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wMB-PROPERTY OF ORDER p IN BANACH SPACES

MANIJEH BAHREINI ESFAHANI[1]

Abstract. In this paper, we introduce a new property of Banach spaces called wMB-property of order p ($1 \leq p < \infty$). A necessary and sufficient condition for a Banach space to have the wMB-property of order p is given. We study p-convergent operators and weakly-p-L-sets. Banach spaces with the wMB-property of order p are characterized. Also, the Dunford-Pettis property of order p and DP^*-property of order p are studied in Banach spaces. Finally we show the relation between Pelczynski's property (V) and wMB-property of order p.

1. Introduction

In 1993, J. M. F. Castillo and F. Sanchez in their fundamental paper [6], extend unconditionally converging operators and completely continuous (or Dunford-Pettis) operators to the general case by introducing p-convergent operators, $1 \leq p \leq \infty$. First the authors introduced weakly-p-summable sequences and weakly-p-convergent sequences, in the case of $p = \infty$, weakly-∞-convergent sequences are simply the weaky convergent sequences. Then they introduced p-convergent operators. The 1-convergent operators are simply the unconditionally converging operators and ∞-convergent operators are simply the completely continuous operators.

Ghenciu in 2018, introduced the concept of weakly-p-L-set in a dual space $1 \leq p < \infty$. A weakly-q-L-set in X^* is a weakly-p-L-set in X^* if $p < q$ [14].

In this paper, p-convergent operators and weakly-p-L-set in X^* are used to introduce a new property of Banach spaces called wMB-property of order p, $1 \leq p < \infty$. We prove that a Banach space X has wMB-property of order p provided every unconditionally converging operator $T : X \to \ell_\infty$ is p-convergent.

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Carrion, Galindo and Lourenco in [5] define and discuss the variant of the classical Dunford-Pettis property, called (DP^*P) in Banach spaces. In [18], E. D. Zeekoei and J. H. Fourie introduced the DP^*-property of order p on Banach spaces (DP^*P_p). Clearly, in case of $p = \infty$, we have $DP^*P = (DP^*P_\infty)$.

In [6], Castillo and Sanchez also introduced the Dunford-Pettis property of order p in Banach spaces in $1 \leq p < \infty$, which is a generalization of the classical Dunford-Pettis property.

Finally, the concepts of the Dunford-Pettis property of order p and DP^*-property of order p are studied in Banach spaces. We prove that (wMB) property of order p implies the Dunford-Pettis property of order p in Banach spaces.

1.1. **Definitions and Notation.** By an operator, we mean a bounded linear operator. Let X, Y be Banach spaces. We denote by $L(X, Y)$ the space of all operators from X into Y. For an operator $T : X \to Y$, the adjoint of T is denoted by T^*. For a Banach space X, the closed unit ball of X is denoted by B_X and the identity map on X is denoted by I_X. We denote by $K(X, Y)$ the space of all compact operators from X into Y and $W(X, Y)$ the space of all weakly compact operators from X into Y.

Our discussion will make use of several definitions from the paper [6]. Let X be a Banach space.

First we recall the definition of the weakly-p-summable sequences.

Let $1 \leq p < \infty$, p^* denotes the conjugate of p. If $p = 1$, ℓ_{p^*} plays the role of c_0. The unit vector basis of ℓ_p is denoted by (e_n).

Let $1 \leq p < \infty$ and X be a Banach space. The set of all p-summable sequences in X with the natural norm

$$\|(x_n)\|_p = \left(\sum_{i=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

is denoted by $\ell_p(X)$.

Let $1 \leq p < \infty$. A sequence (x_n) in a Banach space X is called weakly p-summable if $(x^*x_n) \in \ell_p$, for all $x^* \in X^*$. In other words, a sequence (x_n) in X is weakly p-summable if

$$\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < \infty,$$

for each $x^* \in X^*$ (see [9]). The set of all weakly p-summable sequences in X, endowed with the norm

$$\|(x_n)_n\|_p{}^w = \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}$$

is denoted by $\ell_p{}^w(X)$.

For $p = 1$, weakly 1-summable sequences correspond to weakly unconditionally converging series; For $p = \infty$, weakly ∞-summable sequences are just weakly null sequences in X, i.e., sequences which are in $c_0{}^w(X)$.

Ansari in 1995 showed that for $1 < p < \infty$, we have $L(\ell_{p^*}, X) = K(\ell_{p^*}, X)$ if and only if weakly-p-summable sequences in X are norm null, i.e., if and only if $id_X \in C_p(X, X)$ (see [1]).

Let $1 \leq p \leq \infty$. A sequence (x_n) in a Banach space X is said to be weakly-p-convergent to $x \in X$ if the sequence $(x_n - x)$ is weakly-p-summable in X (see [6]). The weakly-∞-convergent sequences are simply the weakly convergent sequences.

Recall that a series $\sum x_n$ is weakly unconditionally converging if and only if $\sum |x^*(x_n)| < \infty$ for each $x^* \in X^*$ and the series $\sum x_n$ is unconditionally converging if and only if every rearrangement converges in the norm topology of X.

An operator $T : X \to Y$ is unconditionally converging if it maps weakly 1-summable sequences to (unconditionally) 1-summable sequences, i.e., T takes weakly unconditionally converging series in X into unconditionally convergent series in Y. The set of all unconditionally converging operators from X to Y will be denoted by $UC(X, Y)$.

An operator $T : X \to Y$ is completely continuous (Dunford-Pettis) if it maps weakly null sequences to norm null sequences. The set of all completely continuous operators from X to Y will be denoted by $CC(X, Y)$.

In [3], space of unconditionally converging operators and space of completely continuous operators have been studied. Recently in [2], the author investigated how some localized properties can be used to study more global structure properties.

Finally, let us recall the definition of the L- sets and V-sets in dual spaces.

A bounded subset A of X^* is called a V-subset of X^* if

$$\lim_n (\sup\{|x^*(x_n)| \,:\, x^* \in A\}) = 0,$$

for each weakly unconditionally converging series $\sum x_n$ in X.

A Bounded subset A of X^* is called an *L-subset* of X^* if

$$\lim_n (\sup\{|x^*(x_n)| \,:\, x^* \in A\}) = 0,$$

for each weakly null sequence (x_n) in X.

The reader is referred to Diestel [8] or Dunford-Schwartz [10] for undefined notation and terminology.

2. Main Results

We begin this section with a simple, but extremely useful, characterization of p-convergent operators. The concept of p-convergent operators for $1 \leq p \leq \infty$ is introduced in [6]. It is well-known that p-summing operators take weakly p-summable sequences to p-summable sequences.

Let $1 \leq p \leq \infty$. An operator $T : X \to Y$ is called p-convergent if T transforms weakly-p-summable sequences into norm-null sequences. The class of p-convergent operators from X into Y is denoted by $C_p(X, Y)$.

The 1-convergent operators are precisely the unconditionally converging operators and ∞-convergent operators are precisely the completely continuous operators. Obviously, $C_q(X, Y) \subset C_p(X, Y)$, when $p < q$.

In this section, we study weakly-p-L- sets and $1 \leq p < \infty$ and V-sets, and then introduce a new property for Banach spaces.

Ghenciu in [14], introduced the concept of weakly-p-L-set, $1 \leq p < \infty$, in dual space.

Let $1 \leq p < \infty$. A Bounded set A in X^* is called a weakly-p-L-set in X^* if

$$\lim_n (\sup\{|x^*(x_n)| : x^* \in A\}) = 0,$$

for each weakly p-summable sequences (x_n) in X.

The weakly-1-L-set in X^* are precisely the V-subset of X^*. If $p < q$, then a weakly p-summable sequences in X is a weakly q-summable sequences in X, i.e., ${\ell_p}^w(X) \subseteq {\ell_q}^w(X)$. Hence a weakly-$q$-$L$-set in X^* is a weakly-p-L-set in X^*, and thus every weakly-q-L-set in X^* is a V-set of X^* for $1 < q$. In the following, we will give equivalent characterizations of Banach spaces which the converse statement holds.

Now we are ready to give our new property for Banach spaces using the concept of weakly-p-L-sets.

Definition 2.1. Let $1 \leq p < \infty$. A Banach space X has the wMB-property of order p (wMB_p) if every V-set in X^* is a weakly-p-L- set of X^*.

Recall that $T : Y \to X$ is an unconditionally converging operator if and only if $T^*(B_{Y^*})$ is a V-set in X^*. Ghenciu generalized this characterization of unconditionally converging operators to p-convergent operators in terms of weakly-p-L-sets and compact operators [14]. The following two theorems, which give a characterization of p-convergent operators, play an important role in this study.

Theorem 2.1 ([14], Theorem 13)**.** *Let $1 \leq p < \infty$. Let $T : Y \to X$ be an operator. The following are equivalent:*

(i) T is p-convergent;

(ii) for any operator $S : \ell_{p^*} \to X$ if $1 < p < 1$ (resp. $S : c_0 \to X$ if $p = 1$), the operator TS is compact.

Theorem 2.2 ([14], Theorem 14)**.** *Let $1 \leq p < \infty$. Let $T : Y \to X$ be an operator. The following are equivalent:*

(i) $T^*(B_{X^*})$ is a weakly-p-L-set;

(ii) T is p-convergent.

In the next theorem which is our main result, we characterize Banach spaces with wMB-property of order p. A necessary and sufficient condition for a Banach space to have the wMB-property of order p has given. We prove that a Banach space X has wMB-property of order p if and only if for every Banach space Y, every unconditionally converging operator $T : X \to Y$ is p-convergent.

Theorem 2.3. *Let $1 \leq p < \infty$. The following statements are equivalent about a Banach space X.*

(i) X has the wMB-property of order p.

(*ii*) *For every Banach space Y, if $T : X \to Y$ is an unconditionally converging operator, then T is p-convergent.*
(*iii*) *Same as* (*ii*) *with $Y = \ell_\infty$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that Y is a Banach space and $T : X \to Y$ is unconditionally converging. Then $T^*(B_{Y^*})$ is a V-set, and hence is weakly-p-L-set in X^*, since X has the wMB-property of order p. Note that, for all weakly p-summable sequences (x_n) in X

$$\|T(x_n)\| = \sup\{|\langle x_n, T^*(y^*)\rangle| \,:\, y^* \in B_Y^*\}.$$

Thus T is p-convergent.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ Suppose that A is a V-set in X^* and (x_n^*) be a sequence in A. Define $T : \ell_1 \to X^*$ by $T(b) = \sum_n b_n x_n^*$ for $b = (b_n) \in \ell_1$. Note that $T^*_{|X} : X \to \ell_\infty$ and $T^*(x) = (x_i^*(x))$. Let (x_n) be a weakly p-summable sequence in X, then (x_n) is a weakly 1-summable sequence, since $p > 1$. But A is a V-set, hence

$$\|T^*(x_n)\| = \sup_i |x_i^*(x_n)| \longrightarrow 0.$$

Therefore, $T^*_{|X}$ is unconditionally converging, and hence $T^*_{|X}$ is p-convergent. Let (x_n) be a weakly p-summable in X and $y \in B_{\ell_1}$. Then

$$|T(y)(x_n)| = |T^*(x_n)(y)| \leq \|T^*(x_n)\| \longrightarrow 0.$$

Thus, $T(B_{\ell_1})$ is a weakly-p-L-set in X^*, and hence X has wMB-property of order p. □

Our first result gives a characterization of unconditionally converging operators in terms of weakly-p-L-sets.

Corollary 2.1. *Let $1 \leq p < \infty$. The following statements are equivalent about a Banach space X.*

(*i*) *X has the wMB-property of order p.*

(*ii*) *If $T : X \to Y$ is an unconditionally converging operator, for every Banach space Y, then T^* maps bounded sets in Y^* onto weakly-p-L-sets in X^*.*

Another result from Theorem 2.3 gives a characterization of Banach spaces with the wMB-property of order p in terms of weakly p-convergent operators. Let us recall definition of the weakly p-convergent operators from [18].

Let $1 \leq p \leq \infty$. An operator $T : X \to Y$ is called weak p-convergent if $(y_n^*(Tx_n))$ converges to 0 for every sequence $(x_n) \in {\ell_p}^w(X)$ and every weakly null sequence (y_n^*) in Y^*.

Obviously, each p-convergent operator is weak p-convergent. Zeekoei and Fourie showed that, for $1 \leq p < \infty$ and operator $T : Y \to X$, T is weak p-convergent if and only if for every weakly compact operator $S : Y \to Z$, the operator ST is p-convergent. If S be the identity map on Y, then we have the following result.

Corollary 2.2. *Let $1 \leq p < \infty$. Let X and Y be Banach spaces and Y be reflexive. Then X has the wMB-property of order p if and only if every unconditionally converging operator $T : X \to Y$ is weak p-convergent.*

In Theorem 2.3, we showed that if $T : X \to Y$ is p-convergent whenever T is an unconditionally converging, then X has the wMB-property of order p. In the next theorem we extend this result to the second adjoint operators. In fact, we show that X has the wMB-property of order p whenever unconditionally converging operator T implies that the second adjoint operator T^{**} is p-convergent.

Theorem 2.4. *Let $1 \leq p < \infty$. Let X be a Banach space and $T : X \to Y$ is an operator, for every Banach space Y. If T^{**} is p-convergent whenever T is unconditionally converging, then X has the wMB-property of order p.*

Proof. Suppose $T : X \to Y$ is an unconditionally converging operator. Then by assumption, the second adjoint operator $T^{**} : X^{**} \to Y^{**}$ is p-convergent. Now, let $\eta_X : X \to X^{**}$ and $\eta_Y : Y \to Y^{**}$ be the natural embedding of X onto X^{**} and Y^{**}, respectively. Thus, $T^{**}\eta_X$ is p-convergent, and hence $\eta_Y T$ is p-convergent. Let $S : \ell_{p^*} \to X$ if $1 < p < 1$ (resp. $S : c_0 \to X$ if $p = 1$), hence operator $\eta_Y TS$ is compact by Theorem 2.1 , thus TS is compact and again using Theorem 2.1, T is p-convergent. Finally, Theorem 2.3, implies that X has the wMB-property of order p. $\square$

Now we study the relation between the wMB-property of order p and the Dunford-Pettis property of order p.

A Banach space X has the Dunford-Pettis property if for every Banach space Y, every weakly compact operator $T : X \to Y$ transforms weakly compact sets of X into norm compact sets in Y, i.e., $W(X, Y) \subseteq CC(X, Y)$. In other words, for every Banach space Y, every weakly compact operator $T : X \to Y$ is completely continuous.

Castillo and Sanchez introduced a weaker property called Dunford-Pettis property of order p (DPP_p) (see [6]).

Let $1 \leq p \leq \infty$. A Banach space X has the Dunford-Pettis property of order p if every weakly compact operator $T : X \to Y$ is p-convergent, for any Banach space Y, i.e., $W(X, Y) \subseteq C_p(X, Y)$.

Clearly Dunford-Pettis property of order p implies Dunford-Pettis property of order q whenever $p < q$. Also Dunford-Pettis property of order ∞ is precisely Dunford-Pettis property and every Banach space has Dunford-Pettis property of order 1.

Next theorem shows that the wMB-property of order p implies the Dunford-Pettis property of order p.

Theorem 2.5. *Let $1 \leq p < \infty$. Let Banach space X has the wMB-property of order p. Then X has the Dunford-Pettis property of order p.*

Proof. Let Y be any Banach space and $T : X \to Y$ is weakly compact. Then it is easy to see that T is unconditionally converging operator. By using Theorem 2.3, T is p-convergent, since X has wMB-property of order p. $\square$

Finally, the concepts of the Dunford-Pettis property of order p and DP^*-property of order p are studied in Banach spaces.

Carrion, Galindo and Lourenco in [5] define and discuss the variant of the classical Dunford-Pettis property, called (DP^*P) in Banach spaces.

A Banach space X has the DP^*-property (DP^*P) if every weakly compact sets in X are limited. In other words, every w^*-null sequence (x_n^*) in X^*, converges uniformly to 0 on all weakly compact sets in X.

Zeekoei and Fourie in [18], introduced the DP^*-property of order p on Banach spaces (DP^*P_p).

Let $1 \leq p \leq \infty$. A Banach space X has the DP^*-property of order p (DP^*P_p) if every weakly-p-compact sets in X are limited.

Clearly, DP^*P_q implies DP^*P_p if $p < q$. Also the DP^*-property implies DP^*-property of order p for $1 \leq p < \infty$ and $DP^*P = (DP^*P_\infty)$.

We prove that wMB-property of order p implies the Dunford-Pettis property of order p in Banach spaces.

Theorem 2.6. *Let $1 \leq p < \infty$. Let X be a Grothendick space with the wMB-property of order p. Then X has the DP^*-property of order p.*

Proof. Since Banach space X is Grothendick, the Dunford-Pettis property of order p implies the DP^*-property of order p. Now , Theorem 2.5 gives the result. □

Corollary 2.3. *Let $1 \leq p < \infty$. Let X be a reflexive space with the wMB-property of order p. Then X has the DP^*-property of order p.*

The following result gives another characterization of Banach spaces witch have the wMB-property of order p.

Corollary 2.4. *Let $1 \leq p < \infty$. Let Banach space X has the wMB-property of order p. Then every weakly compact operator T from X into c_0 is p-convergent.*

It was shown in [14] that if $1 \leq p < \infty$ and A is a bounded subset of a Banach space X, then A is weakly-p-L-subset of X^* if and only if (x_n) is a weakly p-summable sequence in X and (x_n^*) is a sequence in X^*, then $\lim x_n^*(x_n) = 0$. We generalize this result to weakly p-summable sequences in X and weakly null sequences (x_n^*) in X^* if X has the wMB-property of order p.

Theorem 2.7. *Let $1 \leq p < \infty$. Let Banach space X has the wMB-property of order p. If (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly null sequence in X^*, then $\lim x_n^*(x_n) = 0$.*

Proof. Suppose (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly null sequence in X^*. Let $T : X \to c_0$ such that $Tx = (x_n^*)$. Thus, T^* is weakly compact and T is p-convergent by Corollary 2.4 Therefore, (Tx_n) is norm-null sequence, and hence $\lim x_n^*(x_n) = 0$. □

In our last result we investigate the converse of Theorem 2.5. We show that if a Banach space X has the wMB-property of order p, then X has the Dunford-Pettis property of order p for $1 \leq p < \infty$.

Recall that a Banach space X has Pelczynski's property (V) if every V-subset of X^* is weakly sequentially compact in the weak topology of X^*. Equivalently, X has Pelczynski's property (V) if for every Banach space Y, every unconditionally converging operator $T : X \to Y$ is weakly compact (see [15]).

Theorem 2.8. *Let $1 \leq p < \infty$. Let X be a Banach space with Dunford-Pettis property of order p. If X has Pelczynski's property (V), then X has wMB-property of order p.*

Proof. Suppose Y is any Banach space and operator $T : X \to Y$ is unconditionally converging. Then T is weakly compact, since X has the Pelczynski's property (V). Now as X has the Dunford-Pettis property of order p, then T is p-convergent. Thus, X has wMB-property of order p. □

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[1]Department of Mathematics,
University of Khansar,
Khansar, Iran
Email address: mebahreini@yahoo.com
Email address: mebahreini@khansar-cmc.ac.ir