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CERTAIN PROPERTIES OF APOSTOL-TYPE HERMITE-BASED-FROBENIUS-GENOCCHI POLYNOMIALS

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ABSTRACT. This paper is well designed to set-up some new identities related to generalized Apostol-type Hermite-based-Frobenius-Genocchi polynomials and by applying the generating functions, we derive some implicit summation formulae and symmetric identities. Further a relationship between Array-type polynomials, Apostol-type Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also established.

1. INTRODUCTION

Let $a, b, c \in \mathbb{R}^+$, $a \neq b$ and $x \in \mathbb{R}$. The generalized Apostol-Bernoulli, Euler and Genocchi polynomials with the parameters are given by means of the following generating function as follows (see [1–17]):

$$\left(\frac{t}{\lambda b^t - a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda; a, b, c)\frac{t^n}{n!}, \tag{1.1}$$

where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < 2\pi$,

$$\left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda; a, b, c)\frac{t^n}{n!}, \tag{1.2}$$

where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < \pi$, and

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda; a, b, c)\frac{t^n}{n!}, \tag{1.3}$$

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where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < \pi$.

It is clear from (1.1), (1.2) and (1.3) that $B_n^{(\alpha)}(x;\lambda;1,e,e) = B_n(x;\lambda)$, $E_n^{(\alpha)}(x;\lambda;1,e,e) = E_n(x;\lambda)$ and $G_n^{(\alpha)}(x;\lambda;1,e,e) = G_n(x;\lambda)$.

Recently, Kurt et al. [3] and Simsek (see [13, 14]) introduced the Apostol type Frobenius-Euler polynomials defined as follows.

Let $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x \in \mathbb{R}$. The generalized Apostol type Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u,a,b,c,\lambda)\frac{t^n}{n!}. \tag{1.4}$$

For $x = 0$ and $\alpha = 1$ in (1.4), we get

$$\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} H_n(u,a,b;\lambda)\frac{t^n}{n!},$$

where $H_n(u,a,b;\lambda)$ denotes the generalized Apostol type Frobenius-Euler numbers (see [14, 16, 17]).

On setting $a = 1$, $b = e$, $\lambda = 1$ in (1.4), the result reduces to

$$\left(\frac{1-u}{e^t - u}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u)\frac{t^n}{n!}, \quad \alpha \in \mathbb{Z},$$

where $H_n^{(\alpha)}(x;u)$ is called classical Frobenius-Euler polynomial of order α.

Observe that $H_n^{(1)}(x,u) = H_n(x,u)$ which denotes the Frobenius-Euler polynomials and $H_n^{(\alpha)}(0;u) = H_n^{(\alpha)}(u)$, which denotes the Frobenius-Euler numbers of order α. $H_n(x;-1) = E_n(x)$, which denotes the Euler polynomials, (see [7, 11, 15]).

Very recently, Yaşar and Özarslan [17] introduced Frobenius-Genocchi polynomials defined by means of the following generating relation:

$$\frac{(1-\lambda)t}{e^t - \lambda}e^{xt} = \sum_{n=0}^{\infty} G_n^F(x;\lambda)\frac{t^n}{n!}. \tag{1.5}$$

Taking $\lambda = -1$ in (1.5), we get Genocchi polynomials

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi.$$

Pathan and Khan [10] introduced the generalized Hermite-based Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ of two variables defined by

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!},$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al. [2, page 386, (1.6)] in the form

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}.$$

Definition 1.1. Let $c > 0$. The generalized 2-variable 1-parameter Hermite Kamp'e de Feriet polynomials $H_n(x, y; c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y; c)\frac{t^n}{n!}. \tag{1.6}$$

This is an extended 2-variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ defined by (see [5–7, 10])

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}.$$

Note that $H_n(x, y; e) = H_n(x, y)$. In order to collect the powers of t we expand the left hand side of (1.6) to the representation

$$H_n(x, y; c) = n!\sum_{j=0}^{[\frac{n}{2}]} \frac{(\ln c)^{n-2j} x^{n-2j} y^j}{j!(n-2j)!}. \tag{1.7}$$

Simsek [13] constructed the λ-Stirling type number of second kind $S(n, \nu; a, b; \lambda)$ by mean of the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{S}(n, \nu; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^{\nu}}{\nu!}, \tag{1.8}$$

and the generalized array type polynomials is defined by Simsek (see [13, page 6, (3.1)])

$$\sum_{n=0}^{\infty} \mathcal{S}_{\nu}^{n}(x; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^{\nu}}{\nu!} b^{xt}.$$

Kurt and Simsek [3] introduced the polynomial $Y_n(x; \lambda; a)$, which is given by the following generating function:

$$\frac{t}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x; \lambda; a)\frac{t^n}{n!}, \quad a \geq 1. \tag{1.9}$$

We also note that for $x = 0$, above equation gives a relation as $Y_n(0; \lambda; a) = Y_n(\lambda; a)$ (see [13, 14]). Again if we set $x = 0$ and $a = 1$ in (1.9), we get

$$\frac{t}{\lambda - 1} = \sum_{n=0}^{\infty} Y_n(0, \lambda; 1)\frac{t^n}{n!}.$$

The paper is organized as follows. In Section 2, we introduce generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ and their properties. In Section 3, we derive some implicit summation formulae for generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials. In Section 4, we give general symmetry identities by using different analytical means and applying generating functions and last Section 5, we find relation between λ-type Stirling polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials.

2. Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$

The intent of this section is to define the generalized Apostol-type Hermite-based-Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$ with suitable properties.

Definition 2.1. For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, the generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a.b.c; \lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^n}{n!}. \tag{2.1}$$

Remark 2.1. For $y = 0$ (2.1) reduces to

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!},$$

where $\mathcal{G}_n^{(\alpha)}(x; u; a, b, c; \lambda)$ is known as Apostol-type Frobenius Genocchi polynomials of order α (see [8]).

Remark 2.2. On setting $x = y = 0$ and $\alpha = 1$ in (2.1), we have

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right) = \sum_{n=0}^{\infty} \mathcal{G}_n(u; a, b; \lambda)\frac{t^n}{n!},$$

where $\mathcal{G}_n^{\alpha}(u; a.b.c; \lambda)$ denotes the generalized Apostol-type Frobenius-Genocchi numbers.

Remark 2.3. If we set $a = 1$, $b = c = e$, $u = -1$, then (2.1) immediately reduces to Hermite-based Genocchi polynomials (see [6, 7])

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} {}_HG_n^{(\alpha)}(x, y; \lambda), \quad |t| < \pi.$$

Now we give some properties of the generalized Apostol-type Hermite-based- Frobenius Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$, which are stated in terms of theorems as follows.

Theorem 2.1. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the following result holds true*

$$\begin{aligned}(2u-1)&\sum_{r=0}^{n}\binom{n}{r}{}_H\mathcal{G}_r(x, y; u; a, b, c; \lambda)\,\mathcal{G}_{n-r}(z; 1-u; a, b, c; \lambda)\\ =&n(u-1){}_H\mathcal{G}_{n-1}(x+z, y; u; a, b, c; \lambda) + nu\,{}_H\mathcal{G}_{n-1}(x+z, y; 1-u, a, b, c; \lambda)\\ &+\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\,{}_H\mathcal{G}_r(x+z, y; u; a, b, c; \lambda)\\ &-\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z, y; 1-u, a, b, c; \lambda).\end{aligned} \tag{2.2}$$

Proof. In order to prove (2.2), for $\alpha = 1$, we get

$$
\begin{aligned}
(2.3) \qquad &(2u-1)\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)c^{xt+yt^2}\left(\frac{(a^t-(1-u))t}{\lambda b^t-(1-u)}\right)c^{zt}\\
&=t^2(a^t-u)(a^t-(1-u))c^{(x+z)t+yt^2}\left[\frac{1}{\lambda b^t-u}-\frac{1}{\lambda b^t-(1-u)}\right].
\end{aligned}
$$

Employing the result of (2.1), (2.3) reduces as

$$
\begin{aligned}
(2.4) \qquad &(2u-1)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x,y;u;a,b,c;\lambda)\frac{t^r}{r!}\sum_{n=0}^{\infty}\mathcal{G}_n(z;1-u;a,b,c;\lambda)\frac{t^n}{n!}\\
&=(a^t-(1-u)t)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^r}{r!}-(a^t-u)t\\
&\quad\times\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^r}{r!}.
\end{aligned}
$$

Using [15, page 100, (1)] (2.4) reduces to

(2.5)

$$
\begin{aligned}
&(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}{}_H\mathcal{G}_r(x,y;u;a,b,c;\lambda)\,{}_H\mathcal{G}_{n-r}(z,y;1-u;a,b,c;\lambda)\frac{t^n}{n!}\\
&=(a^t-(1-u)t)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^r}{r!}-(a^t-u)t\\
&\quad\times\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^r}{r!}\\
&=(u-1)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^{r+1}}{r!}+u\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u,a,b,c;\lambda)\frac{t^{r+1}}{r!}\\
&\quad+\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z,y;u;a,b,c;\lambda)\frac{t^n}{n!}\\
&\quad-\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^n}{n!}.
\end{aligned}
$$

On comparing the coefficient of t^n from the above equation, we arrive at our desired result. $\square$

Theorem 2.2. *For $a,b,c\in\mathbb{R}^+$, $a\neq b$, $x,y\in\mathbb{R}$, $\lambda\in\mathbb{C}$, $k\in\mathbb{N}$, $\alpha\in\mathbb{Z}$, the following relationship holds true*

$$
(2.6) \qquad \sum_{k=0}^{n}{}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\,{}_H\mathcal{G}_{(n-k)}^{(\alpha-m)}(x,y;u;a.b.c;\lambda)=\mathcal{G}_n^{(-m)}(u;a,b;\lambda).
$$

Proof. In order to prove (2.6), replacing x with $-x$, y with $-y$ and α with $-\alpha$ in (2.1), we get get

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\alpha)} c^{-(xt+yt^2)}. \tag{2.7}$$

Making use of the above equation in the left-hand side of (2.6), we can write

$$\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha-m)}(x,y;u;a,b,c;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{-m}.$$

We can write the above equation as

$$\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha-m)}(x,y;u;a,b,c;\lambda)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty} \mathcal{G}_n^{(-m)}(u;a,b;\lambda)\frac{t^n}{n!}.$$

Using [15, page 100, (1)] in the above equation and then comparing the coefficients of t^n, we immediately come to our desired result (2.6). □

Theorem 2.3. *For $n \geq 0$, $p,q \in \mathbb{R}$, the following formula for generalized Apostol type Frobenius-Genocchi-Hermite polynomials holds true*

$${}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda) = \sum_{k=0}^{n}\frac{n!}{(n-k)!}{}_H\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)$$
$$\times\sum_{j=0}^{[\frac{k}{2}]}\frac{((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^j}{(k-2j)!j!}.$$

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!}$$
$$=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt+yt^2}c^{(p-1)xt}c^{(q-1)yt^2}$$
$$=\left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x\ln c)^k\frac{t^k}{k!}\right)\times\left(\sum_{j=0}^{\infty}((q-1)y\ln c)^j\frac{t^{2j}}{j!}\right)$$
$$=\left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x\ln c)^k((q-1)y\ln c)^j\frac{t^{k+2j}}{k!j!}\right).$$

Replacing k by $k-2j$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)$$

$$\times\left(\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]}((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}\frac{t^k}{(k-2j)!j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]}{}_{H}\mathcal{G}_{n}^{(\alpha)}(x,y;u,a,b,c;\lambda)((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}\frac{t^{n+k}}{(k-2j)!j!n!}.$$

Again replacing n by $n-k$ in above equation, we have

$$\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}$$
$$\times\frac{t^n}{(k-2j)!j!(n-k)!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result. $\square$

Remark 2.4. By taking $c=e$ in Theorem 2.3, we get the following corollary.

Corollary 2.1. *For $p,q\in\mathbb{R}$, $x,y\in\mathbb{C}$ and $n\geq 0$, we have*

$${}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b;\lambda)$$
$$=\sum_{k=0}^{n}\frac{n!}{(n-k)!}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b;\lambda)\sum_{j=0}^{[\frac{k}{2}]}\frac{((p-1)x)^{k-2j}((q-1)y)^{j}}{(k-2j)!j!}.$$

Theorem 2.4. *For $n\geq 0$, $p,q\in\mathbb{R}$ and $x,y\in\mathbb{C}$, we have*

$$\begin{aligned}&{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)H_k((p-1)x,(q-1)y;c).\end{aligned}\tag{2.8}$$

Proof. In order to proof above result, we set x as px and y as qy in (2.1),

$$\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha}c^{xt+yt^2}c^{(p-1)xt}c^{(q-1)yt^2}$$
$$=\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}H_k((p-1)x,(q-1)y;c)\frac{t^k}{k!}.$$

By assistance of [15] and then on comparing the coefficients of t^n, we have arrive at our result. $\square$

Theorem 2.5. *For $n\geq 0$, $p,q\in\mathbb{R}$ and $x,y\in\mathbb{C}$, we have*

$${}_{H}\mathcal{G}_{n}^{(\alpha+\beta)}(x+z,y+z;u,a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}{}_{H}\mathcal{G}_{k}^{(\alpha)}(x,z;u;a,b,c;\lambda)$$

$$\times {}_H\mathcal{G}_{n-k}^{(\beta)}(z, y; u; a, b, c; \lambda),$$

$$ {}_H\mathcal{G}_n^{(-\alpha)}(2x, 2y; u^2; a^2, b^2, c^2; \lambda^2) = \sum_{k=0}^{n} \binom{n}{k} {}_H\mathcal{G}_k^{(-\alpha)}(x, y; u; a, b, c; \lambda) \times {}_H H_{n-k}^{(-\alpha)}(x, y; -u; a, b, c; \lambda).$$

Proof. Proof of these identities can be solved by making use of (2.1) and (1.5) with some required calculations. □

3. Summation Formulae for Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomials

Here in this section, we provide the implicit formulae for generalized Apostol-type Hermite-based-Frobinis-Genocchi polynomials.

Theorem 3.1. *For $a, b, c \in \mathbb{R}^+, a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the following relation holds true*

$$ {}_H\mathcal{G}_{k+l}^{(\alpha)}(z, y; u; a, b, c; \lambda) = \sum_{n,m=0}^{k,l} \binom{l}{m}\binom{k}{n} (z-x)^{m+n} (\ln c)^{m+n} \times {}_H\mathcal{G}_{k-n+l-m}^{(\alpha)}(x, y; u; a, b, c; \lambda). \tag{3.1}$$

Proof. Replacing t by $t + w$ in (2.1) and then using ([15], page 52, (2)), in the above equation, we get

$$\left(\frac{(a^{(t+w)} - u)(t + w)}{\lambda b^{t+w} - u}\right)^{\alpha} c^y (t + w)^2 = c^{-x(t+w)} \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.2}$$

Replacing x by z and then equating the obtained equation from the above equation (3.2), we get

$$c^{(z-x)(t+w)} \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{(\alpha)}(x; u; a, b, c; \lambda)) \frac{t^k}{k!}\frac{w^l}{l!} = \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{(\alpha)}(z, y; u; a, b, c; \lambda)) \frac{t^k}{k!}\frac{w^l}{l!}.$$

Expanding the exponent part of left-hand side, the above equation converts as

$$\sum_{N=0}^{\infty} \frac{(\ln c)[(z-x)(t+w)]^N}{N!} \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{(\alpha)}(x, y; u; a, b, c; \lambda)) \frac{t^k}{k!}\frac{w^l}{l!} = \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{(\alpha)}(z, y; u; a, b, c; \lambda)) \frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.3}$$

On comparing the coefficients of equal powers of t and w after taking the reference of [15, page 52, (2) and page 100, (1)] to the above equation, we attain our required result. □

Corollary 3.1. *For $l = 0$, the above result reduces to*

$$ {}_H\mathcal{G}_k^{(\alpha)}(z, y; u; a, b, c; \lambda) = \sum_{n=0}^{k} \binom{k}{n} (z-x)^n (\ln c)^n {}_H\mathcal{G}_{k-n}^{(\alpha)}(x, y; u; a, b, c; \lambda).$$

Theorem 3.2. *For $a, b, c \in \mathbb{R}^+, a \neq b,\ x, y \in \mathbb{R},\ \lambda \in \mathbb{C},\ k \in \mathbb{N},\ \alpha \in \mathbb{Z},\ n \geq 0,$ the following relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}^{(\alpha)}(u; a, b; \lambda)\, H_m(x, y; c). $$

Proof. From equation (2.1) and (1.7), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!} &= \left(\frac{(a^t - u)t}{\lambda b^t - u} \right)^{\alpha} c^{xt+yt^2} \\ &= \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(u; a, b) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x, y; c) \frac{t^m}{m!}. \end{aligned} $$

On using [15, page 100, (1)], and then comparing the coefficient of equal powers, we have the required result. □

Theorem 3.3. *For $a, b, c \in \mathbb{R}^+,\ a \neq b,\ x, y \in \mathbb{R},\ \lambda \in \mathbb{C},\ k \in \mathbb{N},\ \alpha \in \mathbb{Z},$ the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} (\ln c)^{n-m}\, {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda). $$

Proof. Replacing x by $x + 1$, (2.1) reduces to

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda) \frac{t^n}{n!} &= \left(\frac{(a^t - u)t}{\lambda b^t - u} \right)^{\alpha} c^{(x+1)t+yt^2} \\ &= \left(\frac{(a^t - u)t}{\lambda b^t - u} \right)^{\alpha} c^{(xt+yt^2)} c^t \\ &= \sum_{m=0}^{\infty} {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^m}{m!} \sum_{n=0}^{\infty} \frac{(\ln c)^n\, t^n}{n!}. \end{aligned} $$

Using [15, page 100, (1)] and on comparing coefficient of t^n, we have the required result. □

Theorem 3.4. *For $a, b, c \in \mathbb{R}^+,\ a \neq b,\ x, y \in \mathbb{R},\ \lambda \in C,\ k \in \mathbb{N},\ \alpha \in \mathbb{Z},$ the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha+1)}(x, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(u; a, b; \lambda)\, {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b; \lambda). $$

Proof. Replacing α by $\alpha + 1$ in (2.1), we have

$$ \begin{aligned} \left(\frac{(a^t - u)t}{\lambda b^t - u} \right)^{\alpha+1} c^{xt+yt^2} &= \left(\frac{(a^t - u)t}{\lambda b^t - u} \right) \left(\frac{(a^t - u)t}{\lambda b^t - u} \right)^{\alpha} c^{xt+yt^2} \\ &= \sum_{n=0}^{\infty} \mathcal{G}_n(u; a, b; \lambda) \frac{t^n}{n!} \sum_{m=0}^{\infty} {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^m}{m!}. \end{aligned} $$

Making use of [15, page 100, (1)] and then on comparing coefficient of t^n, we lead to our required result. □

Theorem 3.5. *For $a,b,c \in \mathbb{R}^+$, $a \neq b$, $x,y \in \mathbb{R}$, $\lambda \in C$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(y,x;u;a,b,c;\lambda) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!\,(n-2k)!} \mathcal{G}_{n-2k}^{(\alpha)}(y,u;a,b,c;\lambda)(x\ln c)^k. $$

Proof. Interchanging x and y in (2.1), we have

$$ \begin{aligned} \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{yt+xt^2} &= \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(y,x;u;a,b,c;\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(y;u;a,b,c;\lambda)\frac{t^n}{n!} \sum_{k=0}^{\infty}(x\ln c)^k \frac{t^{2k}}{k!}. \end{aligned} $$

Making use of [15, page 100, (3))] and then on comparing coefficient of t^n, we lead to our required result. □

4. Symmetric Identities

In this section, we establish symmetric identities for generalized Apostol type Hermite-based Frobenius-Genocchi polynomials by applying the generating function (2.1). Such type of identities have been introduced by many authors namely Khan [6], Khan et al. [5,7] and Pathan and Khan [10–12].

Theorem 4.1. *Let $a,b,c > 0$, $a \neq b$, $x,y \in \mathbb{R}$ and $n \geq 0$, the following relation holds true*

$$ \begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H\mathcal{G}_{n-k}^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda) {}_H\mathcal{G}_k^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda) \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \mathcal{G}_{n-k}^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda) {}_H\mathcal{G}_k^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda). \end{aligned} \tag{4.1} $$

Proof. In order to proof (4.1), we suppose a function $H(t)$ as

$$ H(t) = \left[\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)\right]^{\alpha} c^{2(abxt+a^2b^2yt^2)}. $$

The above expression is symmetric in a and b hence we can write above equation into two ways as follows:

(4.2)
$$ \begin{aligned} H(t) &= \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda)\frac{(at)^n}{n!} \sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda)\frac{(bt)^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H\mathcal{G}_{n-k}^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda) {}_H\mathcal{G}_k^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda)\frac{t^n}{n!}. \end{aligned} $$

Again we can write

(4.3)
$$H(t)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_k^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda)\frac{(at)^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda)\frac{t^n}{n!}.$$

Comparing (4.2) and (4.3), we arrive at our desired result. □

Corollary 4.1. *For $\alpha=1$ in Theorem 4.1, we have the following symmetric identity:*
$$\sum_{k=0}^{n}\binom{n}{k}b^ka^{n-k}{}_H\mathcal{G}_{n-k}(bx,b^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k(ax,a^2y;u;A,B,c;\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}(ax,a^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k(bx,b^2y;u;A,B,c;\lambda).$$

Theorem 4.2. *Let $a,b,c>0$, $a\neq b$, $x,y\in\mathbb{R}$ and $n\geq 0$, the following relation holds true:*
$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{(i+j)}b^ka^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j,b^2y;u;A,B,c;\lambda\right)$$
$$\times\ \mathcal{G}_k^{(\alpha)}(az,0;u;A,B,c;\lambda)$$
$$=\sum_{k=0}^{n}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{(i+j)}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2y;u;A,B,c;\lambda\right)$$
$$\times\ \mathcal{G}_k^{(\alpha)}(bz,0;u;A,B,c;\lambda).$$

Proof. In order to prove above result, we suppose $I(t)$ is
$$I(t)=\left[\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)\right]^{\alpha}\frac{(1+\lambda(-1)^{a+1}c^{abt})^2}{(\lambda c^{at}+1)(\lambda c^{bt}+1)}c^{ab(x+z)t+a^2b^2yt^2}$$
$$=\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)^{\alpha}c^{abxt+a^2b^2yt^2}\sum_{i=0}^{a-1}(-\lambda)^ic^{ibt}\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)^{\alpha}c^{abzt}\sum_{j=0}^{b-1}(-\lambda)^jc^{jat}.$$

Using [15, page 100, (1)] we have
$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}a^{n-k}b^k{}_H\mathcal{G}_{n-k}^{(\alpha)}(bx+\frac{b}{a}i+j,b^2y;u;A,B,c;\lambda)$$
$$\times\ \mathcal{G}_k^{(\alpha)}(az;u;A,B,c;\lambda)\frac{t^n}{n!}.$$

On the other hand, we have
$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}b^{n-k}a^k{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2y;u;A,B,c;\lambda\right)$$

$$\times\, \mathcal{G}_k^{(\alpha)}\left(bz;u;A,B,c;\lambda\right)\frac{t^n}{n!}.$$

On comparing both the results, we have the required relation. ☐

5. Relation Between λ-Type Striling Numbers of Second Kind, Apostol-Bernoulli Polynomial and Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomial

This section deals with some relationships in between Array-type polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Hermite-based Frobenius-Genocchi polynomial.

Theorem 5.1. *For $a,b,c\in\mathbb{R}^+$, $a\neq b$, $x,y\in\mathbb{R}$, $\lambda\in\mathbb{C}$, $k\in\mathbb{N}$, $\alpha\in\mathbb{Z}$ and ν be an integer, then we have*

$$ {}_H\mathcal{G}_{n-2\nu}^{(-\nu)}(x,y;u;a,b,b;\lambda)=\frac{(\nu)!}{(-n)_{2\nu}}\sum_{k=0}^{n}\sum_{m=0}^{l}\binom{m}{k}\binom{n}{m}S\left(k,v,1,b;\frac{\lambda}{u}\right)\times Y_{m-k}^{(\nu)}\left(\frac{1}{u};a\right)H_{l-m}(x,y). \tag{5.1}$$

Proof. In order to proof above result, we replace of c with b and α with $-\nu$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\nu)}b^{xt+yt^2}.$$

On arranging the above equation, we arrive at

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!}=(\nu!)\frac{\left(\frac{\lambda}{u}b^t-1\right)^{\nu}b^{xt+yt^2}}{(\nu!)\left(\frac{a^t}{u}-1\right)^{\nu}t^{\nu}}\frac{t^{\nu}}{t^{\nu}}.$$

By assistance of (1.8) and (1.9), above equation reduces to

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!}=(\nu!)\sum_{k=0}^{\infty}S\left(n,v,1,b;\frac{\lambda}{u}\right)\frac{t^k}{k!}\times\sum_{m=0}^{\infty}Y_m^{(\nu)}\left(\frac{1}{u},1;a\right)\frac{t^m}{m!}\sum_{l=0}^{\infty}H_l(x,y;b)\frac{t^l}{l!}. \tag{5.2}$$

Using Lemma [15, page 100, (1)] we get

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!}=\nu!\sum_{l=0}^{\infty}\sum_{k=0}^{m}\sum_{m=0}^{l}\binom{m}{k}\binom{l}{m}S\left(k,v,1,b;\frac{\lambda}{u}\right)\times Y_{m-k}^{(\nu)}\left(\frac{1}{u},1;a\right)H_{l-m}(x,y;b)\frac{t^l}{l!}.$$

Using [15, page 23, (22) and (23)] and replacing l by n, and then by comparing the coefficients of t^n we arrive at our required result. ☐

Theorem 5.2. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and ν be an integer, we have*

$$ {}_H\mathcal{G}_{n-2\nu}^{(-\nu)}(x,y;u;a,b,b;\lambda) = \frac{(\nu)!}{(-n)_{2\nu}} \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}\left(k,\nu,1,b,\frac{\lambda}{u}\right) \times {}_H\mathcal{B}_{n-k}^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right). $$

Proof. Making replacement of c with b and α with $-\nu$ in (2.1), we get

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t - u}\right)^{(-\nu)} b^{xt+yt^2}. $$

On arranging the above equation, we arrive at

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = (\nu!)\frac{\left(\frac{\lambda}{u}b^t-1\right)^{\nu} b^{xt+yt^2}}{(\nu!)\left(\frac{a^t}{u}-1\right)^{\nu} t^{\nu}}\frac{t^{\nu}}{t^{\nu}}. $$

Using (1.8) and (1.1), the above equation converts into

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!} =& (\nu!)\sum_{k=0}^{\infty} \mathcal{S}\left(k,\nu,1,b;\frac{\lambda}{u}\right)\frac{t^k}{k!} \\ & \times \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right)\frac{t^n}{n!}. \end{aligned} $$

Using [15, page 100, (1)] right-hand side, it converts as follows

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!} =& \nu! \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mathcal{S}\left(k,\nu,1,b,\frac{\lambda}{u}\right) \\ & \times {}_H\mathcal{B}_{n-k}^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right)\frac{t^n}{n!!}. \end{aligned} $$

Using [15, page 23, (22) and (23)] and replacing l with n, then by comparing the coefficients of t^n, we arrive at our required result. □

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