD}(\mathbb{Z}_4 \times \mathbb{Z}_2) = 20$, $\mathcal{N}_{SD}(D_4) = \mathcal{N}_{SD}(\mathbb{Z}_2 \times \mathbb{Z}_2) = 6$, and $\mathcal{N}_{SD}(T_4) = \mathcal{N}_{SD}(\mathbb{Z}_4) = 4$.

Now, we consider Theorem 3.1 in more detail for some particular cases. By putting $k = 0$ (i.e., $n = 2^m$) in Theorem 3.1, we get

$$\mathcal{N}(SD_{2^{m+3}}) = 2\Big(\mathcal{N}_{SD}(\mathbb{Z}_{2^{m+2}}) + \mathcal{N}_{SD}(D_{2^{m+2}}) + \mathcal{N}_{SD}(T_{2^{m+2}}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2^{m+1}})\Big).$$

By Table 1, $\mathcal{N}_{SD}(\mathbb{Z}_{p^m}) = \mathcal{N}(\mathbb{Z}_{p^m}) = 2^m$ and consequently

$$\mathcal{N}(SD_{2^{m+3}}) = 2\Big(\mathcal{N}_{SD}(D_{2^{m+2}}) + \mathcal{N}_{SD}(T_{2^{m+2}})\Big), \tag{3.1}$$

where the values of $\mathcal{N}_{SD}(D_{2^{m+2}})$ and $\mathcal{N}_{SD}(T_{2^{m+2}})$, for $m \geq 1$, can be calculated using Theorems 2.3 and 2.4.

By putting $k = 1$ (i.e., $n = p^m$ or $n = 2^{m_0}p^m$) in Theorem 3.1, we get

$$\begin{aligned}\mathcal{N}(SD_{8n}) =& 2\Big(\mathcal{N}_{SD}(\mathbb{Z}_{4n}) + \mathcal{N}_{SD}(D_{4n}) + \mathcal{N}_{SD}(T_{4n}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2n}) \\ &- \mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p}}) - \mathcal{N}_{SD}(D_{\frac{4n}{p}}) - \mathcal{N}_{SD}(T_{\frac{4n}{p}}) + 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p}}) + \mathcal{N}_{SD}(SD_{\frac{8n}{p}})\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{N}(SD_{8p^m}) =& 2\Big(\mathcal{N}_{SD}(\mathbb{Z}_{4p^m}) + \mathcal{N}_{SD}(D_{4p^m}) + \mathcal{N}_{SD}(T_{4p^m}) \\ &- 2\mathcal{N}_{SD}(\mathbb{Z}_{2p^m}) - \mathcal{N}_{SD}(\mathbb{Z}_{4p^{m-1}}) - \mathcal{N}_{SD}(D_{4p^{m-1}}) \\ &- \mathcal{N}_{SD}(T_{4p^{m-1}}) + 2\mathcal{N}_{SD}(\mathbb{Z}_{2p^{m-1}}) + \mathcal{N}_{SD}(SD_{8p^{m-1}})\Big)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathcal{N}(SD_{2^{m_0+3}p^m}) =& 2\Big(\mathcal{N}_{SD}(\mathbb{Z}_{2^{m_0+2}p^m}) + \mathcal{N}_{SD}(D_{2^{m_0+2}p^m}) \\ &+ \mathcal{N}_{SD}(T_{2^{m_0+2}p^m}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2^{m_0+1}p^m}) \\ &- \mathcal{N}_{SD}(\mathbb{Z}_{2^{m_0+2}p^{m-1}}) - \mathcal{N}_{SD}(D_{2^{m_0+2}p^{m-1}}) \\ &- \mathcal{N}_{SD}(T_{2^{m_0+2}p^{m-1}}) + 2\mathcal{N}_{SD}(\mathbb{Z}_{2^{m_0+1}p^{m-1}}) \\ &+ \mathcal{N}_{SD}(SD_{2^{m_0+3}p^{m-1}})\Big).\end{aligned} \tag{3.3}$$

Using Table 1 and Theorems 2.3 and 2.4, the relations (3.2) and (3.3) transformed into solvable recurrence relations.

Also, in Theorem 3.1, if $m_1 = m_2 = \cdots = m_k = 1$, i.e., $n = p_1p_2\cdots p_k$, then

$$\begin{aligned}\mathcal{N}(SD_{8n}) =& 2\Bigg(\mathcal{N}_{SD}(\mathbb{Z}_{4n}) + \mathcal{N}_{SD}(D_{4n}) + \mathcal{N}_{SD}(T_{4n}) - 2\mathcal{N}_{SD}(\mathbb{Z}_{2n}) \\ &+ \sum_{r=1}^{k}(-1)^r\binom{k}{r}\Big(\mathcal{N}_{SD}(\mathbb{Z}_{\frac{4n}{p_1p_2\cdots p_r}}) + \mathcal{N}_{SD}(D_{\frac{4n}{p_1p_2\cdots p_r}}) + \mathcal{N}_{SD}(T_{\frac{4n}{p_1p_2\cdots p_r}})\end{aligned} \tag{3.4}$$

$$- 2\mathcal{N}_{SD}(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_r}}) - \mathcal{N}_{SD}(SD_{\frac{8n}{p_1p_2\cdots p_r}})\Big)\Big).$$

3.2. **The number of fuzzy subgroups of the group U_{6n}.** In this section, consider the group U_{6n}, where $n \geq 2$ which has the following presentation:

$$U_{6n} = \langle x, y \mid x^{2n} = y^3 = e, yxy = x\rangle, \quad n \geq 2.$$

In fact, for $n = 1$ we have $U_6 \cong D_6$. According to the presentation of U_{6n}, we get $x^{2i}y = yx^{2i}$ and $x^{2i+1}y = y^2x^{2i+1}$. The group U_{6n} is metacyclic, that is, it has a cyclic normal subgroup N such that the quotient U_{6n}/N is also cyclic. Therefore, it is supersolvable, and consequently, all maximal subgroups of U_{6n} have prime index. These maximal subgroups are either of type $\langle xy^i\rangle \cong \mathbb{Z}_{2n}$ or of type $\langle x^p, y\rangle$, where $0 \leq i \leq 2$ and p is a prime divisor of $2n$. In particular, the number of all maximal subgroups of U_{6n} is $\varphi(2n) + 3$ [15]. The distinct fuzzy subgroups of U_{6n} with respect to the natural equivalence relation $\sim$ are enumerated in [12]. In the following, we aim to count the fuzzy subgroups of U_{6n} with respect to $\approx$.

By the properties of automorphism group and the order of elements of U_{6n}, we find that $\mathrm{Aut}(U_{6n}) = \{f_{\alpha,\beta,\gamma} \mid 0 \leq \alpha \leq 2n-1 \text{ and } (\alpha, 2\mathrm{n}) = 1, \beta = 0, 1, 2, \gamma = 1, 2\}$, where $f_{\alpha,\beta,\gamma}(x) = x^\alpha y^\beta$ and $f_{\alpha,\beta,\gamma}(y) = y^\gamma$. This implies that $|\mathrm{Aut}(U_{6n})| = 6\varphi(2n)$. By $\mathrm{Aut}(U_{6n})$, it is clear that three subgroups $M_{1i} = \langle xy^i\rangle \cong \mathbb{Z}_{2n}$, where $0 \leq i \leq 2$, are isomorphic by

$$f_{1,1,\gamma} = \begin{cases} x \to xy, \\ y \to y^\gamma, \end{cases} \quad \text{and} \quad f_{1,2,\gamma} = \begin{cases} x \to xy^2, \\ y \to y^\gamma. \end{cases}$$

Therefore, the number of non-isomorphic maximal subgroups of U_{6n} is $\varphi(2n)+1$ which are $M_1 = \langle x\rangle \cong \mathbb{Z}_{2n}$ and $\langle x^p, y\rangle$, where p is a prime divisor of $2n$.

Let $n = 2^{\alpha_1}3^{\alpha_2}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors. For counting $\mathcal{N}(U_{6n})$, we consider two cases:

Case 1: $\alpha_2 = 0$.

Then, $n = 2^{\alpha_1}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ and non-isomorphic maximal subgroups of U_{6n} are

$$M_1 = \langle x\rangle = \{x^t \mid 1 \leq t \leq 2n\} \cong \mathbb{Z}_{2n},$$
$$M_2 = \langle x^2, y\rangle = \langle x^2y\rangle = \{x^{2t}, x^{2t}y, x^{2t}y^2 \mid 1 \leq t \leq n\} \cong \mathbb{Z}_{3n},$$
$$M_{i+2} = \langle x^{p_i}, y\rangle = \left\{x^{p_it}, x^{p_it}y, x^{p_it}y^2 \mid 1 \leq t \leq \frac{2n}{p_i}\right\} \cong U_{\frac{6n}{p_i}}, \quad \text{where } 1 \leq i \leq k.$$

Therefore, an arbitrary intersection of the non-isomorphic maximal subgroups of U_{6n} is as follows, where $1 \leq r \leq k$, $i_r \in \{3, \ldots, k+2\}$,

$$M_1 \cap M_2 = \langle x^2\rangle \cong \mathbb{Z}_n,$$
$$M_1 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y\rangle \cong \mathbb{Z}_{\frac{3n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$

$$M_1 \cap M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}, y \rangle \cong U_{\frac{6n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}.$$

By Theorem 2.2, we get

(3.5)

$$\begin{aligned}
&\mathcal{N}(U_{6n})\\
=&2\Bigg(\mathcal{N}_U(\mathbb{Z}_{2n}) + \mathcal{N}_U(\mathbb{Z}_{3n}) - \mathcal{N}_U(\mathbb{Z}_n)\\
&- \sum_{i=1}^{k} \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{p_i}}) + \mathcal{N}_U(\mathbb{Z}_{\frac{3n}{p_i}}) - \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_i}}) - \mathcal{N}_U(U_{\frac{6n}{p_i}})\\
&+ \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{p_ip_j}}) + \mathcal{N}_U(\mathbb{Z}_{\frac{3n}{p_ip_j}}) - \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_ip_j}}) - \mathcal{N}_U(U_{\frac{6n}{p_ip_j}})\\
&- \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=j+1}^{k} \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}}) + \mathcal{N}_U(\mathbb{Z}_{\frac{3n}{p_ip_jp_l}}) - \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_ip_jp_l}}) - \mathcal{N}_U(U_{\frac{6n}{p_ip_jp_l}})\\
&+ \cdots + (-1)^k\Big(\mathcal{N}_U(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_k}}) + \mathcal{N}_U(\mathbb{Z}_{\frac{3n}{p_1p_2\cdots p_k}}) - \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_1p_2\cdots p_k}}) - \mathcal{N}_U(U_{\frac{6n}{p_1p_2\cdots p_k}})\Big)\Bigg).
\end{aligned}$$

For every cyclic subgroup $\mathbb{Z}_m$ of U_{6n}, $\mathcal{N}_U(\mathbb{Z}_m) = \mathcal{N}(\mathbb{Z}_m)$. Every non-cyclic subgroup of U_{6n} is of type $H_r = \langle x^r, y\rangle \cong U_{\frac{6n}{r}}$, where r is an odd divisor of n [15]. Every $f \in \mathrm{Aut}(H_r)$ is of type

$$f = \begin{cases} x^r \to x^{r\alpha}y^\beta, & 1 \le \alpha \le \frac{2n}{r}, (\alpha, \frac{2n}{r}) = 1, \beta = 0, 1, 2,\\ y \to y^\gamma, & \gamma = 1, 2, \end{cases}$$

which can be extended to an automorphism of U_{6n} as $f_E(x) = x^\alpha y^\beta$ and $f_E(y) = y^\gamma$. Consequently, $\mathcal{N}_U(H_r) = \mathcal{N}(H_r)$ and $\mathcal{N}_U(U_6) = \mathcal{N}(D_6) = 6$. Now, by rewriting the formula (3.5), we obtain the following theorem.

Theorem 3.2. *Let $n = 2^{\alpha_1}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the prime factorization of n, where 3 does not divide n. Then, the number of all distinct fuzzy subgroups of the group U_{6n}, $n \ge 2$, with respect to $\approx$ is given by the equality*

$$\mathcal{N}(U_{6n}) = 2\Bigg(\mathcal{N}(\mathbb{Z}_{2n}) + \mathcal{N}(\mathbb{Z}_{3n}) - \mathcal{N}(\mathbb{Z}_n) + \sum_{r=1}^{k}\Bigg(\sum_{i_1=1}^{k-r+1}\sum_{i_2=i_1+1}^{k-r+2}\sum_{i_3=i_2+1}^{k-r+3}\cdots\sum_{i_r=i_{r-1}+1}^{k}(-1)^r\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}}\Bigg)\Bigg),$$

where

$$\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} = \mathcal{N}(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) + \mathcal{N}(\mathbb{Z}_{\frac{3n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}(U_{\frac{6n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}),$$

and the above iterated sum is equal to 0 *for* $k = 0$*.*

For studying Theorem 3.2 in detail, put $k = 0$ (i.e., $n = 2^m$). Then,

$$\mathcal{N}(U_{6\cdot 2^m}) = 2\Big(\mathcal{N}(\mathbb{Z}_{2^{m+1}}) + \mathcal{N}(\mathbb{Z}_{3\cdot 2^m}) - \mathcal{N}(\mathbb{Z}_{2^m})\Big).$$

By Table 1, we get

$$\mathcal{N}(U_{6\cdot 2^m}) = 2^{m+1}(3 + m). \tag{3.6}$$

By putting $k = 1$ (i.e., $n = p^m$ or $n = 2^{m_0}p^m$) in Theorem 3.2, we get

$$\mathcal{N}(U_{6n}) = 2\Big(\mathcal{N}(\mathbb{Z}_{2n}) + \mathcal{N}(\mathbb{Z}_{3n}) - \mathcal{N}(\mathbb{Z}_n) - \mathcal{N}(\mathbb{Z}_{\frac{2n}{p}}) - \mathcal{N}(\mathbb{Z}_{\frac{3n}{p}}) + \mathcal{N}(\mathbb{Z}_{\frac{n}{p}}) + \mathcal{N}(U_{\frac{6n}{p}})\Big).$$

In particular,

$$\begin{aligned}\mathcal{N}(U_{6p^m}) =& 2\Big(\mathcal{N}(\mathbb{Z}_{2p^m}) + \mathcal{N}(\mathbb{Z}_{3p^m}) - \mathcal{N}(\mathbb{Z}_{p^m}) - \mathcal{N}(\mathbb{Z}_{2p^{m-1}}) - \mathcal{N}(\mathbb{Z}_{3p^{m-1}})\\ &+ \mathcal{N}(\mathbb{Z}_{p^{m-1}}) + \mathcal{N}(U_{6p^{m-1}})\Big).\end{aligned}$$

By Table 1, we obtain $\mathcal{N}(U_{6p^m}) = 2^m(2m+5)+2\mathcal{N}(U_{6p^{m-1}})$. By solving this recurrence relation, we obtain

$$\mathcal{N}(U_{6p^m}) = 2^m\sum_{i=0}^{m-1}(2(m-i)+5) + 2^m\mathcal{N}(D_6) = 2^m(m^2+6m+6). \tag{3.7}$$

If $m_1 = m_2 = \cdots = m_k = 1$, i.e., $n = 2^{\alpha_1}p_1p_2\cdots p_k$, then

$$\begin{aligned}\mathcal{N}(U_{6n}) =& 2\Bigg(\mathcal{N}(\mathbb{Z}_{2n}) + \mathcal{N}(\mathbb{Z}_{3n}) - \mathcal{N}(\mathbb{Z}_n)\\ &+ \sum_{r=1}^{k}(-1)^r\binom{k}{r}\Big(\mathcal{N}(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_r}}) + \mathcal{N}(\mathbb{Z}_{\frac{3n}{p_1p_2\cdots p_r}}) - \mathcal{N}(\mathbb{Z}_{\frac{n}{p_1p_2\cdots p_r}})\\ &- \mathcal{N}(U_{\frac{6n}{p_1p_2\cdots p_r}})\Big)\Bigg).\end{aligned} \tag{3.8}$$

Case 2: $\alpha_2 \neq 0$.

Then, $n = 2^{\alpha_1}3^{\alpha_2}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ and non-isomorphic maximal subgroups of U_{6n} are $M_1 = \langle x\rangle \cong \mathbb{Z}_{2n}$, $M_2 = \langle x^2, y\rangle \cong \mathbb{Z}_n \times \mathbb{Z}_3$, $M_3 = \langle x^3, y\rangle \cong U_{\frac{6n}{3}}$ and $M_{i+3} = \langle x^{p_i}, y\rangle \cong U_{\frac{6n}{p_i}}$, where $1 \leq i \leq k$. According to the structure of maximal subgroups, every non-cyclic subgroup of U_{6n} is of type $H_r = \langle x^r, y\rangle$, where r is a divisor of $2n$ [15]. Similar to Case 1, if $r \neq 3$ then every automorphism of H_r can be extended to an automorphism of U_{6n} and $\mathcal{N}_U(H_r) = \mathcal{N}(H_r)$.

Intersection of maximal subgroups of U_{6n} is as follows, where $1 \leq r \leq k$, $i_r \in \{3, \ldots, k+3\}$,

$M_1 \cap M_2 = \langle x^2\rangle \cong \mathbb{Z}_n$,

$M_1 \cap M_3 = \langle x^3 \rangle \cong \mathbb{Z}_{\frac{2n}{3}},$
$M_1 \cap M_2 \cap M_3 = \langle x^6 \rangle \cong \mathbb{Z}_{\frac{n}{3}},$
$M_1 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}, y \rangle \cong \mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_3,$
$M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{3p_{i_1}p_{i_2}\cdots p_{i_r}}, y \rangle \cong U_{\frac{6n}{3p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$M_1 \cap M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$M_1 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{3p_{i_1}p_{i_2}\cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{2n}{3p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$M_1 \cap M_2 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{6p_{i_1}p_{i_2}\cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{n}{3p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}, y \rangle \cong U_{\frac{6n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$
$$M_2 \cap M_3 = \begin{cases} \langle x^6 y \rangle \cong \mathbb{Z}_n, & \text{if } \alpha_2 = 1, \\ \langle x^6, y \rangle \cong \mathbb{Z}_{\frac{n}{3}} \times \mathbb{Z}_3, & \text{if } \alpha_2 \geq 2, \end{cases}$$
$$M_2 \cap M_3 \cap M_{i_1} \cap \cdots \cap M_{i_r} = \begin{cases} \langle x^{6p_{i_1}p_{i_2}\cdots p_{i_r}} y \rangle \cong \mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}, & \text{if } \alpha_2 = 1, \\ \langle x^{6p_{i_1}p_{i_2}\cdots p_{i_r}}, y \rangle \cong \mathbb{Z}_{\frac{n}{3p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_3, & \text{if } \alpha_2 \geq 2. \end{cases}$$

Similar to Case 1, by using Theorem 2.2, we get the following theorem.

Theorem 3.3. *Let $n = 2^{\alpha_1} 3^{\alpha_2} p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors and $\alpha_2 \neq 0$. Then, the number of all distinct fuzzy subgroups of the group U_{6n} with respect to $\approx$ is given by the equality*

$$\begin{aligned} \mathcal{N}(U_{6n}) =& 2\Bigg(\mathcal{N}_U(\mathbb{Z}_{2n}) + \mathcal{N}_U(\mathbb{Z}_n \times \mathbb{Z}_3) + \mathcal{N}_U(\mathbb{Z}_{\frac{n}{3}}) + \mathcal{N}_U(U_{\frac{6n}{3}}) \\ & - \mathcal{N}_U(\mathbb{Z}_n) - \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{3}}) - \mathcal{N}_U(A) \\ & + \sum_{r=1}^{k} \Bigg(\sum_{i_1=1}^{k-r+1} \sum_{i_2=i_1+1}^{k-r+2} \sum_{i_3=i_2+1}^{k-r+3} \cdots \sum_{i_r=i_{r-1}+1}^{k} (-1)^r \mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} \Bigg)\Bigg), \end{aligned}$$

where the above iterated sum is equal to 0 for $k = 0$,

$$A = \begin{cases} \mathbb{Z}_n, & \textit{if } \alpha_2 = 1, \\ \mathbb{Z}_{\frac{n}{3}} \times \mathbb{Z}_3, & \textit{if } \alpha_2 \geq 2, \end{cases}$$

and

$$\begin{aligned} \mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} =& \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{p_{i_1}\cdots p_{i_r}}}) + \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_{i_1}\cdots p_{i_r}}} \times \mathbb{Z}_3) + \mathcal{N}_U(\mathbb{Z}_{\frac{n}{3p_{i_1}\cdots p_{i_r}}}) + \mathcal{N}_U(U_{\frac{6n}{3p_{i_1}\cdots p_{i_r}}}) \\ & - \mathcal{N}_U(\mathbb{Z}_{\frac{n}{p_{i_1}\cdots p_{i_r}}}) - \mathcal{N}_U(\mathbb{Z}_{\frac{2n}{3p_{i_1}\cdots p_{i_r}}}) - \mathcal{N}_U(U_{\frac{6n}{p_{i_1}\cdots p_{i_r}}}) - \mathcal{N}_U(B), \end{aligned}$$

with $B = \begin{cases} \mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}, & \text{if } \alpha_2 = 1, \\ \mathbb{Z}_{\frac{n}{3p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_3, & \text{if } \alpha_2 \geq 2. \end{cases}$

In the following, by using Theorem 3.3, we will focus on determining $\mathcal{N}(U_{6\cdot 3^m})$, where $m \geq 1$. First, suppose that $m = 1$. By putting $n = 3$ in Theorem 3.3 (i.e., $\alpha_1 = k = 0$, $\alpha_2 = 1$), we get

$$\mathcal{N}(U_{18}) = 2\Big(\mathcal{N}_U(\mathbb{Z}_6) + \mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3) + \mathcal{N}_U(U_6) - 2\mathcal{N}_U(\mathbb{Z}_3) - \mathcal{N}_U(\mathbb{Z}_2) + 1\Big). \tag{3.9}$$

Based on Table 1, $\mathcal{N}_U(\mathbb{Z}_6) = 6$, $\mathcal{N}_U(\mathbb{Z}_3) = \mathcal{N}_U(\mathbb{Z}_2) = 2$ and $\mathcal{N}_U(U_6) = \mathcal{N}(D_6) = 6$. By the presentation $\langle x, y \mid x^6 = y^3 = e,\ yxy = x\rangle$ and automorphisms of U_{18}, non-isomorphic chains of subgroups of $\langle x^2, y\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ terminating in $\langle x^2, y\rangle$ are as follows:

$$\begin{array}{lll} \langle x^2, y\rangle, & \langle x^2\rangle \subset \langle x^2, y\rangle, & \langle x^2 y\rangle \subset \langle x^2, y\rangle, \\ \langle y\rangle \subset \langle x^2, y\rangle, & \{e\} \subset \langle x^2, y\rangle, & \{e\} \subset \langle x^2\rangle \subset \langle x^2, y\rangle, \\ \{e\} \subset \langle x^2 y\rangle \subset \langle x^2, y\rangle, & \{e\} \subset \langle y\rangle \subset \langle x^2, y\rangle, & \end{array}$$

which imply $\mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3) = 8$. Hence, $\mathcal{N}(U_{18}) = 30$.

For $n = 3^m$, $m \geq 2$, by Theorem 3.3, we have

$$\begin{aligned} \mathcal{N}(U_{6\cdot 3^m}) =& 2\Big(\mathcal{N}_U(\mathbb{Z}_{2\cdot 3^m}) + \mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) - \mathcal{N}_U(\mathbb{Z}_{3^m}) - \mathcal{N}_U(\mathbb{Z}_{2\cdot 3^{m-1}}) \\ & - \mathcal{N}_U(\mathbb{Z}_{3^{m-1}} \times \mathbb{Z}_3) + \mathcal{N}_U(\mathbb{Z}_{3^{m-1}}) + \mathcal{N}_U(U_{6\cdot 3^{m-1}})\Big). \end{aligned}$$

Using Table 1,

$$\mathcal{N}(U_{6\cdot 3^m}) = 2^m(m+2) + 2\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) - 2\mathcal{N}_U(\mathbb{Z}_{3^{m-1}} \times \mathbb{Z}_3) + 2\mathcal{N}_U(U_{6\cdot 3^{m-1}}).$$

By solving the above recurrence relation, we get

$$\begin{aligned} \mathcal{N}(U_{6\cdot 3^m}) =& 2^{m-1}\mathcal{N}_U(U_{18}) + 2\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) - 2^{m-1}\mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3) \\ & + 2^{m-1}(m^2 + 5m - 6) + \sum_{i=1}^{m-2} 2^i \mathcal{N}_U(\mathbb{Z}_{3^{m-i}} \times \mathbb{Z}_3). \end{aligned}$$

Since $\mathcal{N}_U(U_{18}) = \mathcal{N}(U_{18}) = 30$ and $\mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3) = 8$, hence

$$\mathcal{N}(U_{6\cdot 3^m}) = 2^{m-1}(m^2 + 5m + 16) + 2\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) + \sum_{i=1}^{m-2} 2^i \mathcal{N}_U(\mathbb{Z}_{3^{m-i}} \times \mathbb{Z}_3). \tag{3.10}$$

For determining $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3)$, consider the subgroup $\langle x^2, y\rangle \cong \mathbb{Z}_{3^m} \times \mathbb{Z}_3$ of $U_{6\cdot 3^m}$. Non-isomorphic chains relative to automorphisms of $U_{6\cdot 3^m}$ are one of the following types:

$$\begin{aligned} \mathcal{C}_1 &: H_1 \subset \cdots \subset H_{r-1} \subset H_r = \langle x^2\rangle \subset \langle x^2, y\rangle, \\ \mathcal{C}_2 &: K_1 \subset \cdots \subset K_{s-1} \subset K_s = \langle x^2 y\rangle \subset \langle x^2, y\rangle, \\ \mathcal{C}_3 &: L_1 \subset \cdots \subset L_{t-1} \subset L_t = \langle x^6, y\rangle \subset \langle x^2, y\rangle, \\ \mathcal{C}_4 &: L_1 \subset \cdots \subset L_{t-1} \subset \langle x^2, y\rangle. \end{aligned}$$

In fact, every chain in $\mathcal{C}_4$ obtains by deleting $\langle x^6, y\rangle$ of a chain of $\mathcal{C}_3$. Since $\langle x^2\rangle \cong \mathbb{Z}_{3^m}$, $\langle x^2y\rangle \cong \mathbb{Z}_{3^m}$ and $\langle x^6, y\rangle \cong \mathbb{Z}_{3^{m-1}} \times \mathbb{Z}_3$, hence $|\mathcal{C}_1| = |\mathcal{C}_2| = \mathcal{N}_U(\mathbb{Z}_{3^m}) = \mathcal{N}(\mathbb{Z}_{3^m})$ and $|\mathcal{C}_3| = |\mathcal{C}_4| = \mathcal{N}_U(\mathbb{Z}_{3^{m-1}} \times \mathbb{Z}_3)$. On the other hand, $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) = \sum_{i=1}^4 |\mathcal{C}_i|$, which leads to $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) = 2^{m+1} + 2\mathcal{N}_U(\mathbb{Z}_{3^{m-1}} \times \mathbb{Z}_3)$. By solving the obtained recurrence relation, we get $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) = (m-1)\,(2^{m+2} - 2^{m+1}) + 2^{m-1}\mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Since $\mathcal{N}_U(\mathbb{Z}_3 \times \mathbb{Z}_3) = 8$, then $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) = 2^{m+1}(m+1)$.

Now, by putting $\mathcal{N}_U(\mathbb{Z}_{3^m} \times \mathbb{Z}_3)$ in (3.10), we find

$$\begin{aligned}(3.11)\qquad \mathcal{N}(U_{6\cdot 3^m}) &= 2^{m-1}(m^2+5m+16) + 2^{m+2}(m+1) + \sum_{i=1}^{m-2} 2^{m+1}(m-i+1)\\ &= 2^{m-1}(3m^2+15m+12).\end{aligned}$$

3.3. **The number of fuzzy subgroups of the group V_{8n}.** Consider the group V_{8n}, where $n \geq 2$ having the following presentation:

$$V_{8n} = \langle x, y \mid x^{2n} = y^4 = e, xyx = y^{-1}, xy^{-1}x = y\rangle, \quad n \geq 2.$$

In fact, for $n = 1$ we have $V_8 \cong D_8$. The group V_{8n} was presented for first time in [9] in the case that n is odd. For the main properties of the group V_{8n}, when n is an even natural number, we refer reader to [3].

According to the presentation of V_{8n}, we get $x^iy^2 = y^2x^i$ and

$$\begin{cases} x^iy = yx^{-i}, x^iy^3 = y^3x^{-i}, & i \text{ is even},\\ x^iy = y^3x^{-i}, x^iy^3 = y^{-1}x^{-i}, & i \text{ is odd}.\end{cases}$$

Since V_{8n} is supersolvable, hence all maximal subgroups of V_{8n} have prime index and they are of type $\langle x, y^2\rangle$ or $\langle x^p, y^2, x^iy\rangle$, where p is a prime divisor of $2n$ and $0 \leq i \leq p-1$. In particular, the number of all maximal subgroups of V_{8n} is $1+\nu(2n)$, where $\nu(2n)$ is the sum of prime divisors of $2n$ [15].

By the properties of automorphism group and the order of elements of V_{8n}, we find that $\mathrm{Aut}(V_{8n})$ is the set $\{f_{\alpha,\beta} \mid 0 \leq \alpha \leq 2n-1 \text{ and } (\alpha, 2\mathrm{n}) = 1, 0 \leq \beta \leq \mathrm{n}-1\}$ of the size $4n\varphi(2n)$, where

$$f_{\alpha,\beta} = \begin{cases} x \to x^\alpha \text{ or } x^\alpha y^2,\\ y \to x^{2\beta}y \text{ or } x^{2\beta}y^3.\end{cases}$$

Let $n = 2^{m_0}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors. First, we show that for $0 \leq r, s \leq p_i - 1$, two maximal subgroups $H_r = \langle x^{p_i}, y^2, x^ry\rangle$ and $H_s = \langle x^{p_i}, y^2, x^sy\rangle$ are isomorphic. If r and s are both even or both odd, and additionally $r\rangle s$, then $r-s$ is even and two subgroups H_r and K_s are isomorphic by

$$f_{1,r-s} = \begin{cases} x \to x,\\ y \to x^{r-s}y.\end{cases}$$

If r is even and s is odd, then one can easily prove that

$$\langle x^{p_i}, y^2, x^sy\rangle = \langle x^{p_i}, y^2, x^{p_i+s}y\rangle.$$

Since p_i+s is even, an argument similar to the previous case shows that the subgroups $H_s = \langle x^{p_i}, y^2, x^{p_i+s}y\rangle$ and H_r are isomorphic.

Therefore, there are $k+3$ non-isomorphic maximal subgroups that are

$$M_1 = \langle x, y^2\rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2, \quad M_2 = \langle x^2, y^2, y\rangle = \langle x^2y^2, y\rangle, \quad M_3 = \langle x^2, y^2, xy\rangle$$
$$M_{i+3} = \langle x^{p_i}, y^2, y\rangle = \langle x^{p_i}y^2, y\rangle \cong V_{\frac{8n}{p_i}}, \quad \text{where } 1 \le i \le k.$$

In the group V_{8n}, the number of fuzzy subgroups of M_i, $1 \le i \le 3$, under $\approx$ is known only in the case where n is odd. Therefore, we proceed in the following to count the number of fuzzy subgroups of V_{8n} under $\approx$, where n is odd. If n is even, determining $\mathcal{N}(V_{8n})$ involves complex computations and will be the subject of further research.

Now, let the prime factorization of n be $p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes. As mentioned earlier, the non-isomorphic maximal subgroups of V_{8n} are

$$M_1 = \langle x, y^2\rangle = \{x^t, x^ty^2 \mid 1 \le t \le 2n\} \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2,$$
$$M_2 = \langle x^2y^2, y\rangle = \{x^{2t}, x^{2t}y, x^{2t}y^2, x^{2t}y^3 \mid 1 \le t \le n\} \cong T_{4n},$$
$$M_3 = \langle x^2, y^2, xy\rangle = \langle x^2y^2, xy\rangle = \{x^{2t}, x^{2t}y^2, x^{2t+1}y, x^{2t+1}y^3 \mid 1 \le t \le n\} \cong D_{4n},$$
$$M_{i+3} = \langle x^{p_i}y^2, y\rangle = \left\{x^{p_it}, x^{p_it}y, x^{p_it}y^2, x^{p_it}y^3 \mid 1 \le t \le \frac{2n}{p_i}\right\} \cong V_{\frac{8n}{p_i}}, \text{where } 1 \le i \le k.$$

Therefore, an arbitrary intersection of the non-isomorphic maximal subgroups of V_{8n} is as follows, where $1 \le r \le k$, $i_r \in \{4, \ldots, k+3\}$,

$$M_1 \cap M_2 = M_1 \cap M_3 = M_2 \cap M_3 = M_1 \cap M_2 \cap M_3 = \langle x^2, y^2\rangle = \langle x^2y^2\rangle \cong \mathbb{Z}_{2n},$$
$$M_1 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}, y^2\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_2,$$
$$M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y^2, y\rangle \cong T_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y^2, x^{p_{i_1}p_{i_2}\cdots p_{i_r}}y\rangle \cong D_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_1 \cap M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y^2\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_1 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y^2\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_2 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}y^2\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_1 \cap M_2 \cap M_3 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1}p_{i_2}\cdots p_{i_r}}\rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}},$$
$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}, y^2, y\rangle = \langle x^{p_{i_1}p_{i_2}\cdots p_{i_r}}y^2, y\rangle \cong V_{\frac{8n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}.$$

By Theorem 2.2, we get

$$\mathcal{N}(V_{8n}) = 2\Bigg(\mathcal{N}_V(\mathbb{Z}_{2n} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{4n}) + \mathcal{N}_V(D_{4n}) - 2\mathcal{N}_V(\mathbb{Z}_{2n})$$

$$
\begin{aligned}
&-\sum_{i=1}^{k} \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i}} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{\frac{4n}{p_i}}) + \mathcal{N}_V(D_{\frac{4n}{p_i}}) - 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i}}) - \mathcal{N}_V(V_{\frac{8n}{p_i}})\\
&+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_j}} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{\frac{4n}{p_ip_j}}) + \mathcal{N}_V(D_{\frac{4n}{p_ip_j}}) - 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_j}})\\
&- \mathcal{N}_V(V_{\frac{8n}{p_ip_j}}) - \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=j+1}^{k} \Big(\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{\frac{4n}{p_ip_jp_l}})\\
&+ \mathcal{N}_V(D_{\frac{4n}{p_ip_jp_l}}) - 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}}) - \mathcal{N}_V(V_{\frac{8n}{p_ip_jp_l}})\Big)\\
&+ \cdots + (-1)^k\Big(\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_k}} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{\frac{4n}{p_1p_2\cdots p_k}}) + \mathcal{N}_V(D_{\frac{4n}{p_1p_2\cdots p_k}})\\
&- 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_k}}) - \mathcal{N}_V(V_{\frac{8n}{p_1p_2\cdots p_k}})\Big)\Bigg).
\end{aligned}
$$

By rewriting the above formula, we obtain the following theorem.

Theorem 3.4. *Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors, where $p_1, p_2, \ldots, p_k$ are distinct odd prime numbers. Then, the number of all distinct fuzzy subgroups of the group V_{8n}, $n \geq 2$, with respect to $\approx$ is given by the equality*

$$
\begin{aligned}
\mathcal{N}(V_{8n}) =& 2\Bigg(\mathcal{N}_V(\mathbb{Z}_{2n} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{4n}) + \mathcal{N}_V(D_{4n}) - 2\mathcal{N}_V(\mathbb{Z}_{2n})\\
&+ \sum_{r=1}^{k}\Bigg(\sum_{i_1=1}^{k-r+1}\sum_{i_2=i_1+1}^{k-r+2}\sum_{i_3=i_2+1}^{k-r+3}\cdots\sum_{i_r=i_{r-1}+1}^{k} (-1)^r \mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}}\Bigg)\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} =& \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_2) + \mathcal{N}_V(T_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) + \mathcal{N}_V(D_{\frac{4n}{p_{i_1}p_{i_2}\cdots p_{i_r}}})\\
&- 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}_V(V_{\frac{8n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}),
\end{aligned}
$$

and the above iterated sum is equal to 0 *for* $k = 0$.

Note that in Theorem 3.4, the value of $\mathcal{N}_V$ for a cyclic subgroup $\mathbb{Z}_m$ is equal to $\mathcal{N}(\mathbb{Z}_m)$ which is determined in Theorem 2.1. For determining the value $\mathcal{N}_V$ for dihedral and dicyclic groups, suppose that $H_r = \langle x^{2r}, y^2, y\rangle = \langle x^{2r}y^2, y\rangle \cong T_{\frac{4n}{r}}$, $K_r = \langle x^{2r}, y^2, x^r y\rangle = \langle x^{2r}y^2, x^r y\rangle \cong D_{\frac{4n}{r}}$ and $L_r = \langle x^r, y^2, y\rangle = \langle x^r y^2, y\rangle \cong V_{\frac{8n}{r}}$, where r is a proper divisor of n. If $f \in \mathrm{Aut}(H_r)$ and $g \in \mathrm{Aut}(K_r)$, then they are as follows, where $1 \leq \alpha, \beta \leq \frac{2n}{r}$ and $(\alpha, \frac{2n}{r}) = 1$,

$$
f = \begin{cases} x^{2r}y^2 \to (x^{2r}y^2)^\alpha, \\ y \to (x^{2r}y^2)^\beta y, \end{cases} \quad \text{and} \quad g = \begin{cases} x^{2r}y^2 \to (x^{2r}y^2)^\alpha, \\ x^r y \to (x^{2r}y^2)^\beta x^r y, \end{cases}
$$

which can be extended to an automorphism of V_{8n} as follows:

$$f_E = \begin{cases} x \to x^{\alpha}, \\ y \to x^{2r\beta}y, x^{2r\beta}y^3, \end{cases} \quad \text{and} \quad g_E = \begin{cases} x \to x^{\alpha}, \\ y \to x^{2r\beta - r\alpha + r}y, x^{2r\beta - r\alpha + r}y^3. \end{cases}$$

Therefore, $\mathcal{N}_V(H_r) = \mathcal{N}(H_r)$ and $\mathcal{N}_V(K_r) = \mathcal{N}(K_r)$. Note that $\mathcal{N}_V(H_n) = \mathcal{N}_V(T_4) = 4$ and $\mathcal{N}_V(K_n) = \mathcal{N}_V(D_4) = \mathcal{N}_V(\mathbb{Z}_2 \times \mathbb{Z}_2) = 6$. Also, every automorphism of K_r is of type

$$h = \begin{cases} x^r y^2 \to (x^r y^2)^{\alpha}, (x^r y^2)^{\alpha} y^2, & 1 \le \alpha \le \frac{2n}{r},\ (\alpha, \frac{2n}{r}) = 1, \\ y \to x^{2\beta}y, x^{2\beta}y^3, & 1 \le \beta \le \frac{2n}{r}, \end{cases}$$

which can be extended to an automorphism of V_{8n} as $h_E = \begin{cases} x \to x^{\alpha}, x^{\alpha}y^2, \\ y \to x^{2\beta}y, x^{2\beta}y^3, \end{cases}$ and so $\mathcal{N}_V(L_r) = \mathcal{N}(L_r)$. Note that $\mathcal{N}_V(L_n) = \mathcal{N}_V(V_8) = 24$.

In light of the above, the value $\mathcal{N}_V$ of cyclic, dihedral and dicyclic groups can be calculated by Theorems 2.1, 2.3 and 2.4, respectively. In the following, according to the automorphisms of $V_{8n} = \langle x, y \rangle$, we treat to the counting the fuzzy subgroups of $H = \langle x, y^2 \rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$. Assume that $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ is the decomposition of n as a product of odd prime factors. Maximal subgroups of H are $\langle x \rangle \cong \mathbb{Z}_{2n}$, $\langle xy^2 \rangle \cong \mathbb{Z}_{2n}$, $\langle x^2 y^2 \rangle \cong \mathbb{Z}_{2n}$, and $\langle x^{p_i}, y^2 \rangle \cong \mathbb{Z}_{\frac{2n}{p_i}} \times \mathbb{Z}_2$, where $1 \le i \le k$. Two subgroups $\langle x \rangle$ and $\langle xy^2 \rangle$ are isomorphic by the automorphism $f_{1,0}$ of V_{8n}. Therefore, the members of the non-isomorphic maximal subgroups of H are as follows:

$$M_1 = \langle x \rangle = \{x^t \mid 1 \le t \le 2n\}, \quad M_2 = \langle x^2 y^2 \rangle = \{x^{2t}, x^{2t}y^2 \mid 1 \le t \le n\},$$
$$M_{i+2} = \langle x^{p_i}, y^2 \rangle = \left\{ x^{p_i t}, x^{p_i t} y^2 \mid 1 \le t \le \frac{2n}{p_i} \right\}, \quad 1 \le i \le k.$$

Consequently, an arbitrary intersection of the non-isomorphic maximal subgroups of H is as follows, where $1 \le r \le k$, $i_r \in \{3, \dots, k+2\}$,

$$M_1 \cap M_2 = \langle x^2 \rangle \cong \mathbb{Z}_n,$$
$$M_1 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1} p_{i_2} \cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1} p_{i_2} \cdots p_{i_r}}},$$
$$M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1} p_{i_2} \cdots p_{i_r}}, y^2 \rangle = \langle x^{2p_{i_1} p_{i_2} \cdots p_{i_r}} y^2 \rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1} p_{i_2} \cdots p_{i_r}}},$$
$$M_1 \cap M_2 \cap M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{2p_{i_1} p_{i_2} \cdots p_{i_r}} \rangle \cong \mathbb{Z}_{\frac{n}{p_{i_1} p_{i_2} \cdots p_{i_r}}},$$
$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r} = \langle x^{p_{i_1} p_{i_2} \cdots p_{i_r}}, y^2 \rangle \cong \mathbb{Z}_{\frac{2n}{p_{i_1} p_{i_2} \cdots p_{i_r}}} \times \mathbb{Z}_2.$$

By Theorem 2.2, we get

$$\begin{aligned} \mathcal{N}(H) = 2\Bigg(& 2\mathcal{N}_V(\mathbb{Z}_{2n}) - \mathcal{N}_V(\mathbb{Z}_n) - \sum_{i=1}^{k} 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{n}{p_i}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i}} \times \mathbb{Z}_2) \\ & + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i p_j}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{n}{p_i p_j}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_i p_j}} \times \mathbb{Z}_2) \end{aligned}$$

$$-\sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=j+1}^{k} 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{n}{p_ip_jp_l}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_ip_jp_l}} \times \mathbb{Z}_2)$$
$$+ \cdots + (-1)^k\Big(2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_k}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{n}{p_1p_2\cdots p_k}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_1p_2\cdots p_k}} \times \mathbb{Z}_2)\Big)\Big).$$

Since every automorphism of subgroups of H can be extended to a automorphism of H, then by rewriting the above formula, we obtain the following theorem.

Theorem 3.5. *Suppose that $n = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k} \geq 3$ is an odd natural number and $V_{8n} = \langle x, y\rangle$, then the number of all distinct fuzzy subgroups of the subgroup $\langle x, y^2\rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$ under $\approx$ with respect to automorphisms of V_{8n} is given by the equality*

$$\mathcal{N}_V(\mathbb{Z}_{2n} \times \mathbb{Z}_2) = 2\Bigg(2\mathcal{N}_V(\mathbb{Z}_{2n}) - \mathcal{N}_V(\mathbb{Z}_n) + \sum_{r=1}^{k}\Bigg(\sum_{i_1=1}^{k-r+1}\sum_{i_2=i_1+1}^{k-r+2}\sum_{i_3=i_2+1}^{k-r+3}\cdots\sum_{i_r=i_{r-1}+1}^{k}(-1)^r\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}}\Bigg)\Bigg),$$

where

$$\mathcal{N}_{p_{i_1}p_{i_2}\cdots p_{i_r}} = 2\mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}}) - \mathcal{N}_V(\mathbb{Z}_{\frac{2n}{p_{i_1}p_{i_2}\cdots p_{i_r}}} \times \mathbb{Z}_2).$$

4. Conclusion

This paper focuses on the enumeration of distinct fuzzy subgroups of the groups SD_{8n}, U_{6n} and $V_{8(2n-1)}$ under the equivalence relation $\approx$. The equivalence relation $\approx$ is based on chains of subgroups and automorphisms. Therefore, the subgroup lattice, the structure of maximal subgroups, and automorphism of the mentioned groups are studied. By using the inclusion-exclusion principle, recurrence relations for counting fuzzy subgroups are derived. These relations, presented in Theorems 3.1, 3.2, 3.3, and 3.4, are solvable in some special cases. A summary of the obtained results is described in Table 2.

Also, in this paper, suitable tools are provided for the counting of fuzzy subgroups of the group V_{8n} with respect to $\approx$. The value of $\mathcal{N}(V_{8n})$ is obtained for odd values of n, while the determination of $\mathcal{N}(V_{8n})$ for even values of n remains for future work. Furthermore, future research directions can involve computation of the number of fuzzy normal subgroups of SD_{8n}, U_{6n}, and V_{8n} with respect to $\approx$.

TABLE 2. The exact number of distinct fuzzy subgroups of SD_{8n}, U_{6n}, and $V_{8(2n-1)}$ in some particular cases

Group G	$\mathcal{N}(G)$	Relation Used
SD_{16}	56	Equation (3.1)
SD_{32}	144	Equation (3.1)
SD_{64}	352	Equation (3.1)
SD_{8p}	112	Equation (3.2)
SD_{8p^2}	452	Equation (3.2)
SD_{8p^3}	1560	Equation (3.2)
SD_{8p^4}	4896	Equation (3.2)
SD_{16p}	376	Equation (3.3)
SD_{8pq}	792	Equation (3.4)
$U_{6\cdot 2^m}$	$2^{m+1}(m+3)$	Equation (3.6)
$U_{6\cdot 3^m}$	$2^{m-1}(3m^2+15m+12)$	Equation (3.11)
U_{6p^m}, $p \neq 2, 3$	$2^m(m^2+6m+6)$	Equation (3.7)
U_{36}	96	Theorem 3.3
U_{12p}, $p \neq 2, 3$	88	Equation (3.8)
U_{18p}, $p \neq 2, 3$	174	Theorem 3.3
U_{6pq}, $p, q \neq 2, 3$	150	Equation (3.8)
V_{8p}, $p \neq 2$	136	Theorem 3.4
V_{8p^2}, $p \neq 2$	552	Theorem 3.4
V_{8p^3}, $p \neq 2$	1912	Theorem 3.4
V_{8p^4}, $p \neq 2$	6016	Theorem 3.4
V_{8pq}, $p, q \neq 2$	968	Theorem 3.4

REFERENCES

[1] L. Bentea and M. Tărnăuceanu, *A note on the number of fuzzy subgroups of finite groups*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **54**(1) (2008), 209–220.

[2] H. Darabi, F. Saeedi and M. Farrokhi D.G., *The number of fuzzy subgroups of some non-abelian groups*, Iran. J. Fuzzy Syst. **10**(6) (2013), 101–107. https://doi.org/10.22111/ijfs.2013.1333

[3] M. R. Darafsheh and N. S. Poursalavati, *On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups*, SUT J. Math. **37** (2001), 1–17. https://doi.org/10.55937/sut/1017153423

[4] P. S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264–269. https://doi.org/10.1016/0022-247X(81)90164-5

[5] B. Davvaz and L. K. Ardekani, *Counting the number of fuzzy subgroups of a special class of non-abelian groups of order* p^3, ARS Combin. **103** (2012), 175–179.

[6] B. Davvaz and L. K. Ardekani, *Classifying fuzzy subgroups of dicyclic groups*, Journal of Multiple-Valued Logic and Soft Computing **20**(5-6) (2013), 507–525.

[7] B. Davvaz and L. K. Ardekani, *Counting fuzzy subgroups of non-abelian groups of orders* p^3 *and* 2^4, Journal of Multiple-Valued Logic and Soft Computing **21**(5-6) (2013), 479–492.

[8] V. N. Dixit, R. Kumar and N. Ajmal, *Level subgroups and union of fuzzy subgroups*, Fuzzy Sets and Systems **37**(3) (1990), 359–371. https://doi.org/10.1016/0165-0114(90)90032-2

[9] G. James and M. Liebeck, *Representations and Characters of Groups*, Second Edition, Cambridge University Press, New York, 2001. https://doi.org/10.1017/CBO9780511814532
[10] L. K. Ardekani, *Counting fuzzy subgroups of some finite groups by a new equivalence relation*, Filomat **33**(19) (2019), 6151–6160. https://doi.org/10.2298/FIL1919151K
[11] L. K. Ardekani and B. Davvaz, *On the number of fuzzy subgroups of dicyclic groups*, Soft. Computing **24** (2020), 6183–6191.
[12] L. K. Ardekani and B. Davvaz, *On the subgroups lattice and fuzzy subgroups of finite groups* U_{6n}, Fuzzy Information and Engineering **14**(2) (2022), 152–166. https://doi.org/10.1080/16168658.2022.2119828
[13] V. Murali and B. B. Makamba, *On an equivalence of fuzzy subgroups, I*, Fuzzy Sets and Systems **123**(2) (2001), 259–264. https://doi.org/10.1016/S0165-0114(00)00098-1
[14] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517. https://doi.org/10.1016/0022-247X(71)90199-5
[15] H. B. Shelash and A. R. Ashrafi, *Computing maximal and minimal subgroups with respect to a given property in certain finite groups*, Quasigroups Related Systems **27** (2019), 133–146.
[16] M. Tărnăuceanu and L. Bentea, *On the number of fuzzy subgroups of finite abelian groups*, Fuzzy Sets and Systems **159**(9) (2008), 1084–1096. https://doi.org/10.1016/j.fss.2007.11.014
[17] M. Tărnăuceanu, *Classifying fuzzy subgroups of finite nonabelian groups*, Iran. J. Fuzzy Syst. **9**(4) (2012), 31–41. https://doi.org/10.22111/ijfs.2012.131
[18] M. Tărnăuceanu, *On the number of fuzzy subgroups of finite symmetric groups*, Journal of Multiple-Valued Logic and Soft Computing **21**(1-2) (2013), 201–213.
[19] M. Tărnăuceanu, *Classifying fuzzy subgroups for a class of finite p-groups*, Critical Review **7** (2013), 30–39.
[20] M. Tărnăuceanu, *A new equivalence relation to classify the fuzzy subgroups of finite groups*, Fuzzy Sets and Systems **289** (2016), 113–121. https://doi.org/10.1016/j.fss.2015.08.024
[21] A. C. Volf, *Counting fuzzy subgroups and chains of subgroups*, Fuzzy Systems and Artificial Intelligence **10** (2004), 191–200.
[22] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
[23] Y. Zhang and K. Zou, *A note on an equivalence relation on fuzzy subgroups*, Fuzzy Sets and Systems **95**(2) (1998), 243–247. https://doi.org/10.1016/S0165-0114(97)00185-1

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