Kragujevac Journal of Mathematics
Volume 49(4) (2025), Pages 541–553.

STRONGLY EXTENDING MODULAR LATTICES

SHAHABADDIN EBRAHIMI ATANI[1], MEHDI KHORAMDEL[1],
SABOURA DOLATI PISH HESARI[1], AND MAHSA NIKMARD ROSTAM ALIPOUR[1]

Abstract. In this paper, our purpose is to initiate the study of the concept of strongly extending modular lattices based on the similar notion of strongly extending modules. We will prove some basic properties of strongly extending modular lattices and employ this results to give applications to the category of modules with a fixed hereditary torsion class and Grothendieck categories.

1. Introduction

The notion of CC or extending for modules and related notions is an interesting topics for several authors that were extensively studied in the literature ([18]). A module M is said to be an *extending* (or a CS) module provided that every submodule of M is contained in a direct summand of M as an essential submodule. A module M is called a *FI-extending* module provided that each of its fully invariant submodule is essential in a direct summand ([12]). Another interesting related concepts of the extending modules is strongly FI-extending ([13, 15]). The strongly FI-extending property of modules has been used for the existence and description of the FI-extending module hull of any finitely generated projective module over a semiprime ring ([14]). A module M is said to be a *strongly FI-extending* module if each fully invariant submodule is essentially contained in a fully invariant direct summand. In [19], a subclass of extending modules, strongly extending modules, introduced and investigated. A module M is said to be *strongly extending* provided that each submodule is essential in a fully invariant direct summand. Recently, the known conditions on modules (extending,

Key words and phrases. Modular lattice, upper continuous lattice, linear lattice morphism, fully invariant element, strongly extending lattice.
2020 *Mathematics Subject Classification.* Primary: 06C05, 06C99. Secondary: 06B35, 16D80.
DOI
Received: August 02, 2021.
Accepted: July 30, 2022.

FI-extending, strongly FI-extending, etc.) have been introduced and considered in lattices, in order to give some interesting results to Grothendieck categories and the category of modules with hereditary torsion theories [5–7, 9, 10].

When we study the classes of extending, FI-extending, strongly FI-extending lattices, it is an ambition to study the notion of strongly extending in lattices. Also one of the motivations to study this topic is the following questions.

(1) If a lattice L is strongly extending, then is it true that every complement is fully invariant?

(2) Is it true that every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L if and only if every complement of L is fully invariant in L?

This paper is allocated to initiate the strongly extending condition for lattices, and investigate their properties that are similar to results on modules introduced and studied in [19]. We will adopt the results from [19] to strongly extending lattices, however it is not always easy because some theoretical tools and techniques in modules do not work in lattices.

In Section 2, we recall some preliminaries and definitions about lattices from [1–11]. We recall the useful notion of linear morphisms between two lattices introduced by Albu and Iosif [5]. This concept is used in our main results. In Section 3, we define the conditions strongly extending and Abelian for lattices, and some of their structural properties are studied. We will answer the previous questions affirmatively. We will show that every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L if and only if $D(L) \subset FI(L)$. Also, it is shown that, a strongly extending lattice L is extending and every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L. Moreover, if L is complete and strongly extending, then $D(L)$ is a sublattice of L and every its subset has a greatest lower bound. Further, we prove that the strongly extending condition of lattices is preserved by their complement intervals, and consider when direct joins have this property. In Section 3 and Section 4 we exhibit some usage of the results to Grothendieck categories and the category of modules with a fixed hereditary torsion class.

2. Preliminaries

Throughout this paper, by L, we will indicate a modular lattice $(L, \leq, \wedge, \vee, 0, 1)$ that has least element 0 and greatest element 1. For any $l, k \in L$, where $l \leq k$, let k/l denote the interval $\{x \in L \mid l \leq x \leq k\}$. For basic terminology and notation on lattices, we refer the reader to [4, 16, 17, 20] and [21], but particularly to [4]. For a lattice L, by $D(L)$, $P(L)$, $E(L)$ and $C(L)$, we denote the set of all *complement elements* of L, the set of all *pseudo-complement elements* of L, the set of all *essential elements* of L and the set of all *closed elements* of L, respectively.

A lattice L is said to be *extending* or *CC* if, for any $l \in L$, we have l is essential in $k/0$, for some complement interval $k/0$ in L. Also, L is said to be *quasi-continuous* provided that it is extending and for any two complement elements l_1, l_2 of L with $l_1 \wedge l_2 = 0$, we have $l_1 \vee l_2 \in D(L)$ ([8, Definition 1.2]).

By Albu and Iosif [5], a map $\theta : L \to L'$ between two lattices L with greatest element 1_L, least element 0_L and a lattice L' with greatest element $1_{L'}$, least element $0_{L'}$ is called a *linear morphism* provided that there exist $i \in L'$ and $k \in L$ (k is said to be a kernel of θ) such that $\theta(l) = \theta(l \vee k)$, for each $l \in L$, and f induces a lattice isomorphism:

$$\bar{\theta} : 1_L/k \to i/0_{L'}, \quad \bar{\theta}(l) = \theta(l), \quad \text{for all } l \in 1/k.$$

Assume that L is a lattice. By [6, Examples 0.2 (2)], if $c, d \in L$ and $c \wedge d = 0$, then the mapping

$$p_{d,c} : (c \vee d)/0 \to c/0, \quad p_{d,c}(a) := (a \vee d) \wedge c,$$

is said to be the canonical projection on $c/0$, which is a linear morphism (surjective) and its kernel is d. Notice that if L is a modular lattice, then $p_{d,c}(a) = a$, for all $a \in c/0$. In particular, if $k \in L$ is a complement of $l \in L$, we will use the notation $\widetilde{p}_{l,k}$, the linear endomorphism of L obtained by composing $\widetilde{p}_{l,k}$ with the canonical inclusion mapping $i : k/0 \to L$. If there is not any ambiguity about l, the notation $\widetilde{p}_k$ will be used instead of $\widetilde{p}_{l,k}$.

Throughout this paper, $End(L)$ denotes the collection of all linear endomorphisms of a modular lattice L (it is a monoid, with respect to the composition "$\circ$" of functions). We will use the notation fg for the composition $f \circ g$ of two linear morphisms f, g. An element $l \in L$ is said to be a fully invariant element, provided that $\theta(l) \leq l$ for each $\theta \in End(L)$ ([9]). By $FI(L)$, we will indicate the set $\{l \in L \mid l \text{ is fully invariant in } L\}$. A linear endomorphism θ of a modular lattice L is said to be a *left semicentral idempotent* of $End(L)$ (or L) if $\theta^2 = \theta$ and $\theta\psi = \theta\psi\theta$ for all $\psi \in End(L)$ ([10]). We exhibit by $S_l(L)$ the collection of all left semicentral idempotents of L.

It is assumed throughout this paper that a ring R is an associative ring with unity and all modules are unital right R-modules. The notation $Mod - R$ denotes the category of all unital right R-modules. We denote by M_R a unital right R-module M. Let $L(M_R)$ indicate the lattice of all submodules of a module M_R. For submodules T and H of M, $T \leq H$ will denote that T is a submodule of H.

3. Strongly Extending Lattices

This section is allocated to introduce and investigate our main concept, namely, strongly extending lattices and give some properties of this class of lattices and establish some relations between the notion of strongly extending and the other notions in the literature. We begin with the following lemma which is a quite useful in this note.

Lemma 3.1. *Let θ be an idempotent linear endomorphism of L. Then $\theta(1)$ is a complement of* $\ker(\theta)$ *and $\widetilde{p}_{\theta(1)} = \theta$.*

Proof. Let $k := \ker(\theta)$. We claim $\theta(1) \vee k = 1$ and $\theta(1) \wedge k = 0$. As θ commutes with arbitrary joins ([6, Lemma 06]), $\theta(\theta(1) \vee k) = \theta(\theta(1)) \vee \theta(k) = \theta(1)$. Thus, $\overline{\theta}(\theta(1) \vee k) = \overline{\theta}(1)$. As $\overline{\theta}$ is an isomorphism, $\theta(1) \vee k = 1$.

Since $\overline{\theta}$ is an isomorphism, we have $\theta(1) \wedge k = \overline{\theta}(c)$, for some $c \in 1/k$. Thus, $\theta(1) \wedge k = \theta(c)$. Hence,

$$\theta(\theta(1) \wedge k) = \theta(\theta(c)) = \theta(c).$$

As $\theta(1) \wedge k \leq k$, $\theta(\theta(1) \wedge k) = 0$. Therefore, $0 = \theta(c) = \theta(1) \wedge k$, as desired.

Now we show that $\widetilde{p}_{\theta(1)} = h$. As θ commutes with arbitrary joins and $\theta(k) = 0$, $\theta(x \vee k) = \theta(x) \vee \theta(k) = \theta(x)$. Since θ is idempotent,

$$\begin{aligned}\overline{\theta}(x \vee k) =& \theta(x \vee k) = \theta(x) = \theta(\theta(x)) \\ =& \theta(\theta(x) \vee k) = \overline{\theta}(\theta(x) \vee k).\end{aligned}$$

Thus, $x \vee k = \theta(x) \vee k$, because $\overline{\theta}$ is a lattice isomorphism. As L is modular and $\theta(x) \leq \theta(1)$,

$$\begin{aligned}\widetilde{p}_{\theta(1)}(x) =& (x \vee k) \wedge \theta(1) \\ =& (\theta(x) \vee k) \wedge \theta(1) = \theta(x) \vee (\theta(1) \wedge k) \\ =& \theta(x).\end{aligned}$$

It completes the proof. □

Definition 3.1. A lattice L is said to be Abelian, if any idempotent linear endomorphism of L is central in $End(L)$ (i.e., commute with any linear endomorphism of L).

In the following, we provide a characterization for Abelian lattices.

Proposition 3.1. *Let L be a lattice. Then the following statements are equivalent:*
(1) *$D(L) \subseteq FI(L)$;*
(2) *L is Abelian.*

Proof. (1) $\Rightarrow$ (2) Let θ be an idempotent linear endomorphism of L. Put $l := \theta(1)$ and $m = \ker(\theta)$. By Lemma 3.1, $l \wedge m = 0$, $l \vee m = 1$ and $\widetilde{p}_l = \theta$. By (1), $l, m \in FI(L)$. Therefore, we have $\widetilde{p}_l, \widetilde{p}_m \in S_l(End(L))$, by [10, Lemma 2.8] (it is known that if $e \in D(L)$, then $\widetilde{p}_e \in S_l(End(L))$ if and only if $e \in FI(L)$ [10, Lemma 2.8]). Let $\psi \in End(L)$. We will show that $\psi\theta = \theta\psi$. Let $x \in L$. Then $\psi(\theta(x)) = \psi(\widetilde{p}_l(x)) = \psi((x \vee m) \wedge l)$. As $(x \vee m) \wedge l \leq l$ and $l \in FI(L)$, we have

$$\psi((x \vee m) \wedge l) \leq \psi(l) \leq l.$$

Moreover, $m \in FI(L)$ and $(x \vee m) \wedge l \leq x \vee m$, hence

$$\psi((x \vee m) \wedge l) \leq \psi(x \vee m) = \psi(x) \vee \psi(m) \leq \psi(x) \vee m.$$

Thus,

$$\psi(\theta(x)) = \psi(\widetilde{p}_l(x)) = \psi((x \vee m) \wedge l) \leq (\psi(x) \vee m) \wedge l = \widetilde{p}_l(\psi(x)) = \theta(\psi(x)).$$

For the reverse, we have $(x \vee l) \wedge m \leq m \leq x \vee m$. Since L is modular,

$$\begin{aligned}((x \vee m) \wedge l) \vee ((x \vee l) \wedge m) =&(x \vee m) \wedge (l \vee ((x \vee l) \wedge m))\\ =&(x \vee m) \wedge ((l \vee m) \wedge (x \vee l))\\ =&(x \vee m) \wedge (x \vee l).\end{aligned}$$

Thus, we have

$$x \leq (x \vee m) \wedge (x \vee l) =((x \vee m) \wedge l) \vee ((x \vee l) \wedge m) = \widetilde{p}_l(x) \wedge \widetilde{p}_m(x).$$

Hence, $\psi(x) \leq \psi((\widetilde{p}_l(x) \wedge \widetilde{p}_m(x)))$ and

$$\theta(\psi(x)) \leq \theta(\psi((\widetilde{p}_l(x) \wedge \widetilde{p}_m(x)))) = \theta(\psi(\widetilde{p}_l(x))) \vee \theta(\psi(\widetilde{p}_m(x))).$$

Since $\widetilde{p}_l = \theta,\ \widetilde{p}_m \in S_l(End(L))$, $\theta\psi\theta = \psi\theta$ and $\widetilde{p}_m\psi\widetilde{p}_m = \psi\widetilde{p}_m$. Therefore,

$$\theta\psi\theta(x) \vee \theta f\widetilde{p}_m(x) = \psi\theta(x) \vee \theta\widetilde{p}_m\psi\widetilde{p}_m(x).$$

As $\theta(\widetilde{p}_m)(c) = 0$, for each $c \in L$, we have $\theta\psi(x) \leq \psi\theta(x)$. Therefore, $e\theta = \psi\theta$, as desired.

(2) $\Rightarrow$ (1) Let $l \in D(L)$. By (2), $\widetilde{p}_l$ is central and so $\widetilde{p}_l \in S_l(End(L))$, by [10, Lemma 2.8]. Therefore, $l \in FI(L)$ and $D(L) \subseteq FI(L)$. ☐

In the following, we introduce the key definition of this paper.

Definition 3.2. A lattice L is said to be strongly extending, provided that for any $l \in L$, $l \in E(e/0)$ for some $e \in (FI(L) \cap D(L))$.

In the following observation, we give some characterizations of strongly extending lattices.

Theorem 3.1. *Let L be a lattice. Then the following statements are equivalent:*
(1) *L is a strongly extending lattice;*
(2) *L is extending and $C(L) \subseteq FI(L)$;*
(3) *L is extending and $P(L) \subseteq FI(L)$;*
(4) *L is extending and $D(L) \subseteq FI(L)$;*
(5) *L is extending and L is Abelian.*

Proof. (1) $\Rightarrow$ (2) If L is strongly extending, then L is extending. Let $e \in C(L)$. Hence there exists $l \in D(L) \cap FI(L)$ such that $e \in E(l/0)$. Thus, $e = l$, and so $C(L) \subseteq FI(L)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It is clear, because $D(L) \subseteq P(L) \subseteq C(L)$, by [8, Proposition 1.7 (1)].

(4) $\Rightarrow$ (5) It is clear by Proposition 3.1.

(5) $\Rightarrow$ (1) Let $l \in L$. Then $l \in E(k/0)$, for some $k \in D(L)$. By (5), $\widetilde{p}_k \in S_l(End(L))$. Therefore, $k \in FI(L)$, by [10, Lemma 2.8]. Hence, L is strongly extending. ☐

Corollary 3.1. *If L is a uniform lattice, then L is strongly extending.*

The converse of Corollary 3.1 is true, provided that L is indecomposable.

Theorem 3.2. *Let L be a complete strongly extending lattice. Then $D(L)$ is a sublattice of L. Moreover, every subset of $D(L)$ has a greatest lower bound.*

Proof. Let $e, f \in D(L)$ and $e \vee e' = 1, e \wedge e' = 0,\ f \vee f' = 1$ and $f \wedge f' = 0$. We are going to show $f \vee e \in D(L)$. By Theorem 3.1, $D(L) \subset FI(L)$, and so by [9, Lemma 1.8(4)], we have

$$e = (e \wedge f) \vee (e \vee f').$$

Therefore, $e \wedge f \in D(e)$, and hence $e \wedge f \in D(L)$, by [8, Proposition 1.7(3)].

Now, we will show that $e \vee f \in D(L)$. By [9, Lemma 1.8(4)], we have

$$f = (e \wedge f) \vee (e' \vee f).$$

Therefore,

$$e \vee f = e \vee (f \wedge e) \vee (f \wedge e') = e \vee (f \wedge e').$$

By the previous argument, $f \wedge e' \in D(L)$, hence there exits $t \in L$ such that $1 = (f \wedge e') \vee t$ and $(f \wedge e') \wedge t = 0$. Since $e \in FI(L)$, we have

$$e = (e \wedge t) \vee (e \wedge (f \wedge e')) = e \wedge t,$$

by [9, Lemma 1.8 (4)]. Thus, $e \leq t$ and $e \in D(t/0)$. Let $t = e \dot{\vee} h$. Then $1 = (f \wedge e') \vee e \vee h = (e \vee f) \dot{\vee} h$. Hence, $e \vee f \in D(L)$.

Now, suppose that $\{d_i\}_{i \in I} \subset D(L)$, where I is an arbitrary index set. Then $\bigwedge_{i \in I} d_i \in E(a/0)$, for some $a \in D(L) \cap FI(L)$. Let $d'_i \in D(L)$ be such that $d_i \vee d'_i = 1$ for each $i \in I$. Since $a \in FI(L)$, $a = (a \wedge d_i) \vee (a \wedge d'_i)$, by [9, Lemma 1.8 (4)]. Since $\bigwedge_{i \in I} d_i \in E(a/0)$ and $(\bigwedge_{i \in I} d_i) \wedge d'_i = 0$, for each $i \in I$, we have $d'_i \wedge a = 0$. Therefore, $a = a \wedge d_i$, for each $i \in I$, and so $a \leq d_i$, for each $i \in I$. Hence, $a \leq \bigwedge_{i \in I} d_i$ and $a = \bigwedge_{i \in I} d_i \in D(L)$. Hence, every subset of $D(L)$ has a greatest lower bound. □

Corollary 3.2. *Let L be a strongly extending lattice. Then L is quasi-continuous.*

Proof. Assume that L is a strongly extending lattice. Then L satisfies the condition C_1. Moreover, L has C_3 property by Theorem 3.2. □

Next, we give some properties of a strongly extending lattice.

Proposition 3.2. *Let L be a strongly extending lattice. Then the following statements hold.*

(1) *If θ is a linear monomorphism, then $\theta(1) \in E(L)$.*

(2) *If $\theta\psi = 1_{End(L)}$, for some $\psi, \theta \in End(L)$, then $\psi\theta = 1_{End(L)}$.*

Proof. (1) Let θ be a linear monomorphism. Then $\theta(1) \in E(h/0)$, for some $h \in D(L) \cap FI(L)$. Since $h \in D(L)$, $1 = h \dot{\vee} h'$, for some $h' \in L$. Hence, $(\widetilde{p}_{h'} \circ \theta)(1) = 0$. By Theorem 3.1, $\widetilde{p_{h'}}$ is central, therefore $\theta \circ \widetilde{p}_{h'} = \widetilde{p}_{h'} \circ \theta$. Thus, $(\theta \circ \widetilde{p}_{h'})(1) = \theta(h') = 0$. Since θ is a linear monomorphism, $\theta(h') = \theta(0)$ implies that $h' = 0$. Therefore, $h = 1$ and $\theta(1) \in E(L)$.

(2) Let $\theta, \psi \in End(L)$ and $\theta \circ \psi(x) = x$, for each $x \in L$. Then

$$\psi \circ \theta \circ \psi \circ \theta(x) = \psi((\theta \circ \psi)(\theta(x))) = \psi(\theta(x)) = \psi \circ \theta(x).$$

This proves that $\psi\theta$ is an idempotent linear morphism of L. By Theorem 3.1, $\psi\theta$ is central in $End(L)$. Therefore, $\theta \circ (\psi \circ \theta) = (\psi \circ \theta) \circ \theta$. Thus, we have

$$\begin{aligned}\psi \circ \theta(x) =&(\psi \circ \theta)(\theta \circ \psi(x)) = ((\psi \circ \theta) \circ \theta)(\psi(x))\\ =&(\theta \circ (\psi \circ \theta)(\psi(x)) = (\theta \circ \psi)(\theta \circ \psi(x))\\ =&\theta \circ \psi(x) = x.\end{aligned}$$

Therefore, $\psi\theta = 1_{End(L)}$. □

Lemma 3.2. *Let L be a lattice and $1 = c\dot{\vee}d$, for some $c, d \in L$. Then there is not any non-zero linear morphism between $c/0$ and $d/0$ if and only if $c \in FI(L)$.*

Proof. Assume that $c \in FI(L)$. Let $\theta : c/0 \to d/0$ be a linear morphism and λ the composition

$$L \xrightarrow{\widetilde{p}_c} c/0 \xrightarrow{\theta} d/0 \xrightarrow{i} L\,,$$

where $\widetilde{p}_c : L \to c/0$ is the canonical projection $\widetilde{p}_{d,c}$ on $c/0$ and $i : d/0 \to L$ is the mapping of canonical inclusion. Thus, $\lambda \in End(L)$ as a composition of linear morphisms of lattices. Since $c \in FI(L)$, $h(c) \leq c$. It is clear that $\lambda(c) \leq d$. Hence, $\lambda(c) \leq c \wedge d = 0$ and so $\lambda(c) = 0$. This proves $\theta(c) = 0$, and so $\theta = 0$, as desired.

Conversely, assume that there is not any non-zero linear morphism between $c/0$ and $d/0$, for each $i \neq j \in I$. Let $\theta \in End(L)$ and λ be the composition

$$c/0 \xrightarrow{\theta|_{c/0}} L \xrightarrow{p_d} d/0\,,$$

where $\theta|_{c/0}$ is the restriction of θ to $c/0$. Then, by our assumption, $\lambda = 0$. Hence, $p_d(\theta(c)) = 0$. Therefore, $\theta(c) \leq \ker(p_d) = c$, and so c is fully invariant. □

Corollary 3.3. *Let L be a strongly extending lattice and $1 = c\dot{\vee}e$, for some $c, e \in L$. Then there is not any non-zero linear morphism between $c/0$ and $e/0$.*

Proof. It is clear from Theorem 3.1 and Lemma 3.2. □

In the sequel, we show that the strongly extending property of a lattice is preserved by complement intervals and also consider when direct joins have this property.

Proposition 3.3. *Let L be a strongly extending lattice. If $l \in D(L)$, then $l/0$ is strongly extending.*

Proof. Assume that L is strongly extending, $l \in D(L)$ and $x \in l/0$. Then $x \in E(p/0)$, for some $p \in D(L) \cap FI(L)$. As $l, p \in D(L)$, $p \vee q = 1$ and $p \wedge q = 0$, also $l \vee m = 1$ and $l \wedge m = 0$, for some $m, q \in L$. Since $x \in E(p/0)$, $x \in E((p \wedge l)/0)$. We are now going to prove that $p \wedge l \in FI(l/0) \cap D(l/0)$. As $l \vee m = 1$ and $p \in FI(L)$, $p = (p \wedge l) \vee (p \wedge m)$, by [9, Lemma 1.8 (4)]. Therefore, $(p \wedge l) \vee (p \wedge m) \vee q = 1$. By modularity, we have

$$l = l \wedge 1 =d \wedge ((p \wedge l) \vee (p \wedge m) \vee q) = (p \wedge l) \vee (l \wedge ((p \wedge m) \vee q)).$$

Also, $(p \wedge l) \vee (l \wedge ((p \wedge m) \vee q)) \leq l$ and

$$\begin{aligned}(p \wedge l) \wedge (l \wedge ((p \wedge m) \vee q)) =&(p \wedge l) \wedge ((p \wedge m) \vee q)\\ \leq& p \wedge ((p \wedge m) \vee q)\\ =&(p \wedge m) \vee (p \wedge q) = p \wedge m\\ \leq& m.\end{aligned}$$

Therefore,

$$(p \wedge l) \wedge (l \wedge ((p \wedge m) \vee q)) \leq l \wedge m = 0.$$

Hence, we have $p \wedge l \in D(l/0)$. Moreover, $p \wedge l \in FI(l/0)$, by [9, Lemma 1.8 (3)]. This proves that $l/0$ is strongly extending. □

Proposition 3.4. *Let L be a strongly pseudo-complemented lattice and $1 = p\dot{\vee}q$, for some $p, q \in L$. Then the following statements are equivalent:*

(1) *L is strongly extending;*

(2) *each closed element t of L with $t \wedge q = 0$ or $t \wedge p = 0$ is a fully invariant complement.*

Proof. (1) ⇒ (2) It is clear by Theorem 3.1.

(2) ⇒ (1) We will show that, if $t \in C(L)$, then $t \in D(L) \cap FI(L)$. Put $c := t \wedge p$. Then there exists $e \in C(t/0)$ such that $c \in E(e/0)$, because $t/0$ is essentially closed by [8, Lemma 1.6, Lemma 1.14]. As $e \in C(t/0)$ and $t \in C(L)$, we have $e \in C(L)$, by [8, Lemma 1.6, Lemma 1.11]. Since $c \wedge q = 0$ and $c \in E(e/0)$, $e \wedge q = 0$. By (2), $e \in D(L) \cap FI(L)$. Hence,

$$e \vee e' = 1 \quad \text{and} \quad e \wedge e' = 0,$$

for some $e' \in L$. By modularity and $e \leq t$, we have $t = e \vee (e' \wedge t)$. By the previous argument, $e' \wedge t \in C(L)$. Since $c \in E(e/0)$ and $c \wedge e' = 0$, we have $(t \wedge e') \wedge p = 0$. By (2), $t \wedge e' \in D(L) \cap FI(L)$. Hence,

$$1 = (t \wedge e') \vee d \quad \text{and} \quad (t \wedge e') \wedge d = 0,$$

for some $d \in L$. Now, by modularity we have $e' = (t \wedge e') \vee (d \wedge e')$. Therefore,

$$1 = e \vee e' = e \vee (t \wedge e') \vee (d \wedge e') = t \vee (d \wedge e').$$

Moreover,

$$t \wedge (d \wedge e') = (t \wedge e') \wedge d = 0.$$

Thus, $t \in D(L)$. So L is extending by [8, Proposition 1.10 (4)]. Since $e \in FI(L)$ and $e' \wedge t \in FI(L)$, we have $t \in FI(L)$, by [9, Lemma 1.8 (1)]. Therefore, L is strongly extending by Theorem 3.1. □

Theorem 3.3. *Let L be a strongly pseudo-complemented lattice and $1 = m\dot{\vee}n$, for some $m, n \in L$. Then L is strongly extending provided that the following statements hold.*

(1) *$m/0$ and $n/0$ are strongly extending.*

(2) *For each sublattices H_1 of $m/0$, there is not a non-zero linear morphisms from H_1 to $n/0$.*

(3) *For each sublattice H_2 of $n/0$, there is not a non-zero linear morphisms from H_2 to $m/0$.*

Proof. Assume that k is a closed element of L with $k \wedge m = 0$. Let $\widetilde{p}_m : L \to m/0$ and $\widetilde{p}_n : L \to n/0$ be the canonical projections $\widetilde{p}_{n,m}$ and $\widetilde{p}_{m,n}$, respectively. We consider $\widetilde{p}_n|_{k/0} : k/0 \to n/0$, the restriction of $\widetilde{p}_n$ to $k/0$. Let $x = \ker(\widetilde{p}_n|_{k/0})$. Then $x \leq m = \ker(\widetilde{p}_n)$. Therefore, $x = 0$. Thus, $\widetilde{p}_n|_{k/0} : k/0 \to \widetilde{p}_m(k/0)$ is a linear monomorphism by [5, Corollary 1.6]. Therefore, $\widetilde{p}_n|_{k/0} : k/0 \to \widetilde{p}_n(k/0)$ is a lattice isomorphism (by definition of linear monomorphism). Let $\psi : \widetilde{p}_n(k/0) \to k/0$ be the inverse of $\widetilde{p}_n|_{k/0}$. Then we denote by θ the composition

$$\widetilde{p}_n|_{k/0}(k/0) \xrightarrow{\psi} k/0 \xrightarrow{\widetilde{p}_m|_{k/0}} m/0\,.$$

Since $\widetilde{p}_n|_{k/0}(k/0) \subseteq n/0$, we have $\theta = 0$, by our assumption. Therefore,

$$\widetilde{p}_m(\psi(\widetilde{p}_n|_{k/0}(k/0))) = \widetilde{p}_m(k/0) = 0.$$

Hence, $k \leq \ker(\widetilde{p}_m) = n$. Since $k \in C(L)$, $k \in C(n/0)$. Thus by strongly extending property of $n/0$, $k \in FI(n/0) \cap D(n/0)$. By [8, Proposition 1.7 (3)], $k \in D(L)$. By Lemma 3.2, $n \in FI(L)$, therefore $k \in FI(L)$, by [9, Lemma 1.8 (2)]. Hence, by Proposition 3.4, L is strongly extending. □

4. Applications to Grothendieck Categories

This section is allocated to employ the main results in Section 3 to Grothendieck categories. First, we recall some notations and terminology from [1–11]. In this section $\mathcal{G}$ will indicate a Grothendieck category. Let H be an object of $\mathcal{G}$. We will denote by $L(H)$, the upper continuous modular lattice of all subobjects of H ([11], [21, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]). According to [2], for any object H of $\mathcal{G}$, and for each subset $W \subseteq L(H)$, we denote

$$\bigwedge W = \bigcap_{E \in W} E, \quad \bigvee W = \sum_{E \in W} E.$$

We recall the next definition from [2], which is the key definition of this section.

Definition 4.1 ([2])**.** If $\mathbb{P}$ is a condition on lattices, then it is called $H \in \mathcal{G}$ is $\mathbb{P}$, provided that the lattice $L(H)$ satisfies $\mathbb{P}$. Further, a subobject H' of an object $H \in \mathcal{G}$ is $\mathbb{P}$ if the element H' of the lattice $L(H)$ satisfies $\mathbb{P}$.

Now, by Definition 4.1, one can define the concepts of a strongly extending object and fully invariant subobject, etc. Notice that we will use the term direct summand subobject instead of complement subobject.

By [6, Lemma 5.1], it is known that if $H_1, H_2 \in \mathcal{G}$ and $\theta : H_1 \to H_2$ is a morphism, then the canonical mapping $\varphi : L(H_1) \to L(H_2)$ defined by $\varphi(K) := \theta(K)$, for each $K \leq H_1$, is a linear morphism of lattices. Notice that, the notions of linear morphism and morphism are different. For any two objects H_1 and H_2, we denote by $LHom(H_1, H_2)$, the set of all linear morphisms $\psi : L(H_1) \to L(H_2)$.

In the following, we give some results.

Theorem 4.1. *If H is an object of a Grothendieck category $\mathcal{G}$, then H is strongly extending if and only if H is extending and every direct summand of H is fully invariant in H.*

Proposition 4.1. *Let $H = H_1 \oplus H_2$, where $H \in \mathcal{G}$ and H_1, H_2 are subobject of H. If H is strongly extending, then $Hom(H_1, H_2) = 0$ and $Hom(H_2, H_1) = 0$.*

Proof. Assume that $H = H_1 \oplus H_2$ and X is strongly extending. If $\theta : H_1 \to H_2$ is a morphism, then the map $\psi : L(H_1) \to L(H_2)$ defined by $\psi(A) := \theta(A)$, for each $A \leq H_1$, is a linear morphism ([6, Lemma 5.1]). By Corollary 3.3, $\psi = 0$, therefore $\theta = 0$. □

Theorem 4.2. *Assume that H is an object of a Grothendieck category $\mathcal{G}$ and H is strongly extending. Then the intersection of any family of direct summands of H is a direct summand of H.*

Theorem 4.3. *Let $H = H_1 \oplus H_2$, where $H \in \mathcal{G}$ and H_1, H_2 are subobject of H. If H_1 and H_2 are strongly extending and for each subobject K_1 of H_1 and K_2 of H_2, $LHom(K_1, H_2) = 0$ and $LHom(K_2, H_1) = 0$, then H is strongly extending.*

5. Applications to Modules with a Hereditary Torsion Theory

In this section, some applications of the results proved in Sections 3 to the category of modules with a fixed hereditary torsion class are given. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in $Mod-R$, and $\tau(M)$ the τ-torsion submodule of a module M. We recall some notations and terminology from [1–11]. For an R-module M, by $Sat_\tau(M)$, we will denote the set $\{K \mid K \leq M \text{ and } M/K \in \mathcal{F}\}$. Let $K \leq M$. Then by $\overline{K}$, we will denote the τ-saturation of K (in M) defined by $\overline{K}/K = \tau(M/K)$. Let K be submodule of M. Then K is said to be τ-saturated if $\overline{K} = K$. One can prove that $Sat_{\tau(M)} = \{K \mid K \leq M,\ \overline{K} = K\}$. By [21, Chapter 9, Proposition 4.1], it is known that for a right R-module M, $(Sat_\tau(M), \subseteq, \bigwedge, \bigvee, \tau(M), M)$ is an upper continuous modular lattice (the greatest element is M and the least element is $\tau(M)$) and $\bigvee$ and $\bigwedge$ defined as follows:

$$\bigvee_{i\in J} K_i = \overline{\sum_{i\in J} K_i} \quad \text{and} \quad \bigwedge_{i\in J} K_i = \bigcap_{i\in J} K_i.$$

We refer to [21] the reader for the discussion of torsion theoretical concepts and facts.

We recall the next definition from [2], which is the key definition of this section.

Definition 5.1 ([2])**.** Let $\mathbb{C}$ be a condition on lattices. Then it is called a right R-module M is $\tau - \mathbb{C}$ provided that the lattice $Sat_\tau(M)$ satisfies the condition $\mathbb{C}$.

Moreover, it is called a submodule K of a right R-module M is $\tau - \mathbb{C}$, provided that its τ-saturation $\overline{K}$, which is an element of $Sat_\tau(M)$, satisfies the condition $\mathbb{C}$.

Therefore, we can define the notions of a τ-strongly extending module, τ-Abelian module, etc, based on the Definition 5.1. By [2], we have the concepts of a τ-essential submodule of a module, τ-fully invariant submodules, etc. As $\overline{\overline{K}} = K$, we have K is $\tau - \mathbb{P}$ if and only if $\overline{K}$ is $\tau - \mathbb{P}$. It is known that K is τ-essential in M if and only if $H \cap K \in \mathfrak{T}$ implies that $H \in \mathfrak{T}$, for each $H \leq M$, by [2, Proposition 5.3], moreover, K is a τ-direct summand in M if and only if $M/(K+H) \in \mathfrak{T}$ and $K \cap H \in \mathfrak{T}$, for some $H \leq M$. In [6, Lemma 6.6], it is proved that, if $f : M \to N$ is a morphism of right R-modules, then the canonical mapping $f_\tau : Sat_\tau(M) \to Sat_\tau(N)$ defined by $f_\tau(X) = \overline{f(X)}$, for each $X \in Sat_\tau(M)$ is a linear morphism of lattices.

In the following, we give some results on the strongly τ-extending modules.

Theorem 5.1. *An R-module M is τ-strongly extending if and only if M is τ-CS (τ-extending) and every τ-direct summand of M is τ-fully invariant.*

Proof. Assume that M is τ-strongly extending. It suffices to prove that every τ-direct summand of M is τ-fully invariant. Let N be a τ-direct summand of M. Since M is τ-strongly extending, $Sat_\tau(M)$ is a strongly extending lattice. Hence, $\overline{N}$ is τ-essential in L, where L is fully invariant in lattice $Sat_\tau(M)$. As $\overline{N}$ is closed in $Sat_\tau(M)$, $\overline{N} = L$. Hence, N is τ-fully invariant in M. The converse is clear. □

Proposition 5.1. *Each τ-direct summand a τ-strongly extending module is τ-strongly extending.*

Theorem 5.2. *Suppose that M is a τ-strongly extending R-module and $H_1, H_2 \leq M$ ($H_1, H_2 \notin \mathfrak{T}$) such that $H_1 \cap H_2 \in \mathfrak{T}$, $M = H_1 + H_2$. If $f : H_i \to H_j$ is an R-homomorphism ($1 \leq i \neq j \leq 2$), then $f(H_i) \in \mathfrak{T}$.*

Proof. Since $M = H_1 + H_2$, we have

$$M = H_1 + H_2 \subseteq \overline{H_1} + \overline{H_2} \subseteq \overline{\overline{H_1} + \overline{H_2}}.$$

Therefore, $M = \overline{\overline{H_1} + \overline{H_2}}$. As $H_1 \cap H_2 \in \mathfrak{T}$, we have $\overline{H_1} \wedge \overline{H_2} = \overline{H_1} \cap \overline{H_2} = \tau(M)$. Therefore, $M = \overline{H_1} \dot{\vee} \overline{H_2}$. Let $f : H_1 \to H_2$ be a homomorphism of R-modules H_1 and H_2. Then the canonical mapping $f_\tau : Sat_\tau(H_1) \to Sat_\tau(H_2)$ defined by $f_\tau(X) = \overline{f(X)}$, for each $X \in Sat_\tau(H_1)$ is a linear morphism of lattices. By [3,4], there exist lattice isomorphisms $h : Sat_\tau(H_1) \to Sat_\tau(\overline{H_1})$ and $g : Sat_\tau(H_2) \to Sat_\tau(\overline{H_2})$. By [5, Proposition 2.2(2)], h, g are linear morphisms. Take $\varphi := g \circ f_\tau \circ h^{-1}$. By Corollary 3.3, $\varphi = 0$, thus $\overline{f(H_1)} = 0$, in $Sat_\tau(M)$. Thus, $f(H_1) \in \mathfrak{T}$. Similarly, if $f : H_2 \to H_1$ is a homomorphism between two R-modules H_2 and H_1, then we have $f(H_2) \in \mathfrak{T}$. □

Acknowledgements. The authors would like to thank the referee for the helpful comments which definitely help to improve the quality of the paper.

References

[1] T. Albu, *The Osofsky-Smith theorem for modular lattices and applications. I*, Comm. Algebra **39** (2011), 4488–4506. https://doi.org/10.1080/00927872.2011.616427

[2] T. Albu, *The Osofsky-Smith theorem for modular lattices, and applications. II*, Comm. Algebra **42** (2014), 2663–2683. https://doi.org/10.1080/00927872.2013.770520

[3] T. Albu, *Topics in Lattice Theory with Applications to Rings, Modules, and Categories*, in: *Lecture Notes, XXIII Brazilian Algebra Meeting, Maringa, Parana,* Akadémiai Kiadó, Brasil, 2014, page 80.

[4] T. Albu, *Chain Conditions in Modular Lattices with Applications to Grothendieck Categories and Torsion Theories*, Monograph Series of Parana's Mathematical Society No. 1, Sociedade Paranaense de Matematica, Maringa, Parana, Brasil, 2015.

[5] T. Albu and M. Iosif, *The category of linear modular lattices*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **56** (2013), 33–46.

[6] T. Albu and M. Iosif, *Lattice preradicals with applications to Grothendieck categories and torsion theories*, J. Algebra **444** (2015), 339–366.

[7] T. Albu and M. Iosif, *Modular C_{11} lattices and lattice preradicals*, J. Algebra Appl. **16** (2017), Article ID 1750116, 19 pages. https://doi.org/10.1142/S021949881750116X

[8] T. Albu, M. Iosif and A. Tercan, *The conditions (C_i) in modular lattices, and applications*, J. Algebra Appl. **15** (2016), 19 pages, Article ID 1650001. https://doi.org/10.1142/S0219498816500018

[9] T. Albu, Y. Kara and A. Tercan, *Fully invariant-extending modular lattices, and applications I*, J. Algebra **517** (2019), 207–222. https://doi.org/10.1016/j.jalgebra.2018.08.036

[10] T. Albu, Y. Kara and A. Tercan, *Strongly fully invariant-extending modular lattices*, Quaest. Math. (2020), 1–11. https://doi.org/10.2989/16073606.2020.1861488

[11] T. Albu and C. Năstăsescu, *Relative Finiteness in Module Theory*, Vol. 84, Marcel Dekker, Inc., New York, 1984.

[12] G. F. Birkenmeier, B. J. Müller and S. T. Rizvi, *Modules in which every fully invariant submodule is essential in a direct summand*, Comm. Algebra **30** (2002), 1395–1415. https://doi.org/10.1080/00927870209342387

[13] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, *Modules with fully invariant submodules essential in fully invariant summands*, Comm. Algebra **30** (2002), 1833–1852. https://doi.org/10.1080/00927870209342387

[14] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, *Modules with FI-extending hulls*, Glasg. Math. J. **51** (2009), 347–357. https://doi.org/10.1017/S0017089509005023

[15] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, *Extensions of Rings and Modules*, Birkhäuser, Springer, New York, 2013. https://doi.org/10.1007/978-0-387-92716-9

[16] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, 1967.

[17] P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[18] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics Series **313**, Longman Scientific Technical, Harlow, copublished in the United States with John Wiley Sons, Inc., New York, 1994. https://doi.org/10.1201/9780203756331

[19] S. Ebrahimi Atani, M. Khoramdel and S. Dolati Pish Hesari, *On strongly extending modules*, Kyungpook Math. J. **54** (2014), 237–247. https://doi.org/10.5666/KMJ.2014.54.2.237

[20] G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, 2003.

[21] B. Stenström, *Rings of Quotients. An Introduction to Methods of Ring Theory*, Lecture Notes in Mathematics **237**, Springer-Verlag, Berlin, New York, 1975.

[1]Department of Mathematics,
University of Guilan,
P.O.Box 1914,Rasht, Iran
Email address: ebrahimi@guilan.ac.ir
Email address: mehdikhoramdel@gmail.com
Email address: saboura_dolati@yahoo.com
Email address: mhs.nikmard@gmail.com