Kragujevac Journal of Mathematics
Volume 51(5) (2027), Pages 875–885.

HYPERSTRUCTURAL COMPLETENESS: HYPERMODULES WITH SUPPLEMENTS IN EVERY EXTENSION

BURCU NİŞANCI TÜRKMEN

Abstract. In this work, we introduce and investigate the concept of Krasner hypermodules that have the properties (E) and (EE). The motivation for this study stems from the fundamental observation that the concept of Krasner hypermodule is closed under direct summand and extension. So, we establish several fundamental results concerning Krasner hypermodules have the properties (E) and (EE). Furthermore, we provide a characterization for a linearly compact Krasner hypermodule, the Noetherian property is equivalent to the vanishing of all its radical subhypermodules, thereby characterizing the Noetherian hypermodule in terms of radical. These results not only generalize classical module theory concepts to the hypermodule theory but also deepen the understanding of the interplay between normal injectivity, extension, and semisimplicity in hypercompositional algebra.

1. Introduction

For a clearer exposition, we begin by recalling several fundamental definitions from hypercompositional algebra, as presented in survey articles. Let H be a nonempty set and let $\mathcal{P}(H)$ denote the power set of H. The pair $(H, \circ)$ is called a hypergroupoid if the hyperoperation on H is a function $\circ : H \times H \to \mathcal{P}(H)$. For any subsets X and Y of H, one defines $X \circ Y = \cup_{x\in X,\, y\in Y} x \circ y$. We simply write $a \circ X$ and $X \circ a$ instead of $\{a\} \circ X$ and $X \circ \{a\}$, respectively, for any $a \in H$ and any nonempty subset X of H [12]. A hypergroupoid $(H, \circ)$ is called a semihypergroup if the hyperoperation $\circ$ is associative, i.e., for every $a, b, c \in H$, we have $a \circ (b \circ c) = (a \circ b) \circ c$.

Key words and phrases. Krasner hyperring, Krasner hypermodule, supplement, extension
2020 *Mathematics Subject Classification.* Primary: 20N20. Secondary: 20J99.
DOI
Received: July 26, 2025.
Accepted: January 12, 2026.

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if the reproduction law holds, i.e., for every $x \in H$, $x \circ H = H = H \circ x$. If the hypergroupoid $(H, \circ)$ is a semihypergroup and a quasihypergroup, then it is called a hypergroup. A nonempty subset S of a hypergroup $(H, \circ)$ is called a subhypergroup of H if, for every $a \in S$, $a \circ S = S = S \circ a$ [12]. A canonical hypergroup is a hypergroup $(H, \circ)$ satisfying the following conditions:

(i) it is commutative, i.e., for every $a, b \in H$, $a \circ b = b \circ a$;

(ii) there exists $e \in H$ such that $\{a\} = (a \circ e) \cap (e \circ a)$ for every $a \in H$ (such an element e is called an identity of the hypergroup);

(iii) for every $a \in H$ there exists a unique $a^{-1} \in H$ such that $e \in a \circ a^{-1}$ (the element a^{-1} is called the inverse of a);

(iv) for every $a, b, c \in H$, if $c \in a \circ b$, then $a \in c \circ b^{-1}$ and $b \in a^{-1} \circ c$ [5].

In, [11] an algebraic system $(R, +, \cdot)$ is called a Krasner hyperring, if

(i) $(R, +)$ is a canonical hypergroup;

(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $a \cdot 0 = 0 = 0 \cdot a$ for any $a \in R$;

(iii) the multiplication distributes over the addition on both sides, i.e., for any $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

A hyperring R is called commutative, if it is commutative with respect to the multiplication. If $a \in a \cdot 1_R$ for every $a \in R$, then the element 1_R is called a unit (identity) element of the hyperring R [6].

In [11], a left R-hypermodule is defined as an algebraic system $(H, +, \cdot)$ endowed with an external multivalued operation $\cdot$, i.e., $\circ : R \times H \to \mathcal{P}(H)$ such that, for every $x, y \in R$ and $a, b \in H$, the following statements hold:

(i) $x \circ (a + b) = x \circ a + x \circ b$;
(ii) $(x + y) \circ a = x \circ a + y \circ a$;
(iii) $(x \cdot y) \circ a = x \circ (y \circ a)$;
(iv) $a \in 1_R \circ a$.

Similarly, the concept of right R-hypermodule is defined and we say that $(H, +, \circ)$ is an R-hypermodule if it is a left and right one. Some authors call this hypercompositional structure a general hypermodule. A nonempty subset N of an R-hypermodule H is called a subhypermodule of H, if N is an R-hypermodule under the same hyperoperations of H and we denote this as $N \leq H$. In other words, N is a subhypermodule of H if and only if $x \circ a \subseteq N$ and $a - b \in N$ for every $x \in R$ and $a, b \in N$ [11]. If we consider a Krasner hyperring R, then we may endow a canonical hypergroup $(H, +)$ with an external operation $\cdot : R \times H \to H$ defined as $(r, a) \mapsto r \cdot a \in H$. If, for every $x, y \in R$ and $a, b \in H$, the following statements hold:

(i) $x \cdot (a + b) = x \cdot a + x \cdot b$;
(ii) $(x + y) \cdot a = x \cdot a + y \cdot a$;
(iii) $(x \cdot y) \cdot a = x \cdot (y \cdot a)$;
(iv) $a = 1_R \cdot a$;

(v) $x \cdot 0_=0_R$;

then H is called a Krasner left R-hypermodule [14].

It is well known that a module M is injective if and only if it is a direct summand of every extension N of M. Since every direct summand is a supplement, Zöschinger studied in [24] modules that have a supplement in every extension and termed these modules the property (E) as a generalization of injective modules. In particular, he proved in [24, Lemma 1.2] that every submodule of a module M has the property (E) if and only if M has ample supplements in every extension, namely the property (EE). It is obvious that the class of modules with the property (EE) contains properly artinian modules. Now we generalize this concept to Krasner hypermodules.

In the study unless otherwise stated, we denote as R, a Krasner hyperring with the identity element 1_R. A nonempty subset J of a commutative hyperring R is called a hyperideal, if $x - y \subseteq J$ and $a \cdot x \in J$ for every $a \in R$ and $x, y \in J$. It is well known that every hyperideal J of a hyperring R is a subhypermodule of the R-hypermodule R. Let H be a left Krasner hypermodule over a Krasner hyperring R and K be a subhypermodule of H. Consider the set $\frac{H}{K} = \{ a + K \mid a \in H \}$. Then $\frac{H}{K}$ is a left Krasner hypermodule over R under the hyperoperation defined as $+ : \frac{H}{K} \times \frac{H}{K} \to \mathcal{P}(\frac{H}{K})$ and the external operation $\circ : R \times \frac{H}{K} \to \frac{H}{K}$ defined as $(a + K) + (a^{'} + K) = \{ b + K \mid b \in a + a^{'} \}$ and $x \circ (a + K) = \{ b + K \mid b \in x \cdot a \}$ for every $a, a^{'}, b \in H$ and $x \in R$. The hypermodule $\frac{H}{K}$ is called the quotient (factor) hypermodule of the hypermodule H. Note that $a + K = K$ if and only if $a \in K$ [8]. A nonzero R-hypermodule H is called simple, if the only subhypermodules of H are $\{0_H\}$ and H itself in [22]. We denote by $Soc(H)$ the set of all simple subhypermodules of the R-hypermodule H. Let H and $H^{'}$ be two left Krasner R-hypermodules. A function $f : H \to H^{'}$ is called a homomorphism if for every $a, b \in H$ and $r \in R$, it holds $f(a+b) \subseteq f(a)+f(b)$ and $f(r \circ a) = r \circ f(a)$, while it is called a normal homomorphism if $f(a + b) = f(a) + f(b)$ and $f(r \circ a) = r \circ f(a)$. For any subhypermodule N of a left Krasner R-hypermodule H, the image $f(N)$ is a subhypermodule of $H^{'}$ and the kernel $ker(f) = \{ a \in H \mid f(a) = 0_{H'} \}$ is a subhypermodule of H [9]. Two subhypermodules N and $N^{'}$ of a left Krasner R-hypermodule H are called independent, if $N \cap N^{'} = \{0_H\}$ and in this case their sum $N + N^{'}$ is denoted by $N \oplus N^{'}$ and called direct sum. Moreover, a subhypermodule N of H is called a direct summand of H if $H = N \oplus K$ for some subhypermodule K of H [20]. A subhypermodule N of a left Krasner R-hypermodule H is called a small subhypermodule of H and denoted by $N \ll H$, if $N + L \neq H$ for every proper subhypermodule L of H. We refer the reader to [20] for basic properties related to small subhypermodules.

If $f : H \to H^{'}$ is a normal epimorphism and $\ker(f) \ll H$, then f is called a small normal epimorphism. $Rad(H)$ is defined as the sum of all small subhypermodules of the left Krasner R-hypermodule H, i.e., $Rad(H) = \sum_{L \ll H} L$. If H has no small subhypermodules of H, then it is denoted by $Rad(H) = H$. Notice that $Rad(H)$ is always a subhypermodule of the left Krasner R-hypermodule H and H is local if and only if H is hollow and $Rad(H) \neq H$ [8]. A Krasner R-hypermodule H is called normal projective if for every normal epimorphism $g : T \to K$ and every normal homomorphism $f : H \to K$ there exists $\psi : H \to T$ such that $g \circ \psi = f$. A Krasner R-hypermodule H is called normal injective if for every normal monomorphism $g : T \to K$ and every $f : T \to K$ and every $f : T \to H$, there exists $\phi : K \to H$ such that $\phi \circ g = f$ [2]. Let H be a left Krasner R-hypermodule and U, V be subhypermodules of H. V is called a supplement of U in H, if it is a minimal element in the set $\{L \leq H \mid U + L = H\}$, equivalently $U + V = H$ and $U \cap V \ll V$.

H is called supplemented if every subhypermodule of H has a supplement in H [8].

A subhypermodule U of a left Krasner R-hypermodule H has ample supplements in H if every subhypermodule V such that $U + V = H$ contains a supplement of U in H.

The left Krasner R-hypermodule H is called ample supplemented if every subhypermodule of H has ample supplements in H.

These notions were first introduced for general hypermodules in [8], where several illustrative examples can also be found.

2. Krasner Hypermodules Satisfying Properties (E) and (EE)

The aim of the present paper is to investigate several generalizations of injective hypermodules formulated in terms of the existence of supplements in every extension. These generalizations arise naturally within the category theory of canonical hypergroups and Krasner hypermodules, which was established in [1]. The notion of supplements, as well as that of ample supplemented Krasner hypermodules, was introduced and systematically studied in [8]. Furthermore, the concepts of normal projective and normal injective Krasner hypermodules were characterized in detail in [3]. More recently, the interplay between supplement (and ample supplement) conditions and normal π-projective Krasner hypermodules has been examined in [18]. Building upon these foundational results, the present work aims to extend and unify these approaches, thereby providing a broader perspective on normal injectivity properties in the theory of Krasner hypermodules. The categorical framework of canonical hypergroups and Krasner hypermodules was first systematically established in [1], providing a foundation of a hyperstructure for the study of hypercompositional algebra. Within the notion of supplements and ample supplements for Krasner hypermodules was introduced in [8], extending classical module theory to hypermodule theory. Subsequently, the notions of normal projective and normal injective Krasner hypermodules were thoroughly investigated and characterized in [3], highlighting their homological significance and structural properties. More recently, the interaction between supplement (and ample supplement) conditions and normal π-projective Krasner hypermodules has been explored in [18], revealing deep connections between supplement notion and π-projectivity in Krasner hypermodules. These developments collectively form the theoretical foundation upon which the present study is built.

Now consider a minimal element in the following partially ordered set: $\{ U \mid U \subset H, U + V = H \}$.

A Krasner hypermodule H is said to be Artinian (respectively, Noetherian) if it satisfies either of the following equivalent conditions.

(i) Every nonempty family of subhypermodules of H contains a minimal (respectively, maximal) element with respect to set inclusion.

(ii) Every descending (respectively, ascending) chain of subhypermodules of H stabilizes.

Moreover, a Krasner hypermodule H is Artinian (respectively, Noetherian) if and only if it is Artinian (respectively, Noetherian) when regarded as a hypermodule over its Krasner hyperring. The notion of a Noetherian hyperring and its fundamental properties have been further investigated in [4].

For Artinian Krasner hypermodule V, it is even stronger: if $X + V = H$, then there is a supplement (minimal element which satisfy the equality $X + V = H$) of V in H that is contained in X, i.e., V has ample supplements in H.

In what follows, we introduce and investigate the following external supplement properties for a Krasner hypermodule V:

(i) has a supplement in every extension;
(ii) has ample supplements in every extension.

Definition 2.1. Let V be a Krasner hypermodule. We call V that satisfies the first property is said to satisfy property (E), while a hypermodule that satisfies the second property is said to satisfy property (EE).

In discrete topology, the notion of a linearly compact module over a topological ring plays a fundamental role in module theory and topological algebra. More precisely, a module is said to be linearly compact if it is a Hausdorff module endowed with a linear topology, meaning that it possesses a neighborhood basis of the zero element consisting entirely of open submodules. In addition to this topological structure, the defining feature of linear compactness is a compactness condition: every family of closed cosets that satisfies the finite intersection property namely, every finite subfamily has a nonempty intersection must itself have a nonempty intersection. This condition ensures a strong form of completeness and allows linear compact modules to be viewed as a natural generalization of compactness in the context of modules equipped with linear topologies.

The classical notion of linear compactness for modules over topological rings relies essentially on the interaction between algebraic structure and linear topology. In order to extend this concept to the setting of hypermodules, where the underlying operations are multivalued, a careful methodological adaptation is required. The key idea is to replace single valued algebraic operations with their hyperstructural counterparts while preserving extension properties that underlie linear compactness. In particular, open submodules are naturally generalized to open subhypermodules, and cosets are interpreted in the sense of hyperadditive translations. The finite intersection property, which characterizes compactness type behavior, remains meaningful in this broader context when formulated in terms of inverse families of subhypermodules. Consequently, linear compactness for Krasner hypermodules is defined by requiring that intersections of inverse systems of subhypermodules commute with hyperadditive extensions. This approach ensures that the hypermodule analogue retains the essential completeness and minimality properties of the classical theory, thereby providing a coherent and robust framework for extending supplement and normal injectivity concepts to hypercompositional algebra.

Example 2.1. Let $R = \{0, 1, 2\}$. Define the hyperoperation "+" and the multiplication "·" by the following tables:

+	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	R
2	$\{2\}$	R	$\{2\}$

and

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

.

Then, $A = \{0, 1\}$ and $C = \{0, 2\}$ are only proper subhypermodules of the R-hypermodule R. Therefore, $A \oplus C = R$ and so, by [21, Theorem 3], every left R-hypermodule is (ample) supplementing.

Definition 2.2. Let V be a Krasner hypermodule over a hyperring R with identity element 1_R. The hypermodule V is said to be linearly compact if, whenever $V \subseteq H$ and $\{H_i \mid i \in I\}$ is an inverse family of subhypermodules of H, the equality

$$\left(\bigcap_{i\in I} H_i\right) + V = \bigcap_{i\in I}(H_i + V)$$

holds. In particular, for every subhypermodule X satisfying $X + V = H$, the set

$$\{\, T \mid T \subseteq X,\ T + V = H \,\}$$

is inductive. Hence, by Zorn's Lemma, it admits a minimal element.

Remark 2.1. Every linearly compact Krasner hypermodule has the property (EE).

Example 2.2. (See [10, Example 3.4]) Consider the normal injective $\mathbb{N}$-hypermodule $\mathbb{N}_{\mathfrak{p}^\infty}$. Since $\mathbb{N}_{\mathfrak{p}^\infty}$ is artinian, then it is linearly compact. By Remark 2.1, $\mathbb{N}_{\mathfrak{p}^\infty}$ has the property (EE).

It is clear that every hypermodule satisfying the property (EE) has the property (E). However, the converse does not hold in general; in particular, a hypermodule satisfying the property (E) need not be normal injective.

Lemma 2.1. *Every simple Krasner hypermodule has the property* (E).

Proof. Let V be a simple Krasner hypermodule and let H be an arbitrary extension of V. Since V is simple, either $V \ll H$ or there exists a subhypermodule K of H such that $H = V \oplus K$. If $V \ll H$, then H itself is a supplement of V in H. Otherwise,

K is a supplement of V in H. In both cases, V has a supplement in every extension, and hence V has the property (E). □

Example 2.3 (See [10]). Let $\mathbb{N}$ denote the set of all non-negative integers. Let $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{N}$. As in [21, Construction], we define the hyperoperation "$\oplus$" on $\mathbb{N}$ as follows: for any $m, n \in \mathbb{N}$,

$$m \oplus n = \{m + n,\, k \mid \min\{m,\, n\} + k = \max\{m,\, n\} \text{ for some } k \in \mathbb{N}\}.$$

Thus the hyperstructure $(\mathbb{N}, \oplus, \cdot)$ is a principal hyperideal domain. It follows from [21] that the hyperring $(\mathbb{N}, \oplus, \cdot)$, considered as an $\mathbb{N}$-hypermodule and the factor hypermodule $\frac{\mathbb{N}}{p\mathbb{N}}$ is simple for all $p \in \mathbb{P}$. By [21], $\frac{\mathbb{N}}{p\mathbb{N}}$ has the property (EE).

Proposition 2.1. *Every direct summand of a Krasner hypermodule has the property (E) has itself the property (E).*

Proof. Let H be a Krasner hypermodule satisfying the property (E) and A be a direct summand of H. Suppose that

$$H = A \oplus B$$

is a direct sum decomposition of hypermodules and assume that H is supplementing. We show that A has the property (E).

Let N be any extension of A, i.e., $A \subseteq N$. Consider the hypermodule

$$E = N \oplus B.$$

Since $H = A \oplus B$, we may regard H as a subhypermodule of E.

Because H has the property (E) and E is an extension of H, there exists a subhypermodule $T \leq E$ such that

$$E = H + T \quad \text{and} \quad H \cap T \ll T.$$

Let $\pi : E = N \oplus B \to N$ be the canonical projection and $L = \pi(T) \leq N$.

(1) $N = A + L$. Let $n \in N$. Then, $(n, 0) \in E = H + T$. Hence, there exist $a \in A$, $b \in B$, and $t \in T$ such that $(n, 0) \in (a, b) + t$. Applying π, we obtain $n \in a + \pi(t)$. Thus, $n \in A + L$, and therefore $N = A + L$.

(2) $A \cap L \ll L$. Since $H \cap T \ll T$ and π is a normal epimorphism, smallness is preserved under projection. Hence, $A \cap L \ll L$. Therefore, L is a supplement of A in N. Since N was arbitrary, A has the property (E). Thus, every direct summand of a hypermodule has the property (E) has the property (E). □

The next result shows that the class of Krasner hypermodules satisfying the property (EE) is closed under extensions.

Theorem 2.1. *Let $T \subseteq H$ be Krasner hypermodules. If both T and $\frac{H}{T}$ have the property (EE), then H has the property (EE).*

Proof. Let K be an arbitrary extension of H. Since $T \subseteq H \subseteq K$, we have the normal isomorphism

$$\frac{K}{H} \simeq \frac{\frac{K}{T}}{\frac{H}{T}},$$

which shows that $\frac{K}{T}$ is an extension of $\frac{H}{T}$.

By hypothesis, the subhypermodule $\frac{H}{T}$ has a supplement, say $\frac{L}{T}$, in $\frac{K}{T}$. Thus,

$$\frac{H}{T} + \frac{L}{T} = \frac{K}{T} \quad \text{and} \quad \frac{H}{T} \cap \frac{L}{T} = \frac{H \cap L}{T} \ll \frac{L}{T},$$

and consequently $K = H + L$.

Furthermore,

$$\frac{L/H}{(H \cap L)/H} \cong \frac{L}{H \cap L} \cong \frac{H + L}{H} = \frac{K}{H}.$$

Since T has the property (EE), there exists a subhypermodule $T^{'} \subseteq L$ such that

$$T + T^{'} = L \quad \text{and} \quad T \cap T^{'} \ll T^{'}.$$

Hence,

$$K = H + L = H + (T + T^{'}) = H + T^{'}.$$

Suppose that $H + T^{''} = K$ for some subhypermodule $T^{''} \subseteq T^{'}$. Then, $T + T^{''} \subseteq L$. Since $\frac{L}{T}$ is a supplement of $\frac{H}{T}$ in $\frac{K}{T}$, it follows that $T + T^{''} = L$. By the minimality of $T^{'}$, we obtain $T^{''} = T^{'}$. Therefore, H has the property (EE). □

As a direct consequence of Theorem 2.1, every finitely generated semisimple Krasner hypermodule has the property (EE).

Corollary 2.1. *Let H be a Krasner hypermodule and let T be a maximal subhypermodule of H. If T has the property (EE), then H has the property (EE). In particular, any Krasner hypermodule containing a simple maximal subhypermodule has the property (EE).*

Proof. Let T be Krasner hypermodule satisfying the property (EE). Since simple Krasner hypermodules have the property (EE), the factor hypermodule $\frac{H}{T}$ has also the property (EE). The assertion now follows directly from Theorem 2.1. □

Now, we show that the class of Krasner hypermodules satisfying the property (EE) is closed under finite direct sums.

Proposition 2.2. *Let $\{H_i \mid i = 1, \ldots, n\}$ be a finite family of subhypermodules of a Krasner hypermodule T satisfying the property (EE) and*

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n.$$

Then, H has the property (EE).

Proof. We proceed by induction on n. For $n = 2$, let $H = H_1 \oplus H_2$. Then, $H_2 \cong \frac{H}{H_1}$. By hypothesis and Theorem 2.1, it follows that H has the property (EE). The general case follows by an obvious induction argument. □

Definition 2.3. A Krasner hypermodule V is called radical (respectively, socle-free) if $Rad(V) = V$ (respectively, $Soc(V) = \{0\}$).

Proposition 2.3. *Let V be a linearly compact Krasner hypermodule. Then, V is Noetherian if and only if every radical subhypermodule of V is zero.*

Proof. ($\Rightarrow$) Assume that V is Noetherian. Then every subhypermodule of V is Noetherian. Let W be a radical subhypermodule of V. Since $Rad(V)$ is the intersection of all maximal subhypermodules of V, and V is Noetherian, it follows that $Rad(V) = 0$. Hence $W = 0$.

($\Leftarrow$) Conversely, suppose that V has no nonzero radical subhypermodules. Then, $Rad(V) \ll V$. Indeed, let W be a supplement of $Rad(V)$ in V, and let T be a supplement of W in V. Then, both T and V/W are radical hypermodules. By hypothesis, this implies $T = 0$ and hence $W = V$. Therefore, $Rad(V)$ is small in V.

Moreover, the factor hypermodule $V/Rad(V)$ is semisimple and linearly compact, and hence finitely generated. It follows that V itself is finitely generated. Since every subhypermodule of V is finitely generated, V is Noetherian. ☐

Proposition 2.4. *Let J be a hyperideal of a hyperring R and set $\overline{R} = R/J$. If a Krasner R-hypermodule V has the property (E) with $V \cdot J = \{0\}$, then the Krasner $\overline{R}$-hypermodule V has the property (E).*

Proof. By assumption, $V \cdot J = \{0\}$, and hence V naturally inherits the structure of a Krasner $\overline{R}$-hypermodule. Let H be any extension of V regarded as a Krasner $\overline{R}$-hypermodule, so that $V \subseteq {}_{\overline{R}}H$.

Since a Krasner R-hypermodule V has the property (E), there exists a subhypermodule T of H such that $V + T = H$ and $V \cap T \ll T$. It follows immediately that T, viewed as a $\overline{R}$-subhypermodule of H, is a supplement of V in H. Therefore, the Krasner $\overline{R}$-hypermodule V has the property (E). ☐

3. Conclusion

In this paper, we have introduced and systematically studied the extension of supplement properties (E) and (EE) for Krasner hypermodules as natural generalizations of injectivity within the framework of hypercompositional algebra. These properties were formulated in terms of the existence of supplements and ample supplements in every extension, extending classical results from module theory to the setting of Krasner hypermodules.

We established several fundamental closure properties of hypermodules satisfying (EE), showing that this class is stable under direct summands, finite direct sums, and extensions. In particular, we proved that every linearly compact Krasner hypermodule satisfies property (EE) and obtained a characterization of Noetherian linearly compact Krasner hypermodules in terms of the vanishing of their radical subhypermodules. This result provides a hypermodule theory analogue of classical characterizations of Noetherian modules via radicals and highlights the structural role played by small subhypermodules in hypercompositional algebra.

Furthermore, we demonstrated that simple Krasner hypermodules satisfy property (E), while property (EE) strictly strengthens (E) in general, thereby clarifying the hierarchy between these notions and normal injectivity. Several examples were provided to illustrate the theory and to show that the developed concepts are nontrivial and widely applicable. For future research include the investigation of properties (E) and (EE) in broader classes of hypermodules, such as π-projective Krasner hypermodules, as well as their interaction with homological invariants and categorical constructions. Another promising direction is the study of these properties over noncommutative hyperrings and their applications to the representation theory of hyperstructures. We believe that the results obtained here contribute to a deeper understanding of extension theory and supplement conditions in hypermodule theory and open new avenues for further developments in hypercompositional algebra.

Acknowledgements. This research was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK, Project number: 125F181).

References

[1] R. Ameri, *On categories of hypergroups and hypermodules*, Journal of Discrete Mathematical Sciences-Cryptography **2**(3) (2003), 121–132. https://doi.org/10.1080/09720529.2003.10698084

[2] R. Ameri and H. Shojaei, *Projective and injective Krasner hypermodules*, J. Algebra Appl. **20**(10) (2021), 443–458. https://doi.org/10.1142/S0219498821501863

[3] H. Bordbar and I. Cristea, *About the normal projectivity and injectivity of Krasner hypermodules*, Axioms **10**(83) (2021), 443–458. https://doi.org/10.3390/axioms10020083

[4] H. Bordbar, I. Cristea and M. Novak, *Height of hyperideals in noetherian Krasner hyperrings*, U.P.B. Sci. Bull., Series A **79**(2) (2017), 31–42.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, Italy, 1993.

[6] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press: Palm Harbor, FL, USA, 2007.

[7] A. R. M. Hamzekolaee and M. Norouzi, *A hyperstructural approach to essentially*, Commun. Algebra **46** (2018), 4954–4964. https://doi.org/10.1080/00927872.2018.1448845

[8] A. R. M. Hamzekolaee, M. Norouzi and V. Leoreanu-Fotea, *A new approach to smallness in hypermodules*, Algebraic Structures and Their Applications **8**(1) (2021), 131–145.

[9] Z. Jianming, B. Davvaz and K. P. Shum, *Isomorphism theorems of hypermodules*, Acta Math. Sinica (Chin. Ser.) **50**(4) (2007), 909–914.

[10] E. Kaynar, B. N. Türkmen and E. Türkmen, *A note on hyperrings and hypermodules*, Italian Journal of Pure and Applied Mathematics **53** (2025), 104–116.

[11] M. Krasner, *A class of hyperrings and hyperfields*, Int. J. Math. Sci. **6**(2) (1983), 307–312.

[12] F. Marty, *Sur une généralisation de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45–49.

[13] Ch. G. Massouros, *Methods of constructing hyperfields*, Intern. J. Math. Sci. **8**(4) (1985), 725–728.

[14] Ch. G. Massouros, *Free and cyclic hypermodules*, Ann. Math. Pure Appl. **159** (1988), 153–166.

[15] Ch. G. Massouros and C. Massouros, *Hypercompositional algebra, computer sciences and geometry*, Mathematics **8** (2021), 1–31. https://doi.org/10.3390/math8081338

[16] C. Massouros and Ch. G. Massouros, *An overview of the foundations of the hypergroup theory*, Mathematics **9**(9) (2021), 1–41. https://doi.org/10.3390/math9091054

[17] B. N. Türkmen, H. Bordbar and I. Cristea, *Supplements related to normal π-projective hypermodules*, Mathematics **10** (2022), 1–15. https://doi.org/10.3390/math10010015

[18] B. N. Türkmen and B. Davvaz, *On τ-supplemented Krasner hypermodules*, Italian Journal of Pure and Applied Mathematics **53** (2025), 185–199.

[19] M. De Salvo, *Hyperrings and hyperfields*, Annales Scientifiques de l'Université de Clermont-Ferrand II, Mathématiques **22** (1984), 89–107.

[20] B. Talaee, *Small subhypermodules and their applications*, Rom. J. Math. Comput. Sci. **4**(3) (2013), 5–14.

[21] E. Türkmen, B. N. Türkmen and H. Bordbar, *A characterization of normal injective and normal projective hypermodules*, Axioms **13**(6) (2024), 1–11. https://doi.org/10.3390/axioms13060410

[22] E. Türkmen, B. N. Türkmen and H. Bordbar, *A hyperstructural approach to semisimplicity*, Axioms **13**(2) (2024), 1–16. https://doi.org/10.3390/axioms13020081

[23] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Düsseldorf, Springer, 1991.

[24] H. Zöschinger, *Moduln, die in jeder erweiterung ein komplement haben*, Math. Scand. **35** (1974), 267–287.

Department of Mathematics,
Amasya University, Faculty of Science and Arts,
İpekköy Campuse, Amasya, Turkey
Email address: burcu.turkmen@amasya.edu.tr