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INEQUALITIES FOR STRONGLY r-CONVEX FUNCTIONS ON TIME SCALES

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Abstract. In this paper, first we establish the Hermite-Hadamard type inequality based on diamond-α integral for a subset of strongly r-convex functions. Then we prove several new inequalities for n-times continuously differentiable strongly r-convex functions on time scales by virtue of some techniques and introducing new quantities.

1. Introduction

The analysis on time scales is a relatively new area of mathematics that unifies and generalizes discrete and continuous theories. Moreover, it is a crucial tool in many computational and numerical applications.

The differential calculus on time scales generalizes classical both continuous and discrete differential calculus depending on the structure of the time scale under consideration. There are several common generalizations of classical derivative to time scales. For example, one of them is the so-called Δ-derivative, which simultaneously generalizes the forward divided difference of the first order, while the first-order backward divided difference is generalized by the ∇-derivative. There is also the so-called diamond-α dynamic derivative or, shortly, $\Diamond_\alpha$ -derivative being, in turn, the linear combination of Δ and ∇-derivatives with the coefficients α and $1-\alpha$, respectively, for some $\alpha \in [0,1]$. For each type of derivatives on time scales there is its own notion of the integral. Thus, the diamond-α integral corresponds to the $\Diamond_\alpha$-derivative.

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The main purpose of this discussions is to reflect some certain inequalities for strongly r-convex functions and it is inspired by the papers [3,7–10] where the authors focused on to obtain several new integral inequalities for different class of convex functions which are n-times differentiable on an interval in $\mathbb{R}$. Since many continuous models in biology, physics, chemistry and etc. have discrete analogues, our aim in this paper is to unify these inequalities in the discrete and continuous case.

This paper is organized as follows. In the next section, we briefly recall key notions and notations on time scales and then we introduce diamond-α derivatives by recalling the basic property of this combined dynamic derivatives. We also present definition of diamond-α integral and several theorems concerning the properties of it. In Section 3, which is devoted to our main results, we deduce some integral inequalities by applying the definition of strongly r-convexity and the integral identity which we prove in the sequel. We also introduce some new quantities and use well-known inequalities to present our results.

2. Time Scales Revisited

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The forward jump and backward jump operators σ and ρ can be defined respectively by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Note that for any $t \in \mathbb{T}$, $\sigma(t) \geq t$ and $\rho(t) \leq t$. Moreover, for $t \in \mathbb{T}$, we say the graininess function $\mu : \mathbb{T} \to [0, +\infty)$ to be as follows

$$\mu(t) = \sigma(t) - t.$$

We define the interval $[a, b]_{\mathbb{T}}$ in $\mathbb{T}$ as follows

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half open intervals etc. are defined accordingly.

For $t \in \mathbb{T}$, we have the following cases.

- If $\sigma(t) > t$, then we say that t is right-scattered.
- If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.
- If $\rho(t) < t$, then we say that t is left-scattered.
- If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.

We define $\mathbb{T}^k = \mathbb{T}$ if $\sup \mathbb{T}$ is left-dense and $\mathbb{T}^k = \mathbb{T} \setminus \{\sup \mathbb{T}\}$ if $\sup \mathbb{T}$ is left-scattered. Similarly, we define $\mathbb{T}_k = \mathbb{T}$ if $\inf \mathbb{T}$ is right-dense and $\mathbb{T}_k = \mathbb{T} \setminus \{\inf \mathbb{T}\}$ if $\inf \mathbb{T}$ is right-scattered. We denote $\mathbb{T}^k \cap \mathbb{T}_k = \mathbb{T}^k_k$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. We define $f^{\Delta}(t)$ to be a number, provided it exists, as follows: for any $\epsilon > 0$ there is a neighbourhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We say $f^{\Delta}(t)$ is the delta or Hilger derivative of f at t. Also, we say f is delta differentiable on $\mathbb{T}^k$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. Similarly, we say that a function f defined on $\mathbb{T}$ is ∇ differentiable at $t \in \mathbb{T}_k$ if for $\varepsilon > 0$ there is a neighborhood V of t such that for some γ the following inequality holds:

$$|f(\rho(t)) - f(s) - \gamma(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in V$ and in this case, we write $f^{\nabla}(t) = \gamma$. We say that f is ∇ differentiable on $\mathbb{T}_k$ if $f^{\nabla}(t)$ exists for any $t \in \mathbb{T}_k$.

Definition 2.1 ([12, 13])**.** Let $\alpha \in [0, 1]$ and $f : \mathbb{T} \to \mathbb{R}$ be Δ and ∇ differentiable at $t \in \mathbb{T}$. Define the diamond-α dynamic derivative $f^{\Diamond_\alpha}$ of f at t as follows

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha)f^{\nabla}(t).$$

Thus, f is diamond-α-differentiable at $t \in \mathbb{T}$ if and only if f is Δ and ∇ differentiable at t. When $\alpha = 1$, we have

$$f\Diamond_\alpha(t) = f^{\Delta}(t)$$

and for $\alpha = 0$, we have

$$f\Diamond_\alpha(t) = f^{\nabla}(t).$$

In [12], they proved the following criteria for $\diamond_\alpha$-differentiability of a function.

Theorem 2.1 ([12])**.** *Let* $\alpha \in [0, 1]$.

(a) *If* $t \in \mathbb{T}$ *is dense and* $f'(t)$ *exists, then*

$$f^{\diamond_\alpha}(t) = f^{\Delta}(t) = f^{\nabla}(t) = f'(t).$$

(b) *If* $t \in \mathbb{T}$ *is isolated, then* $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1 - \alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

(c) *If* $t \in \mathbb{T}$ *is left-scattered and right-dense, and*

$$f'(t^+) = \lim_{h \to 0+} \frac{f(t + h) - f(t)}{h}$$

exists, then $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \alpha f'(t^+) + (1 - \alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

(d) *If* $t \in I$ *is right-scattered and left-dense, and*

$$f'(t^-) = \lim_{h \to 0-} \frac{f(t + h) - f(t)}{h}$$

exists, then $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \alpha\frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1 - \alpha)f'(t^-).$$

Below we will list some of the properties of the diamond-α derivative. Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$.

Theorem 2.2 ([12,13])**.** *$f+g$ is diamond-α differentiable at t and*

$$(f+g)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) + g^{\Diamond_\alpha}(t).$$

Theorem 2.3 ([12,13])**.** *For any $c \in \mathbb{C}$, we have cf is diamond-α differentiable at t and*

$$(cf)^{\Diamond_\alpha}(t) = cf^{\Diamond_\alpha}(t).$$

Theorem 2.4 ([12,13])**.** *fg is diamond-α differentiable at t and*

$$\begin{aligned}(fg)^{\Diamond_\alpha}(t) &= f^{\Diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t)\\ &= f(t)g^{\Diamond_\alpha}(t) + \alpha f^\Delta(t)g^\sigma(t) + (1-\alpha)f^\nabla(t)g^\rho(t).\end{aligned}$$

Definition 2.2 ([2])**.** Let $a, t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ derivative of f provided that $F^\Delta(t) = f(t)$ holds for $t \in \mathbb{T}$. We define the Δ integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

Let $g : \mathbb{T} \to \mathbb{R}$. A function $G : \mathbb{T} \to \mathbb{R}$ is called a ∇ derivative of g provided that $G^\nabla(t) = g(t)$ holds for $t \in \mathbb{T}$. We define the ∇ integral of g by

$$\int_a^t g(s)\nabla s = G(t) - G(a), \quad t \in \mathbb{T}.$$

Definition 2.3 ([12,13])**.** Let $\alpha \in [0,1]$, $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Define diamond-$\alpha$ integral of h as follows

$$\int_a^t h(s)\Diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s.$$

Remark 2.1. Note that

$$\begin{aligned}\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Diamond_\alpha} =& \alpha\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Delta} + (1-\alpha)\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\nabla}\\ =& \alpha\left(\alpha\int_a^t f(s)\Delta s + (1-\alpha)\int_a^t f(s)\nabla s\right)^{\Delta}\\ &+ (1-\alpha)\left(\alpha\int_a^t f(s)\Delta s + (1-\alpha)\int_a^t f(s)\nabla s\right)^{\nabla}\\ =& \alpha^2 f(t) + \alpha(1-\alpha)f(\sigma(t)) + \alpha(1-\alpha)f(\rho(t)) + (1-\alpha)^2 f(t)\\ =& \left(2\alpha^2 - 2\alpha + 1\right)f(t) + \alpha(1-\alpha)(f(\sigma(t)) + f(\rho(t))), \quad t \in \mathbb{T}.\end{aligned}$$

Thus, in the general case we do not have

$$\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Diamond_\alpha} = f(t).$$

In [11], they proved the following criteria for $\diamond_\alpha$-integrability of a function.

Theorem 2.5 ([11])**.** (a) *Every monotone function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a, b]_{\mathbb{T}}$ *is* $\diamond_\alpha$-*integrable on* $[a, b]_{\mathbb{T}}$.

(b) *Every continuous function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a, b]_{\mathbb{T}}$ *is* $\diamond_\alpha$-*integrable on* $[a, b]_{\mathbb{T}}$.

(c) *Every regulated function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a, b]_{\mathbb{T}}$ *is* $\diamond_\alpha$-*integrable on* $[a, b]_{\mathbb{T}}$.

Below we suppose that $f, g : \mathbb{T} \to \mathbb{R}$ are diamond-α integrable over $[a, b]_{\mathbb{T}}$.

Theorem 2.6 ([12, 13])**.** *For any* $c \in \mathbb{C}$, *the function* cf *is diamond-*α *integrable over* $[a, b]_{\mathbb{T}}$ *and*

$$\int_a^b (cf)(s)\Diamond_\alpha s = c\int_a^b f(s)\Diamond_\alpha s.$$

Theorem 2.7 ([12, 13])**.** $f + g$ *is diamond-*α *integrable over* $[a, b]_{\mathbb{T}}$ *and*

$$\int_a^b (f + g)(s)\Diamond_\alpha s = \int_a^b f(s)\Diamond_\alpha s + \int_a^b g(s)\Diamond_\alpha s.$$

Theorem 2.8 ([12, 13])**.** *We have*

$$\int_a^b f(s)\Diamond_\alpha s = \int_a^t f(s)\Diamond_\alpha s + \int_t^b f(s)\Diamond_\alpha s,$$

for any $t \in [a, b]_{\mathbb{T}}$.

For $a, b \in \mathbb{T}$, $a < b$, denote

$$\begin{aligned}
x_\alpha &= \frac{1}{b-a}\int_a^b t \diamond_\alpha t,\\
x_{\alpha,\alpha} &= \frac{1}{b-a}\int_a^b t^2 \diamond_\alpha t,\\
x_{\alpha,r,-} &= \left(\frac{1}{b-a}\int_a^b (b-t)^{\frac{1}{r}} \diamond_\alpha t\right)^r,\\
x_{\alpha,r,+} &= \left(\frac{1}{b-a}\int_a^b (t-a)^{\frac{1}{r}} \diamond_\alpha t\right)^r,\\
h_0(x, a) &= 1,\\
h_k(x, a) &= \int_a^x h_{k-1}(\tau, a)\Delta\tau, \quad k \in \mathbb{N}, x \in [a, b]_{\mathbb{T}}.
\end{aligned}$$

We have

$$h_n(x, a) \le \frac{(x-a)^n}{n!}, \quad n \in \mathbb{N}, x \in [a, b]_{\mathbb{T}}.$$

Definition 2.4 ([2])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{C}_{rd}(\mathbb{T})$.

The set of functions $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ that are n-times rd-continuously Δ-differentiable on $[a, b]$ is denoted by $\mathcal{C}^n_{rd}([a, b]_{\mathbb{T}})$.

For some of our main results we will use Taylor's formula.

Theorem 2.9 ([2], Taylor's formula). *Let* $f \in \mathcal{C}_{rd}^{n+1}([a,b]_{\mathbb{T}})$. *Then*

$$f(x) = \sum_{k=0}^{n} h_k(x,a) f^{\Delta^k}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) f^{\Delta^{n+1}}(\tau)\Delta\tau, \quad x \in [a,b]_{\mathbb{T}}.$$

We also need the following well-known inequality for proving our results.

Theorem 2.10 ([4], Hölder's inequality). *Let* $a, b \in \mathbb{T}$, $a < b$. *For rd-continuous functions* $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *we have*

$$\int_a^b |f(t)g(t)|\Delta\tau \leq \left(\int_a^b |f(t)|^p \Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Delta t\right)^{\frac{1}{q}},$$

where $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main Results

In this section, we attempt to establish several new inequalities for strongly r-convex functions on time scales by virtue of some notions and results and by introducing new quantities.

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds, for all $x, y \in I$ and $\lambda \in [0,1]$. Also, for the convex function $f : [a,b] \to \mathbb{R}$ the following inequality is known as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Definition 3.1. Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \to \mathbb{R}$ is called strongly r-convex function with modulus c, if

$$f(\lambda x + (1-\lambda)y) \leq (\lambda (f(x))^r + (1-\lambda)(f(y))^r)^{\frac{1}{r}} - c\lambda(1-\lambda)(x-y)^2,$$

for any pair of $x, y \in I$, $t \in [0,1]$ and $r \neq 0$. If we take $c = 0$, we have the definition of r-convexity of the function f.

We can extend the above definition on any time scale $\mathbb{T}$. From now on, we suppose that $[a,b]_{\mathbb{T}}$ is an interval in $\mathbb{T}$. Note that, if $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is positive strongly r-convex function with modulus c, we have

$$f(t) \leq \left(\frac{b-t}{b-a}(f(a))^r + \frac{t-a}{b-a}(f(b))^r\right)^{\frac{1}{r}} - c(b-t)(t-a), \quad t \in [a,b]_{\mathbb{T}}. \tag{3.1}$$

Now we are in a position to present our first result.

Theorem 3.1. *Suppose that* $0 < r \leq 1$ *and* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is a positive strongly* r*-convex function and* $\diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$. *Then*

$$\frac{1}{b-a}\int_a^b f(t) \diamond_\alpha t \leq \frac{1}{(b-a)^{\frac{1}{r}}}\Big(x_{\alpha,r,-}\big(f(a)\big)^r + x_{\alpha,r,+}(f(b))^r\Big)^{\frac{1}{r}}$$

$$+ c\left(ab - (a+b)x_\alpha + x_{\alpha,\alpha}\right).$$

Proof. By taking the diamond-α integral side by side in (3.1), we have

$$\begin{aligned}
\int_a^b f(t)\diamond_\alpha t \leq& \int_a^b \left(\frac{b-t}{b-a}(f(a))^r + \frac{t-a}{b-a}(f(b))^r\right)^{\frac{1}{r}}\diamond_\alpha t \\
&- c\int_a^b (b-t)(t-a)\diamond_\alpha t \\
=& \left(\int_a^b \left(\left(\left(\frac{b-t}{b-a}\right)^{\frac{1}{r}} f(a)\right)^r + \left(\left(\frac{t-a}{b-a}\right)^{\frac{1}{r}} f(b)\right)^r\right)^{\frac{1}{r}}\diamond_\alpha t\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a)\diamond_\alpha t.
\end{aligned}$$

Now, by applying Minkowski's inequality, we find

$$\begin{aligned}
\int_a^b f(t)\diamond_\alpha t \leq& \left(\left(\int_a^b \left(\frac{b-t}{b-a}\right)^{\frac{1}{r}} f(a)\diamond_\alpha t\right)^r + \left(\int_a^b \left(\frac{t-a}{b-a}\right)^{\frac{1}{r}} f(b)\diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a)\diamond_\alpha t \\
=& \left((f(a))^r\left(\int_a^b \left(\frac{b-t}{b-a}\right)^{\frac{1}{r}}\diamond_\alpha t\right)^r + (f(b))^r\left(\int_a^b \left(\frac{t-a}{b-a}\right)^{\frac{1}{r}}\diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a)\diamond_\alpha t \\
=& \frac{1}{(b-a)^{\frac{1}{r}}}\left((f(a))^r\left(\int_a^b (b-t)^{\frac{1}{r}}\diamond_\alpha t\right)^r + (f(b))^r\left(\int_a^b (t-a)^{\frac{1}{r}}\diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a)\diamond_\alpha t \\
=& \frac{b-a}{(b-a)^{\frac{1}{r}}}\left(x_{\alpha,r,-}(f(a))^r + x_{\alpha,r,+}(f(b))^r\right)^{\frac{1}{r}} - c\int_a^b \left(bt - ab - t^2 + at\right)\diamond_\alpha t \\
=& \frac{b-a}{(b-a)^{\frac{1}{r}}}\left(x_{\alpha,r,-}(f(a))^r + x_{\alpha,r,+}(f(b))^r\right)^{\frac{1}{r}} \\
&- (b-a)c\left(-ab + (a+b)x_\alpha - x_{\alpha,\alpha}\right),
\end{aligned}$$

whereupon we obtain the desired inequality. This completes the proof. □

Before we establish the next result, we provide the following integral identity.

Lemma 3.1. *Let* $f \in \mathcal{C}_{rd}^n([a,b]_{\mathbb{T}})$. *Then*

$$\int_a^b f(s)\Delta s = \sum_{k=1}^n h_k(b,a) f^{\Delta^{k-1}}(a) + \int_a^{\rho^n(b)} h_n(b,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau. \tag{3.2}$$

Proof. Let

$$g(x) = \int_a^x f(s)\Delta s, \quad x \in [a,b]_{\mathbb{T}}.$$

By Taylor's formula, we have

$$g(x) = \sum_{k=0}^{n} h_k(x,a) g^{\Delta^k}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) g^{\Delta^{n+1}}(\tau)\Delta\tau,$$

for $x \in [a,b]_{\mathbb{T}}$. Hence,

$$\int_a^x f(s)\Delta s = \sum_{k=1}^{n} h_k(x,a) f^{\Delta^{k-1}}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau,$$

for $x \in [a,b]_{\mathbb{T}}$. By the last equality, for $x = b$, we find (3.2). This completes the proof. □

Set

$$I(a,b,n,f) = \int_a^b f(s)\Delta s - \sum_{k=1}^{n} h_k(b,a) f^{\Delta^{k-1}}(a),$$

$$y_\alpha = \frac{1}{b-a}\int_a^b t\Delta t,$$

$$y_{\alpha,\alpha} = \frac{1}{b-a}\int_a^b t^2\Delta t,$$

$$y_{\alpha,r,-} = \left(\frac{1}{b-a}\int_a^b (b-t)^{\frac{1}{r}}\Delta t\right)^r,$$

$$y_{\alpha,r,+} = \left(\frac{1}{b-a}\int_a^b (t-a)^{\frac{1}{r}}\Delta t\right)^r.$$

Theorem 3.2. *Let $f \in \mathcal{C}^n_{rd}([a,b]_{\mathbb{T}})$, $r > 0$, $q > 1$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$. Then*

$$\begin{aligned}\frac{1}{b-a}|I(a,b,n,f)| \le & \frac{(b-a)^{n-\frac{1}{rq}}}{n!} 2^{\frac{1}{q}} \Bigg(2^{\frac{1}{rq}} |f^{\Delta^n}(b)| y_{\alpha,r,+}^{\frac{1}{rq}} + 2^{\frac{1}{rq}} |f^{\Delta^n}(a)| y_{\alpha,r,-}^{\frac{1}{rq}} \\ & + c^{\frac{1}{q}} (b-a)^{\frac{1}{rq}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{q}} \Bigg).\end{aligned}$$

Proof. Since $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$, applying (3.1), we have

(3.3)

$$\left|f^{\Delta^n}(x)\right|^q \le \left(\frac{b-x}{b-a}\left|f^{\Delta^n}(a)\right|^{qr} + \frac{x-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}} - c(b-x)(x-a), \quad x \in [a,b]_{\mathbb{T}}.$$

Hence, by Lemma 3.1 and the inequality

$$(x+y)^k \le 2^k(x^k + y^k), \quad x, y, k > 0,$$

we get

$$
\begin{aligned}
|I(a,b,n,f)| =& \left|\int_a^{\rho^n(b)} h_n(b,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau\right| \\
\leq& \int_a^{\rho^n(b)} |h_n(b,\sigma(\tau))| \left|f^{\Delta^n}(\tau)\right| \Delta\tau \quad \text{(By Hölder's inequality)} \\
\leq& \left(\int_a^{\rho^n(b)} |h_n(b,\sigma(\tau))|^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^{\rho^n(b)} \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& \left(\int_a^{\rho^n(b)} \left(\frac{(b-\sigma(\tau))^n}{n!}\right)^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^{\rho^n(b)} \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left(\left(\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right.\right.\right. \\
&\left.\left.\left. + \frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}\right)^{\frac{1}{r}} - c(b-\tau)(\tau-a)\right)\Delta\tau\right)^{\frac{1}{q}} \\
=& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left(\left(\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right.\right.\right. \\
&\left.\left.\left. + \frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}\right)^{\frac{1}{r}} - c(b\tau-ab-\tau^2+a\tau)\right)\Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \frac{(b-a)^n}{n!} \left(\frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}} \int_a^b (\tau-a)^{\frac{1}{r}}\Delta\tau\right. \\
& + \left.\frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}} \int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau\right)^{\frac{1}{q}} \\
& \left. + c\int_a^b \tau^2\Delta\tau - c(a+b)\int_a^b \tau\Delta\tau + abc(b-a)\right)^{\frac{1}{q}} \\
=& (b-a)^{\frac{1}{q}} \frac{(b-a)^n}{n!} \left(\frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a) y_{\alpha,r,+}^{\frac{1}{r}}\right. \\
& \left. + \frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a) y_{\alpha,r,-}^{\frac{1}{r}} + c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab)\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}+\frac{1}{q}} \frac{(b-a)^n}{n!} 2^{\frac{1}{q}} \left(\frac{2^{\frac{1}{rq}}\left|f^{\Delta^n}(b)\right|}{(b-a)^{\frac{1}{rq}}} y_{\alpha,r,+}^{\frac{1}{rq}}\right.
\end{aligned}
$$

$$+\frac{2^{\frac{1}{rq}}|f^{\Delta^n}(a)|}{(b-a)^{\frac{1}{rq}}}y_{\alpha,r,-}^{\frac{1}{rq}}+c^{\frac{1}{q}}|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|^{\frac{1}{q}}\Bigg),$$

whereupon we get the desired result. This completes the proof. □

The next result reads as follows.

Theorem 3.3. *Let $r>0$, $q\geq 1$, $f\in \mathcal{C}^n_{rd}([a,b]_{\mathbb{T}})$, $\left|f^{\Delta^n}\right|\geq 1$ on $[a,b]_{\mathbb{T}}$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$. Then*

$$\frac{1}{b-a}|I(a,b,n,f)|\leq\frac{(b-a)^n}{n!}\Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\left(\left|f^{\Delta^n}(b)\right|^q y_{\alpha,r,+}^{\frac{1}{r}}+\left|f^{\Delta^n}(a)\right|^q y_{\alpha,r,-}^{\frac{1}{r}}\right)$$
$$+c\left(y_{\alpha,\alpha}-(a+b)y_\alpha+ab\right)\Bigg).$$

Proof. Because $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$, the inequality (3.3) holds. Hence,

$$\begin{aligned}|I(a,b,n,f)|=&\left|\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau\right|\\
\leq&\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\int_a^{b}h_n(b,\tau)\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\leq h_n(b,a)\int_a^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\frac{(b-a)^n}{n!}\int_a^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\frac{(b-a)^n}{n!}\left(\int_a^b\left(\frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}+\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}}\Delta\tau\right.\\
&\left.-c\int_a^b(b-\tau)(\tau-a)\Delta\tau\right).\end{aligned}$$

Now, using the inequality

$$(x+y)^k\leq 2^k\left(x^k+y^k\right),\quad x,y>0,k>0,$$

we have

$$\begin{aligned}|I(a,b,n,f)|\leq&\frac{(b-a)^n}{n!}\left(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}\int_a^b(b-\tau)^{\frac{1}{r}}\Delta\tau+2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}\int_a^b(\tau-a)^{\frac{1}{r}}\Delta\tau\right.\\
&\left.-c\int_a^b\left(-ab+(a+b)\tau-\tau^2\right)\Delta\tau\right)\\
=&\frac{(b-a)^n}{n!}\left(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a)y_{\alpha,r,-}^{\frac{1}{r}}+2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a)y_{\alpha,r,+}^{\frac{1}{r}}\right.\end{aligned}$$

$$+(b-a)c\left(y_{\alpha,\alpha}-(a+b)y_\alpha+ab\right)\Bigg).$$

From here,

$$\frac{1}{b-a}|I(a,b,n,f)|\leq\frac{(b-a)^n}{n!}\Bigg(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}}+2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}$$
$$+c\left(y_{\alpha,\alpha}-(a+b)y_\alpha+ab\right)\Bigg).$$

This completes the proof. □

Now we present another inequality for this class of functions by different approach.

Theorem 3.4. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $r>0$, $f\in\mathcal{C}_{rd}^n([a,b]_{\mathbb{T}})$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$. Then*

$$\frac{1}{b-a}|I(a,b,n,f)|\leq\left(\frac{1}{p}-r_0\right)\left(\frac{(b-a)^n}{n!}\right)^p+\left(\frac{1}{q}-r_0\right)\Bigg(2^{\frac{1}{r}}\Bigg(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}}$$
$$+\frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\Bigg)+c(y_{\alpha,\alpha}-(a+b)y_\alpha+ab)\Bigg)$$
$$+2^{\frac{3}{2}}\frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}}\Bigg(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}}\left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}}+|f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}}\right)^{\frac{1}{2}}$$
$$+c^{\frac{1}{2}}|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|^{\frac{1}{2}}\Bigg),$$

where $r_0=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Since $\left|f^{\Delta^n}\right|^q$ is strongly r-convex with modulus c on $[a,b]_{\mathbb{T}}$, we have (3.3). Now, using the refinement of Young inequality

$$xy\leq\frac{x^p}{p}+\frac{y^q}{q}-r_0(x^{\frac{p}{2}}-y^{\frac{q}{2}})^2,\quad x,y\geq 0,$$

we have

$$\begin{aligned}|I(a,b,n,f)|=&\left|\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau\right|\\ \leq&\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|\Delta\tau\\ \leq&\int_a^b h_n(b,\tau)\left|f^{\Delta^n}(\tau)\right|\Delta\tau\\ \leq&\frac{1}{p}\int_a^b(h_n(b,\tau))^p\,\Delta\tau+\frac{1}{q}\int_a^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\ &-r_0\int_a^b\left(((h_n(b,\tau))^{\frac{p}{2}}-\left|f^{\Delta^n}(\tau)\right|^{\frac{q}{2}}\right)^2\Delta\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)\int_a^b \left(\frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr} + \frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}}\Delta\tau \\
&\quad - c\left(\frac{1}{q} - r_0\right)\int_a^b (b-\tau)(\tau-a)\Delta\tau + 2r_0\int_a^b (h_n(b,\tau))^{\frac{p}{2}}|f^{\Delta^n}(\tau)|^{\frac{q}{2}}\Delta\tau \\
&\leq \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}\int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau\right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}\int_a^b (\tau-a)^{\frac{1}{r}}\Delta\tau\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\left(\int_a^b (h_n(b,\tau))^p\Delta\tau\right)^{\frac{1}{2}}\left(\int_a^b |f^{\Delta^n}(\tau)|^q\Delta\tau\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}(b-a)\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\left(\int_a^b \left(\frac{b-\tau}{b-a}|f^{\Delta^n}(a)|^{qr} + \frac{\tau-a}{b-a}|f^{\Delta^n}(b)|^{qr}\right)^{\frac{1}{r}}\right. \\
&\quad \left. - c\int_a^b (b-\tau)(\tau-a)\Delta\tau\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}(b-a)\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\left(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\left(|f^{\Delta^n}(a)|^q\int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ |f^{\Delta^n}(b)|^q \int_a^b (\tau - a)^{\frac{1}{r}} \Delta\tau \Bigg) \\
&+ c(b-a)|y_{\alpha,\alpha} - (a+b)y_\alpha + ab| \Bigg)^{\frac{1}{2}} \\
= &\left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&+ \left(\frac{1}{q} - r_0\right) 2^{\frac{1}{r}} (b-a) \left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,+}^{\frac{1}{r}} \right) \\
&+ \left(\frac{1}{q} - r_0\right) c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&+ 2r_0 \frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}} \Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}} \Bigg(|f^{\Delta^n}(a)|^q (b-a) y_{\alpha,r,-}^{\frac{1}{r}} \\
&+ |f^{\Delta^n}(b)|^q (b-a) y_{\alpha,r,+}^{\frac{1}{r}} \Bigg) \\
&+ c(b-a)|y_{\alpha,\alpha} - (a+b)y_\alpha + ab| \Bigg)^{\frac{1}{r}} \\
= &\left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&+ \left(\frac{1}{q} - r_0\right)(b-a)\Bigg(2^{\frac{1}{r}} \left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,+}^{\frac{1}{r}} \right) \\
&+ c(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \Bigg) \\
&+ 2r_0 \frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}} \Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}} \Bigg(|f^{\Delta^n}(a)|^q (b-a) y_{\alpha,r,-}^{\frac{1}{r}} \\
&+ |f^{\Delta^n}(b)|^q (b-a) y_{\alpha,r,+}^{\frac{1}{r}} \Bigg) \\
&+ c(b-a)|y_{\alpha,\alpha} - (a+b)y_\alpha + ab| \Bigg)^{\frac{1}{2}} \\
\leq &\left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&+ \left(\frac{1}{q} - r_0\right)(b-a)\Bigg(2^{\frac{1}{r}} \left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,+}^{\frac{1}{r}} \right) \\
&+ c(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2^{\frac{3}{2}} r_0 \frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q (b-a) y_{\alpha,r,-}^{\frac{1}{r}} \right.\right. \\
&\left. + |f^{\Delta^n}(b)|^q (b-a) y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \\
&\left. + c^{\frac{1}{2}} (b-a)^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).
\end{aligned}
$$

This completes the proof. □

Remark 3.1. According to Theorem 3.4 we conclude that:

- If $p > q$, then we have

$$
\begin{aligned}
\frac{1}{b-a} |I(a,b,n,f)| \leq & \frac{p-q}{pq} \left(2^{\frac{1}{r}} \left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,-}^{\frac{1}{r}} \right.\right. \\
& \left.\left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,+}^{\frac{1}{r}} \right) + c(y_{\alpha,\alpha} - (a+b) y_\alpha + ab) \right) \\
& + 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right. \\
& \left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).
\end{aligned}
$$

- If $p < q$, then we obtain

$$
\begin{aligned}
\frac{1}{b-a} |I(a,b,n,f)| \leq & \frac{q-p}{pq} \left(\frac{(b-a)^n}{n!} \right)^p \\
& + 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right. \\
& \left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).
\end{aligned}
$$

- If $p = q = 2$, then we have

$$
\begin{aligned}
\frac{1}{b-a} |I(a,b,n,f)| \leq & 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right. \\
& \left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).
\end{aligned}
$$

Remark 3.2. If $\mathbb{T} = \mathbb{R}$, then the delta and nabla derivatives coincide with the classical derivative. Hence, diamond-α integral from a to t of f will be reduced to classical integral.

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