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ZERO-ANNIHILATOR GRAPHS OF COMMUTATIVE RINGS

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Abstract. Assume that R is a commutative ring with nonzero identity. In this paper, we introduce and investigate *zero-annihilator graph of* R denoted by $\mathtt{ZA}(R)$. It is the graph whose vertex set is the set of all nonzero nonunit elements of R and two distinct vertices x and y are adjacent whenever $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \{0\}$.

1. Introduction

Throughout this paper all rings are commutative with nonzero identity. In [6], Beck associated to a ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0), and two distinct vertices x and y are adjacent if $xy = 0$. Later, in [3], Anderson and Livingston studied the subgraph $\Gamma(R)$ of $G(R)$ (whose vertices are the nonzero zero-divisors of R). In the recent years, several researchers have done interesting and enormous works on this field of study. For instance, see [4, 5, 9]. The concept of co-annihilating ideal graph of a ring R, denoted by $\mathcal{A}_R$ was introduced by Akbari et al. in [1]. As in [1], *co-annihilating ideal graph of* R, denoted by $\mathcal{A}_R$, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\mathrm{Ann}_R(I) \cap \mathrm{Ann}_R(J) = \{0\}$. In the present paper, we introduce *zero-annihilator graph of* R denoted by $\mathtt{ZA}(R)$. It is the graph whose vertex set is the set of all nonzero nonunit elements of R and two distinct vertices x and y are adjacent whenever $\mathrm{Ann}_R(Rx + Ry) = \mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \{0\}$. Note that $\mathtt{ZA}(R)$ is an induced subgraph of $\mathcal{A}_R$.

Let G be a simple graph with the vertex set $\mathrm{V}(G)$ and edge set $\mathrm{E}(G)$. For every vertex $v \in \mathrm{V}(G)$, $\mathrm{N}_G(v)$ is the set $\{u \in \mathrm{V}(G) \mid uv \in \mathrm{E}(G)\}$. The *degree* of a vertex v is defined as $\deg_G(v) = |\mathrm{N}_G(v)|$. The *minimum degree* of G is denoted

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by $\delta(G)$. Recall that a graph G is *connected* if there is a path between every two distinct vertices. For distinct vertices x and y of a connected graph G, let $\mathrm{d}_G(x,y)$ be the length of the shortest path from x to y. The *diameter* of a connected graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}_G(x,y) \mid x \text{ and } y \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\mathrm{girth}(G)$, is defined as the length of the shortest cycle in G and $\mathrm{girth}(G) = \infty$ if G contains no cycles. A *bipartite graph* is a graph all of whose vertices can be partitioned into two parts U and V such that every edge joins a vertex in U to a vertex in V. A *complete bipartite graph* is a bipartite graph with parts U, V such that every vertex in U is adjacent to every vertex in V. A graph in which all vertices have degree k is called a *k-regular graph*. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Also, if a graph G contains one vertex to which all other vertices are joined and G has no other edges, is called a *star graph*. A *clique* in a graph G is a subset of pairwise adjacent vertices and the number of vertices in a maximum clique of G, denoted by $\omega(G)$, is called the *clique number* of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously, $\chi(G) \geq \omega(G)$.

2. Some Properties of $\mathtt{ZA}(R)$

Recall that, an *empty graph* is a graph with no edges. A *Bézout ring* is a ring in which all finitely generated ideals are principal.

Theorem 2.1. *Let R be a ring. If $\mathtt{ZA}(R)$ is an empty graph, then R is a local ring and $\mathrm{Ann}_R(x) \neq \{0\}$ for every nonunit element $x \in R$. The converse is true if R is a Bézout ring.*

Proof. Assume that $\mathtt{ZA}(R)$ is empty. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two distinct maximal ideals of R. Then $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ implies that there exist $x \in \mathfrak{m}_1$ and $x_2 \in \mathfrak{m}_2$ such that $x + y = 1$. So x and y are adjacent, which is a contradiction. Hence R is a local ring. Let $\mathfrak{m}$ be the maximal ideal of R and x be an element of $\mathfrak{m}$. Suppose that $\mathrm{Ann}_R(x) = \{0\}$. Then $\{x^n \mid n \in \mathbb{N}\}$ is an infinite clique in $\mathtt{ZA}(R)$ that is a contradiction. So $\mathrm{Ann}_R(x) \neq \{0\}$.

Suppose that R is a local Bézout ring and $\mathrm{Ann}_R(x) \neq \{0\}$ for every nonunit element $x \in R$. Let x, y be two vertices in $\mathtt{ZA}(R)$. Then $x, y \in \mathfrak{m}$. Hence $Rx + Ry = Rz$ for some nonzero nonunit element $z \in R$. So x, y are not adjacent which shows that $\mathtt{ZA}(R)$ is empty. □

Remark 2.1. Suppose that R has a nontrivial idempotent element e. Then $e+(1-e) = 1$ implies that e and $1-e$ are adjacent. Hence $\deg_{\mathtt{ZA}(R)}(e) \geq 1$ and so $\mathtt{ZA}(R)$ is not an empty graph.

Remark 2.2. Let R be a ring. Notice that if R is an Artinian ring or a Boolean ring, then $\dim(R) = 0$. By [2, Theorem 3.4], $\dim(R) = 0$ if and only if for every $x \in R$ there exists a positive integer n such that x^{n+1} divides x^n. Therefore, every nonzero nonunit element of a zero-dimensional ring has a nonzero annihilator. Hence, if R is a zero-dimensional chained ring, then $\mathtt{ZA}(R)$ is an empty graph.

Let $\mathrm{Z}^*(R)$ denote the zero divisors of R and $\mathrm{Z}(R) = \mathrm{Z}^*(R) \cup \{0\}$.

Theorem 2.2. *Let R be a ring and S be a multiplicative closed subset of R such that $S \cap \mathrm{Z}(R) = \{0\}$. Then $\mathtt{ZA}(R) \simeq \mathtt{ZA}(R_S)$.*

Proof. Define the vertex map $\Phi : \mathrm{V}(\mathtt{ZA}(R)) \to \mathrm{V}(\mathtt{ZA}(R_S))$ by $x \mapsto \frac{x}{1}$. We can easily verify that $x = y$ if and only if $\frac{x}{1} = \frac{y}{1}$. Also, it is easy to see that $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \{0\}$ if and only if $\mathrm{Ann}_{R_S}(\frac{x}{1}) \cap \mathrm{Ann}_{R_S}(\frac{y}{1}) = \{\frac{0}{1}\}$. ☐

Theorem 2.3. *Let R be a Bézout ring with $|\mathrm{Max}(R)| < \infty$ such that $\delta(\mathtt{ZA}(R)) > 0$. Then $\mathtt{ZA}(R)$ is a finite graph if and only if every vertex of $\mathtt{ZA}(R)$ has finite degree.*

Proof. The "only if" part is evident.

Suppose that each vertex of $\mathtt{ZA}(R)$ has finite degree. If $\mathrm{Ann}_R(x) = \{0\}$ for some nonzero nonunit element $x \in R$, then x is adjacent to all vertices of $\mathtt{ZA}(R)$ that implies $\mathtt{ZA}(R)$ is a finite graph. Assume that $\mathrm{Ann}_R(x) \neq \{0\}$ for each nonzero nonunit element $x \in R$. We claim that $\mathrm{Jac}(R) = \{0\}$. On the contrary, assume that there exists a nonzero element $a \in \mathrm{Jac}(R)$. Since $\mathtt{ZA}(R)$ has no isolated vertex, a is adjacent to another vertex, say b. Since R is a Bézout ring, $Ra + Rb$ is generated by a nonzero nonunit element c of R and so $\mathrm{Ann}_R(Ra + Rb) = \mathrm{Ann}_R(c) \neq \{0\}$, which is impossible. So $\mathrm{Jac}(R) = \{0\}$. Hence by Chinese Remainder Theorem we have $R \simeq F_1 \times F_2 \times \cdots \times F_n$, where F_i's are fields and $n = |\mathrm{Max}(R)|$. Let $0 \neq u \in F_1$. Then $(u, 0, \ldots, 0)$ and $(0, 1, \ldots, 1)$ are adjacent. Since $(0, 1, \ldots, 1)$ has finite degree, so F_1 is a finite field. Similarly we can show that F_i's are finite fields. Consequently R has finitely many nonzero nonunit elements and the proof is complete. ☐

Theorem 2.4. *Let R be a Bézout ring with $|\mathrm{Max}(R)| < \infty$. Then the following conditions are equivalent:*

(a) $\mathtt{ZA}(R)$ *is a bipartite graph with* $\delta(\mathtt{ZA}(R)) > 0$*;*
(b) $\mathtt{ZA}(R)$ *is a complete bipartite graph;*
(c) $R \simeq F_1 \times F_2$ *where* F_1 *and* F_2 *are two fields.*

Proof. (a)⇒(c) Suppose that $\mathtt{ZA}(R)$ is a bipartite graph with $\delta(\mathtt{ZA}(R)) > 0$. If $\mathrm{Ann}_R(x) = \{0\}$ for some nonzero nonunit element x of R, then $\{x^n \mid n \in \mathbb{N}\}$ is an infinite clique that is a contradiction. Then, for every nonzero nonunit element x of R we have $\mathrm{Ann}_R(x) \neq \{0\}$. Similar to the proof of Theorem 2.3 we can show that $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i's are fields and $n = |\mathrm{Max}(R)|$. Clearly $n \neq 1$. If $n \geq 3$, then $\{(0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1)\}$ is a clique in $\mathtt{ZA}(R)$, a contradiction. So $R \simeq F_1 \times F_2$.

(c)⇒(b) Suppose that $R \simeq F_1 \times F_2$ where F_1 and F_2 are two fields. Every vertex in $\mathtt{ZA}(R)$ is of the form $(u, 0)$ or $(0, v)$ where $0 \neq u \in F_1$ and $0 \neq v \in F_2$. Also, two vertices $(u, 0)$ and $(0, v)$ are adjacent. On the other hand, every two vertices $(u_1, 0), (u_2, 0)$ cannot be adjacent.

(b)⇒(a) is clear. □

Theorem 2.5. *Let R be a ring and $n \geq 2$ be a natural number. Then*

$$\mathrm{girth}(\mathtt{ZA}(M_n(R))) = 3.$$

Proof. For $n = 2$, the following matrices are pairwise adjacent in $\mathtt{ZA}(\mathrm{M}_2(R))$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $n \geq 3$, the following matrices are pairwise adjacent in $\mathtt{ZA}(\mathrm{M}_n(R))$:

$$\begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 \cdots 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 \cdots 1 \end{pmatrix}.$$

□

3. When is $\mathtt{ZA}(R)$ Connected?

A ring R is called *semiprimitive* if $\mathrm{Jac}(R) = 0$, [7]. A ring R is semiprimitive if and only if it is a subdirect product of fields, [8, p. 179].

Theorem 3.1. *Let R be a semiprimitive ring. If at least one of the maximal ideals of R is principal, then $\mathtt{ZA}(R)$ is a connected graph with* $\mathrm{diam}(\mathtt{ZA}(R)) \leq 4$.

Proof. Suppose that $\mathfrak{m}$ is a maximal ideal of R where $\mathfrak{m} = Rt$ for some $t \in R$. Let x, y be two different nonzero nonunit elements of R. Consider the following cases.

Case 1. Let $x, y \notin \mathfrak{m}$. Then $Rx + \mathfrak{m} = R$ and $Ry + \mathfrak{m} = R$. Hence x, y are adjacent to t. So $d_{\mathtt{ZA}(R)}(x, y) \leq 2$.

Case 2. Let $x \in \mathfrak{m}$ and $y \notin \mathfrak{m}$. Notice that y is adjacent to t. Since $\mathrm{Jac}(R) = \{0\}$, there exists a maximal ideal $\mathfrak{m}'$ different from $\mathfrak{m}$ such that $x \notin \mathfrak{m}'$. So $Rx + \mathfrak{m}' = R$, and thus there exist elements $r \in R$ and $z \in \mathfrak{m}'$ such that $rx + z = 1$. Therefore $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(z) = \{0\}$. So x is adjacent to z. Clearly $z \notin \mathfrak{m}$. Then z is adjacent to t. Hence $d_{\mathtt{ZA}(R)}(x, y) \leq 3$.

Case 3. Let $x, y \in \mathfrak{m}$. A manner similar to Case 2 shows that $d_{\mathtt{ZA}(R)}(x, t) \leq 2$ and $d_{\mathtt{ZA}(R)}(y, t) \leq 2$. Therefore $d_{\mathtt{ZA}(R)}(x, y) \leq 4$.

Consequently $\mathtt{ZA}(R)$ is a connected graph with $\operatorname{diam}(\mathtt{ZA}(R)) \leq 4$. □

Theorem 3.2. *Let R be a Bézout ring. If $\mathtt{ZA}(R)$ is connected, then one of the following conditions holds:*

(a) *there exists a nonzero nonunit element x of R such that $\operatorname{Ann}_R(x) = \{0\}$;*
(b) $\operatorname{Jac}(R) = \{0\}$*;*
(c) $\operatorname{Jac}(R) = \{0, x\}$ *where x is the only nonzero nonunit element of R.*

Proof. Assume that for every nonzero nonunit element x of R, $\operatorname{Ann}_R(x) \neq \{0\}$ and also $\operatorname{Jac}(R) \neq \{0\}$. Let x be a nonzero element in $\operatorname{Jac}(R)$. Suppose that $\mathtt{ZA}(R)$ has a vertex y different from x. Thus $Rx + Ry = Rz$ for some $z \in R$, because R is a Bézout ring. Notice that $y \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Hence z is nonzero nonunit and so by assumption $\operatorname{Ann}_R(z) \neq \{0\}$, which shows that x and y are not adjacent. This contradiction implies that $|V(\mathtt{ZA}(R))| = 1$, and so $\operatorname{Jac}(R) = \{0, x\}$. □

As a direct consequence of Theorem 3.1 and Theorem 3.2 we have the following result.

Corollary 3.1. *Let R be a Bézout ring such that at least one of the maximal ideals of R is principal. Then $\mathtt{ZA}(R)$ is connected if and only if one of the following conditions holds:*

(a) *there exists a nonzero nonunit element x of R such that $\operatorname{Ann}_R(x) = \{0\}$;*
(b) $\operatorname{Jac}(R) = \{0\}$*;*
(c) $\operatorname{Jac}(R) = \{0, x\}$ *where x is the only nonzero nonunit element of R.*

Theorem 3.3. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ where F_i's are fields. Then $\mathtt{ZA}(R)$ is a connected graph with*

$$\operatorname{diam}(\mathtt{ZA}(R)) = \begin{cases} 1, & \text{if } n = 2 \text{ and } |F_1| = |F_2| = 2, \\ 2, & \text{if } n = 2 \text{ and either } |F_1| > 2 \text{ or } |F_2| > 2, \\ 3, & \text{if } n \geq 3. \end{cases}$$

Proof. Let $n = 2$. In this case every vertex in $\mathtt{ZA}(R)$ is of the form $(u, 0)$ or $(0, v)$ where $u \neq 0$ and $v \neq 0$. Furthermore, two vertices $(u, 0)$ and $(0, v)$ are adjacent.

In the case when $n = 2$ and $|F_1| = |F_2| = 2$, we have $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. So $\mathtt{ZA}(R) \simeq K_2$.

Let $n = 2$ and $|F_1| > 2$. In this case, every two different vertices $(u_1, 0)$ and $(u_2, 0)$ cannot be adjacent. On the other hand $(u_1, 0)$ and $(u_2, 0)$ are adjacent to $(0, 1)$. So $d_{\mathtt{ZA}(R)}((u_1, 0), (u_2, 0)) = 2$. Hence $\operatorname{diam}(\mathtt{ZA}(R)) = 2$.

Now, let $n \geq 3$. Assume that $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are two different vertices. There exist two indexes i, j such that $u_i \neq 0$ and $v_j \neq 0$. So $u = (u_1, u_2, \ldots, u_n)$ is adjacent to $(1, \ldots, 1, \overbrace{0}^{i\text{–th}}, 1, \ldots, 1)$. Also $v = (v_1, v_2, \ldots, v_n)$ is adjacent to $(1, \ldots, 1, \overbrace{0}^{j\text{–th}}, 1, \ldots, 1)$. If $i \neq j$, then the vertex $(1, \ldots, 1, \overbrace{0}^{i\text{–th}}, 1, \ldots, 1)$

is adjacent to $(1,\dots,1,\overbrace{0}^{j\text{-th}},1,\dots,1)$. Thus $\mathtt{ZA}(R)$ is connected and $d_{\mathtt{ZA}(R)}(u,v)\leq 3$. In special case, we have the following path

$$(0,1,0,\dots,0)-(1,0,1,\dots,1)-(0,1,\dots,1)-(1,0,\dots,0).$$

Consequently $\operatorname{diam}(\mathtt{ZA}(R))=3$. □

4. When is $\mathtt{ZA}(R)$ Star?

Lemma 4.1. *Let R be a ring. If $\mathtt{ZA}(R)$ is a star, then $|\mathrm{Max}(R)|\leq 2$.*

Proof. Suppose that $\mathtt{ZA}(R)$ is a star. If $\mathfrak{m}$ and $\mathfrak{m}'$ are two different maximal ideals of R, then for every $x\in\mathfrak{m}\backslash\mathfrak{m}'$ we have $Rx+\mathfrak{m}'=R$. Hence there exist elements $r\in R$ and $y\in\mathfrak{m}'\backslash\mathfrak{m}$ such that $rs+y=1$. Therefore $\mathrm{Ann}_R(x)\cap\mathrm{Ann}_R(y)=\{0\}$. So x and y are adjacent. Let $\mathfrak{m}_1,\mathfrak{m}_2$ and $\mathfrak{m}_3$ be three different maximal ideals of R. Then there are elements $a\in\mathfrak{m}_1\backslash(\mathfrak{m}_2\cup\mathfrak{m}_3)$, $b\in\mathfrak{m}_2\backslash(\mathfrak{m}_1\cup\mathfrak{m}_3)$ and $c\in\mathfrak{m}_3\backslash(\mathfrak{m}_1\cup\mathfrak{m}_2)$. Then either a,b,c are pairwise adjacent or there exist at least two disjoint edges in $\mathtt{ZA}(R)$, which is a contradiction. Consequently $|\mathrm{Max}(R)|\leq 2$. □

Theorem 4.1. *Let R be a Bézout ring that is not a field. Then $\mathtt{ZA}(R)$ is a star if and only if one of the following conditions holds:*

(a) *$(R,\mathfrak{m})$ when $\mathfrak{m}=\{0,x\}$ in which x is a nonzero element of R with $x^2=0$;*
(b) *$R\simeq\mathbb{Z}_2\times F$ where F is a field.*

Proof. ($\Rightarrow$) Suppose that $\mathtt{ZA}(R)$ is a star. Hence $|\mathrm{Max}(R)|\leq 2$, by Lemma 4.1. Notice that if $\mathrm{Ann}_R(t)=\{0\}$ for some element t of a maximal ideal $\mathfrak{m}$, then $\{t^n\mid n\in\mathbb{N}\}$ is an infinite clique that is impossible. Consider the following cases:

Case 1. $\mathrm{Max}(R)=\{\mathfrak{m}\}$. Let x be a nonzero element in $\mathfrak{m}$. Then by Theorem 2.1, $\mathtt{ZA}(R)$ is empty and so $\mathfrak{m}=\{0,x\}$. On the other hand, by Nakayama's Lemma we have that $x^2=0$.

Case 2. $\mathrm{Max}(R)=\{\mathfrak{m}_1,\mathfrak{m}_2\}$. Since $\mathfrak{m}_1+\mathfrak{m}_2=R$, there exist $x\in\mathfrak{m}_1$ and $y\in\mathfrak{m}_2$ such that $x+y=1$. Hence x and y are adjacent. Now, if there exists $0\neq z\in\mathfrak{m}_1\cap\mathfrak{m}_2$, then z is not adjacent to x and y, because R is a Bézout ring and $\mathrm{Ann}_R(t)=\{0\}$ for every nonzero nonunit element t of R. This contradiction shows that $\mathfrak{m}_1\cap\mathfrak{m}_2=\{0\}$. Hence by Chinese Remainder Theorem we deduce that $R\simeq R/\mathfrak{m}_1\oplus R/\mathfrak{m}_2$. If there exist nozero elements $a_1,a_2\in R/\mathfrak{m}_1$ and $b_1,b_2\in R/\mathfrak{m}_2$, then we have the following path

$$(a_1,0)-(0,b_1)-(a_2,0)-(0,b_2),$$

a contradiction. Hence we can assume that $R/\mathfrak{m}_1=\mathbb{Z}_2$.

($\Leftarrow$) If (a) holds, then clearly $\mathtt{ZA}(R)$ is a star. Assume that (b) holds. Notice that $(1,0)$ is adjacent to all vertices $(0,u)$ where u is a nonzero element of F. Also, for every two different elements $u_1,u_2\in F$, $(0,u_1)$ and $(0,u_2)$ are not adjacent. Consequently $\mathtt{ZA}(R)$ is a star. □

5. When is $\mathtt{ZA}(R)$ Complete?

Proposition 5.1. *Let R be a ring. If $\mathtt{ZA}(R)$ is a complete graph, then $\mathcal{A}_R$ is a complete graph.*

Proof. Assume that $\mathtt{ZA}(R)$ is a complete graph. Let I, J be two nonzero proper ideals of R. Then there are two different nonzero nonunit elements $x, y \in R$ such that $x \in I$ and $y \in J$. Hence $\mathrm{Ann}_R(I) \cap \mathrm{Ann}_R(J) \subseteq \mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \{0\}$. Therefore I and J are adjacent. □

The following remark shows that the converse of Proposition 5.1 is not true.

Remark 5.1. Consider the ring $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. By [1, Theorem 6], $\mathcal{A}_R (= K_2)$ is a complete graph. But $\mathtt{ZA}(R)$ is a 4-regular graph that is not a complete graph.

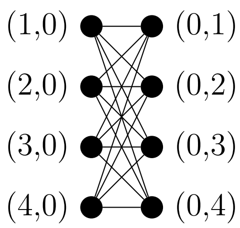

FIGURE 1. $\mathtt{ZA}(R)$

Theorem 5.1. *Let R be a ring. Then $\mathtt{ZA}(R)$ is a complete graph if and only if one of the following conditions holds:*

(a) *R has exactly one nonzero nonunit element;*
(b) *R is an integral domain;*
(c) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. ($\Rightarrow$) Assume that $\mathtt{ZA}(R)$ is a complete graph. Then, by Proposition 5.1, $\mathcal{A}_R$ is a complete graph. Suppose that R is not an integral domain. So there exists a nonzero nonunit element $x \in R$ such that $\mathrm{Ann}_R(x) \neq \{0\}$. Therefore, [1, Theorem 6] implies that either R has exactly one nonzero proper ideal or R is a direct product of two fields. Suppose that the former case holds. If y is a nonzero nonunit element of R different from x, then $Rx = Ry$. So $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \mathrm{Ann}_R(x) \neq \{0\}$, which is a contradiction. Therefore R has exactly one nonzero nonunit element. Now, let R be a direct product of two fields, say $R = F_1 \times F_2$. If there exist two different nonzero elements u, v in F_1, then $(u, 0)$ and $(v, 0)$ cannot be adjacent. Hence $F_1 = \mathbb{Z}_2$. Similarly, we can show that $F_2 = \mathbb{Z}_2$. Consequently $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.

($\Leftarrow$) Clearly, if (a) or (b) holds, then $\mathtt{ZA}(R)$ is a complete graph. Assume that (c) holds. Then $\mathtt{ZA}(R) \simeq K_2$ and we are done. □

6. Chromatic Number and Clique Number of $\mathtt{ZA}(R)$

Recall that, a ring R is said to be *reduced* if it has no nonzero nilpotent elements.

Theorem 6.1. *If R is a reduced Noetherian ring, then the chromatic number of $\mathtt{ZA}(R)$ is infinite or R is a direct product of finitely many fields.*

Proof. The proof is similar to that of [1, Theorem 16]. □

Lemma 6.1. *Let P_1 and P_2 be two prime ideals of a ring R with $P_1 \cap P_2 = \{0\}$. Then every two nonzero elements $x \in P_1$ and $y \in P_2$ are adjacent.*

Proof. Suppose that $r \in \mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y)$. Since $rx = 0 \in P_2$ and $x \notin P_2$, then $r \in P_2$. Similarly it turns out that $r \in P_1$. Hence $r \in P_1 \cap P_2 = \{0\}$. □

Theorem 6.2. *Let R be a ring and $n \geq 2$ be a natural number. If either $|\mathrm{Min}(R)| = n$ or $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i's are rings, then $\omega(\mathtt{ZA}(R)) \geq n$.*

Proof. Assume that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ where $\mathfrak{p}_i$'s are nonzero. So, by Lemma 6.1, $n \leq \omega(\mathtt{ZA}(R))$. Now, suppose that $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i's are rings. Then $\{(1, \ldots, 1, \overbrace{0}^{i\text{–th}}, 1, \ldots, 1) \mid 1 \leq i \leq n\}$ is a clique in $\mathtt{ZA}(R)$ and the result follows. □

7. When is $\mathtt{ZA}(R)$ k-regular?

Recall that a finite field of order q exists if and only if the order q is a prime power p^s. A finite field of order p^s is denoted by $\mathbb{F}_{p^s}$.

Theorem 7.1. *Let R be a Bézout ring with $|\mathrm{Max}(R)| < \infty$. Then $\mathtt{ZA}(R)$ is a k-regular graph $(0 < k < \infty)$ if and only if $R \simeq \mathbb{F}_{k+1} \times \mathbb{F}_{k+1}$.*

Proof. The "if" part has a routine verification. Let $\mathtt{ZA}(R)$ be a k-regular graph $(0 < k < \infty)$. If $\mathrm{Ann}_R(x) = \{0\}$ for some nonzero nonunit element x of R, then $\{x^n \mid n \in \mathbb{N}\}$ is an infinite clique that is a contradiction. Then, for every nonzero nonunit element x of R we have $\mathrm{Ann}_R(x) \neq \{0\}$. Similar to the manner that described in the proof of Theorem 2.3, we have $R \simeq F_1 \times F_2 \times \cdots \times F_n$ where F_i's are fields and $n = |\mathrm{Max}(R)|$. Since $\mathrm{Ann}_R((1, 0, \ldots, 0)) = 0 \times F_2 \times F_3 \times \cdots \times F_n$ and $\mathrm{Ann}_R((0, 1, 0, \ldots, 0)) = F_1 \times 0 \times F_3 \times \cdots \times F_n$, then

$$\mathrm{N}_{\mathtt{ZA}(R)}((1, 0, \ldots, 0)) = \{(0, u_2, \ldots, u_n) | u_i \in F_i \backslash \{0\} \text{ for } 2 \leq i \leq n\}$$

and

$$\mathrm{N}_{\mathtt{ZA}(R)}((0, 1, 0, \ldots, 0)) = \{(u_1, 0, u_3, \ldots, u_n) | u_i \in F_i \backslash \{0\} \text{ for } 1 \leq i \leq n,\ i \neq 2\}.$$

So

$$(|F_2| - 1)(|F_3| - 1) \cdots (|F_n| - 1) = (|F_1| - 1)(|F_3| - 1) \cdots (|F_n| - 1),$$

because $\mathtt{ZA}(R)$ is k-regular. Hence $|F_1| = |F_2|$. Similarly we can show that $|F_1| = |F_2| = \cdots = |F_n|$. Let $n \geq 3$. Note that $\mathrm{N}_{\mathtt{ZA}(R)}((1, 1, 0, \ldots, 0))$ is the union of the following sets

$$\{(u_1, 0, u_3, \ldots, u_n) \mid u_i \in F_i \backslash \{0\} \text{ for } 1 \leq i \leq n,\ i \neq 2\},$$

$$\{(0, u_2, \ldots, u_n) \mid u_i \in F_i \backslash \{0\} \text{ for } 2 \leq i \leq n\}$$

and

$$\{(0, 0, u_3, \ldots, u_n) \mid u_i \in F_i \backslash \{0\} \text{ for } 3 \leq i \leq n\}.$$

Therefore,

$$(|F_1| - 1)^{n-1} = 2(|F_1| - 1)^{n-1} + (|F_1| - 1)^{n-2},$$

since $\mathtt{ZA}(R)$ is k-regular. Thus $|F_1| = 0$ which is a contradiction. Consequently $n = 2$. If there exist two different nonzero elements u, u' in F_1, then $(u, 0)$ and $(u', 0)$ cannot be adjacent. On the other hand for every nonzero elements $u \in F_1$ and $v \in F_2$, $(u, 0)$ and $(0, v)$ are adjacent. So $\deg_{\mathtt{ZA}(R)}((u, 0)) = |F_1| - 1 = k$. Therefore $R \simeq \mathbb{F}_{k+1} \times \mathbb{F}_{k+1}$. $\square$

Corollary 7.1. *Let R be a Bézout ring with $|\mathrm{Max}(R)| < \infty$. If $\mathtt{ZA}(R)$ is a k-regular graph $(0 < k < \infty)$, then $k + 1$ is a prime power.*

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