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ON SOME STATISTICAL APPROXIMATION PROPERTIES OF GENERALIZED LUPAŞ-STANCU OPERATORS

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Abstract. The purpose of this paper is to introduce Stancu variant of generalized Lupaş operators whose construction depends on a continuously differentiable, increasing and unbounded function ρ. Depending on the selection of γ and δ, these operators are more flexible than the generalized Lupaş operators while retaining their approximation properties. For these operators we give weighted approximation, Voronovskaya type theorem, quantitative estimates for the local approximation. Finally, we investigate the statistical approximation property of the new operators with the aid of a Korovkin type statistical approximation theorem.

1. Introduction

Approximation theory rudimentary deals with approximation of functions by simpler functions or more facilely calculated functions. Broadly it is divided into theoretical and constructive approximation. Inspired by the binomial probability distribution, in 1912 S. N. Bernstein [3] was the first to construct sequence of positive linear operators to provide a constructive proof of prominent Weierstrass approximation theorem [33] using probabilistic approach. One can find a detailed monograph about the Bernstein polynomials in [19, 21].

In order to obtain more flexibility, Stancu [32] applied another technique for choosing nodes. He observed that the distance between two successive nodes and between 0 and first node and similarly between last and 1 goes to zero when $m \to \infty$. After

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these observation Stancu introduced the following positive linear operators

$$\left(P_m^{(\gamma,\delta)}f\right)(u)=\sum_{k=0}^{m}\binom{m}{k}u^k(1-u)^{m-k}f\left(\frac{k+\gamma}{m+\delta}\right) \tag{1.1}$$

converge to continuous function $f(u)$ uniformly in $[0,1]$ for each real γ,δ such that $0\leq\gamma\leq\delta$. For more recent literatures on Stancu type operators on can see [1,4,7,15–17,23–31].

In another development in approximation theory Cárdenas et al. [5], in 2011 defined the Bernstein type operators by $B_m(f\circ\tau^{-1})\circ\tau$ and showed that its Korovkin set is $\{e_0,\tau,\tau^2\}$ instead of $\{e_0,e_1,e_2\}$. Recently, Aral et al. [18] in 2014 defined a similar modification of Szász-Mirakyan type operators obtaining approximation properties of these operators on the interval $[0,\infty)$.

Very recently motivated by the above work İlarslan et al. [14] introduced a new modification of Lupaş operators [22] using a suitable function ρ, which satisfies following properties:

(ρ_1) ρ be a continuously differentiable function on $[0,\infty)$;
(ρ_2) $\rho(0)=0$ and $\inf_{u\in[0,\infty)}\rho^{'}(u)\geq 1$.

The generalized Lupaş operators are defined as

$$\mathcal{L}_m^{\rho}(f;u)=2^{-m\rho(u)}\sum_{\ell=0}^{\infty}\frac{(m\rho(u))_\ell}{2^\ell\ell!}\left(f\circ\rho^{-1}\right)\left(\frac{\ell}{m}\right), \tag{1.2}$$

for $m\geq 1$, $u\geq 0$, and suitable functions f defined on $[0,\infty)$. If $\rho(u)=u$, then (1.2) reduces to the Lupaş operators defined in [22].

The purpose of this paper is to define the Stancu type variant of operators (1.2) which depend on ρ. The present work is organized as follows. In the Section 2, we give the definition of a new family of the generalized Lupaş-Stancu operators and calculate its moments and central moments. In the Section 3, we study convergence properties of new constructed operators in the light of weighted space. In Section 4, we obtain the order of approximation of generalized Lupaş-Stancu operators associated with the weighted modulus of continuity. In Section 5, a Voronovskaya type result is obtained. In Section 6, we obtain some local approximation results related to $\mathcal{K}$-functional also we define a Lipschitz-type functions, as well as related results. Finally, in last section, we investigate the statistical approximation property of the new operators with the aid of a Korovkin type statistical approximation theorem

2. Construction of the Generalized Lupaş-Stancu Operators

Persuaded by the above mentioned work, we introduce Stancu variant of operators (1.2), which depend on a suitable function ρ as follows.

Definition 2.1. Let $0 \leq \gamma \leq \delta$ and $m \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$, we define generalized Lupaş-Stancu operators as

$$\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) = 2^{-m\rho(u)} \sum_{\ell=0}^{\infty} \frac{m\rho(u))_\ell}{2^\ell \ell!} \left(f \circ \rho^{-1}\right) \left(\frac{\ell+\gamma}{m+\delta}\right), \tag{2.1}$$

where $(m\rho(u))_l$ is the rising factorial defined as:

$$\begin{aligned} (m\rho(u))_0 =&1, \\ (m\rho(u))_l =&(m\rho(u))(m\rho(u)+1)(m\rho(u)+2)\cdots(m\rho(u)+l-1), \quad l \geq 0. \end{aligned}$$

The operators (2.1) are linear and positive. For $\gamma = \delta = 0$, the operators (2.1) turn out to be generalized Lupaş operators defined in (1.2). Next, we prove some auxiliary results for (2.1).

Lemma 2.1. *Let $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ be given by (2.1). Then for each $u \geq 0$ and $m \in \mathbb{N}$ we have*

(i) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(1;u) = 1$;

(ii) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho;u) = \frac{m}{m+\delta}\rho(u) + \frac{\gamma}{m+\delta}$;

(iii) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2;u) = \frac{m^2}{(m+\delta)^2}\rho^2(u) + \frac{2\gamma m+2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2}$;

(iv) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^3;u) = \frac{m^3}{(m+\delta)^3}\rho^3(u) + \frac{6m^2+3\gamma m^2}{(m+\delta)^3}\rho^2(u) + \frac{6m+6\gamma m+3\gamma^2 m}{(m+\delta)^3}\rho(u) + \frac{\gamma^3}{(m+\delta)^3}$;

(v)

$$\begin{aligned} \mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^4;u) =&\frac{m^4}{(m+\delta)^4}\rho^4(u) + \frac{12m^3+4\gamma m^3}{(m+\delta)^4}\rho^3(u) + \frac{36m^2+6\gamma m^2 m^2+24\gamma m^2}{(m+\delta)^4}\rho^2(u) \\ &+ \frac{12\gamma^2 m+24\gamma m+26m}{(m+\delta)^4}\rho(u) + \frac{\gamma^4}{(m+\delta)^4}. \end{aligned}$$

Corollary 2.1. *For $n = 1,2,3,4$ the n^{th} order central moments of $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ defined as $\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^n;u)$, we have*

(i) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho(w)-\rho(u);u) = \left(\frac{m}{m+\delta}-1\right)\rho(u) + \frac{\gamma}{m+\delta}$;

(ii)

$$\begin{aligned} &\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^2;u) \\ =&\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m+2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2} = \sigma_m(u); \end{aligned}$$

(iii)

$$\begin{aligned} &\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^3;u) \\ =&\left(\frac{m^3}{(m+\delta)^3} - \frac{3m^2}{(m+\delta)^2} + \frac{3m}{m+\delta} - 1\right)\rho^3(u) + \left(\frac{6m^2+3\gamma m^2}{(m+\delta)^3} - \frac{6\gamma m+6m}{(m+\delta)^2}\right. \\ &\left.+\frac{3\gamma}{(m+\delta)}\right)\rho^2(u) + \left(\frac{6m+6\gamma m+3\gamma^2 m}{(m+\delta)^3}\right)\rho(u) + \frac{\gamma^3}{(m+\delta)^3}; \end{aligned}$$

(*iv*)

$$
\begin{aligned}
&\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^4;u)\\
=&\left(\frac{m^4}{(m+\delta)^4}-\frac{4m^3}{(m+\delta)^3}+\frac{6m^2}{(m+\delta)^2}-\frac{4m}{(m+\delta)}+1\right)\rho^4(u)\\
&+\left(\frac{12m^3+4m^3\gamma}{(m+\delta)^4}+\frac{12m^2\gamma+24m^2}{(m+\delta)^3}+\frac{12m\gamma+12m}{(m+\delta)^2}-\frac{4\gamma}{(m+\delta)}\right)\rho^3(u)\\
&+\left(\frac{36m^2+6m^2\gamma^2+24m^2\gamma}{(m+\delta)^4}-\frac{24m\gamma m+24m+12m\gamma^2}{(m+\delta)^3}+\frac{6\gamma^2}{(m+\delta)^2}\right)\rho^2(u)\\
&+\left(\frac{26m+12m\gamma^2+24m\gamma}{(m+\delta)^4}\right)\rho(u)+\frac{\gamma^4}{(m+\delta)^4}-\frac{4\gamma^3}{(m+\delta)^3}.
\end{aligned}
$$

Remark 2.1. It is observed from Lemma 2.1 and Corollary 2.1 that for $\gamma=\delta=0$, we get the moments and central moments of generalized Lupaş operators [14].

3. Weighted Approximation

We start by noting that ρ not only defines a Korovkin-type set $\{1,\rho,\rho^2\}$ but also characterizes growth of the functions which are approximated.

Let $\phi(u)=1+\rho^2(u)$ be a weight function satisfying the conditions (ρ_1) and (ρ_2) given above let $\mathcal{B}_\phi[0,\infty)$ be the weighted space defined by

$$\mathcal{B}_\phi[0,\infty)=\{f:[0,\infty)\to\mathbb{R}\mid |f(u)|\le\mathcal{K}_f\phi(u),u\ge 0\},$$

where $\mathcal{K}_f$ is a constant which depends only on f. $\mathcal{B}_\phi[0,\infty)$ is a normed linear space equipped with the norm

$$\|f\|_\phi=\sup_{u\in[0,\infty)}\frac{|f(u)|}{\phi(u)}.$$

Also, we define the following subspaces of $\mathcal{B}_\phi[0,\infty)$ as

$$
\begin{aligned}
\mathcal{C}_\phi[0,\infty)=&\{f\in\mathcal{B}_\phi[0,\infty): f \text{ is continuous on } [0,\infty)\},\\
\mathcal{C}^*_\phi[0,\infty)=&\left\{f\in\mathcal{C}_\phi[0,\infty):\lim_{u\to\infty}\frac{f(u)}{\phi(u)}=\mathcal{K}_f\right\},
\end{aligned}
$$

where $\mathcal{K}_f$ is a constant depending on f and

$$U_\phi[0,\infty)=\left\{f\in\mathcal{C}_\phi[0,\infty):\frac{f(u)}{\phi(u)}\text{ is uniformly continuous on }[0,\infty)\right\}.$$

Obviously,

$$\mathcal{C}^*_\phi[0,\infty)\subset U_\phi[0,\infty)\subset\mathcal{C}_\phi[0,\infty)\subset\mathcal{B}_\phi[0,\infty).$$

For the weighted uniform approximation by linear positive operators acting from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$, we state the following results due to Gadjiev in [12] and [9].

Lemma 3.1 ([12]). *Let $(\mathcal{A}_m)_{m\geq 1}$ be a sequence of positive linear operators which acts from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$ if and only if the inequality*

$$|\mathcal{A}_m(\phi;u)| \leq \mathcal{K}_m \phi(u), \quad u \geq 0,$$

holds, where $\mathcal{K}_m > 0$ is a constant depending on m.

Theorem 3.1 ([9]). *Let $(\mathcal{A}_m)_{m\geq 1}$ be a sequence of positive linear operators, acting from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$ and satisfying*

$$\lim_{m\to\infty} \|\mathcal{A}_m \rho^i - \rho^i\|_\phi = 0, \quad i = 0,1,2.$$

Then we have

$$\lim_{m\to\infty} \|\mathcal{A}_m(f) - f\|_\phi = 0, \quad \textit{for any } f \in C^*_\phi[0,\infty).$$

Remark 3.1. It is clear from Lemma 2.1 and Lemma 3.1 that the operators $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ act from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$.

Theorem 3.2. *Let $0 \leq \gamma \leq \delta$ and for each function $f \in C^*_\phi[0,\infty)$ we have*

$$\lim_{m\to\infty} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f) - f\|_\phi = 0.$$

Proof. By Lemma 2.1 (i) and (ii), it is clear that

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(1;u) - 1\|_\phi = 0,$$

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho;u) - \rho\|_\phi = \left(\frac{m}{m+\delta} - 1\right) \sup_{u\in[0,\infty)} \frac{\rho(u)}{1+\rho^2(u)} + \frac{\gamma}{m+\delta} \leq \frac{\gamma-\delta}{m+\delta}.$$

Again by Lemma 2.1 (iii), we have

$$\begin{aligned}\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^2;u) - \rho^2\|_\phi =& \left(\frac{m^2}{(m+\delta)^2} - 1\right) \sup_{u\in[0,\infty)} \frac{\rho^2(u)}{1+\rho^2(u)} \\ &+ \frac{2\gamma m + 2m}{(m+\delta)^2} \sup_{u\in[0,\infty)} \frac{\rho(u)}{1+\rho^2(u)} + \frac{\gamma^2}{(m+\delta)^2} \\ \leq& \frac{\gamma^2 - \delta^2 - 2m\delta + 2m\gamma + 2m}{(m+\delta)^2}.\end{aligned} \tag{3.1}$$

Then from Lemma 2.1 and (3.1) we get $\lim_{m\to\infty} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^i) - \rho^i\|_\phi = 0$, $i = 0,1,2$. Hence, the proof is completed. $\square$

4. Rate of Convergence

In this section, we determine the rate of convergence for $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ by weighted modulus of continuity $\omega_\rho(f;\sigma)$ which was recently considered by Holhoş [13] as follows:

$$\omega_\rho(f;\sigma) = \sup_{u,\zeta\in[0,\infty),|\rho(\zeta)-\rho(u)|\leq\sigma} \frac{|f(\zeta) - f(u)|}{\phi(\zeta) + \phi(u)}, \quad \sigma > 0, \tag{4.1}$$

where $f \in \mathcal{C}_\phi[0,\infty)$, with the following properties:

(i) $\omega_\rho(f;0)=0$;
(ii) $\omega_\rho(f;\sigma)\geq 0$, $\sigma\geq 0$ for $f\in \mathcal{C}_\phi[0,\infty)$;
(ii) $\lim_{\sigma\to 0}\omega_\rho(f;\sigma)=0$ for each $f\in U_\phi[0,\infty)$.

Theorem 4.1 ([13]). *Let $\mathcal{A}_m:\mathcal{C}_\phi[0,\infty)\to\mathcal{B}_\phi[0,\infty)$ be a sequence of positive linear operators with*

$$\|\mathcal{A}_m(\rho^0)-\rho^0\|_{\phi^0}=a_m, \tag{4.2}$$
$$\|\mathcal{A}_m(\rho)-\rho\|_{\phi^{\frac{1}{2}}}=b_m, \tag{4.3}$$
$$\|\mathcal{A}_m(\rho^2)-\rho^2\|_{\phi}=c_m, \tag{4.4}$$
$$\|\mathcal{A}_m(\rho^3)-\rho^3\|_{\phi^{\frac{3}{2}}}=d_m, \tag{4.5}$$

where the sequences (a_m), (b_m), (c_m) and (d_m) converge to zero as $m\to\infty$. Then

$$\|\mathcal{A}_m(f)-f\|_{\phi^{\frac{3}{2}}}\leq (7+4a_m+2c_m)\omega_\rho(f;\sigma_m)+\|f\|_\phi a_m, \tag{4.6}$$

for all $f\in\mathcal{C}_\phi[0,\infty)$, where

$$\sigma_m=2\sqrt{(a_m+2b_m+c_m)(1+a_m)}+a_m+3b_m+3c_m+d_m.$$

Theorem 4.2. *Let for each $f\in\mathcal{C}_\phi[0,\infty)$, with $0\leq\gamma\leq\delta$. Then we have*

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f)-f\|_{\phi^{\frac{3}{2}}}\leq\left(7+\frac{2\gamma^2-2\delta^2-4m\delta+4m\gamma+4m}{(m+\delta)^2}\right)\omega_\rho(f;\sigma_m),$$

where ω_ρ is the weighted modulus of continuity defined in (4.1) *and*

$$\begin{aligned}\sigma_m=&2\sqrt{\frac{2\gamma-2\delta}{m+\delta}+\frac{\gamma^2-\delta^2-2m\delta+2m\gamma+2m}{(m+\delta)^2}}\\&+\frac{3\gamma-3\delta}{m+\delta}+\frac{3\gamma^2-3\delta^2-6m\delta+6m\gamma+6m}{(m+\delta)^2}\\&+\frac{6m^2+3\gamma m^2+6m+6\gamma m+3\gamma^2m+\gamma^3-\delta^3-3m^2\delta-3m\delta^2}{(m+\delta)^3}.\end{aligned}$$

Proof. If we calculate the sequences (a_m), (b_m), (c_m) and (d_m), then by using Lemma 2.1, clearly we have

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^0)-\rho^0\|_{\phi^0}=0=a_m,$$
$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho)-\rho\|_{\phi^{\frac{1}{2}}}\leq\frac{\gamma-\delta}{m+\delta}=b_{m,q},$$

and

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^2)-\rho^2\|_{\phi}\leq\frac{\gamma^2-\delta^2-2m\delta+2m\gamma+2m}{(m+\delta)^2}=c_m.$$

Finally,

$$\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^3)-\rho^3\|_{\phi^{\frac{3}{2}}} \tag{4.7}$$

$$\leq \frac{6m^2 + 3\gamma m^2 + 6m + 6\gamma m + 3\gamma^2 m + \gamma^3 - \delta^3 - 3m^2\delta - 3m\delta^2}{(m+\delta)^3} = d_{m,q}.$$

Thus the conditions (4.1)–(4.2) are satisfied. Now, by Theorem 4.1, we obtain the desired result. □

Remark 4.1. For $\lim_{\delta\to 0} \omega_\rho(f;\delta) = 0$ in Theorem 4.2, we get

$$\lim_{m\to\infty} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f) - f\|_{\phi^{\frac{3}{2}}} = 0, \quad \text{for } f \in U_\phi[0,\infty).$$

5. Voronovskaya Type Theorem

In this section, by using a technique which is developed in [5] by Cardenas-Morales, Garrancho and Raşa, we prove pointwise convergence of $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ by obtaining Voronovskaya-type theorems.

Theorem 5.1. *Let $f \in \mathcal{C}_\phi[0,\infty)$, $u \in [0,\infty)$, with $0 \leq \gamma \leq \delta$ and suppose that $(f\circ\rho^{-1})'$ and $(f\circ\rho^{-1})''$ exist at $\rho(u)$. If $(f\circ\rho^{-1})''$ is bounded on $[0,\infty)$, then we have*

$$\lim_{m\to\infty} m\left[\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) - f(u)\right] = \rho(u)\left(f\circ\rho^{-1}\right)'\gamma + \rho(u)\left(f\circ\rho^{-1}\right)''\rho(u).$$

Proof. By using Taylor expansion of $f\circ\rho^{-1}$ at $\rho(u)\in[0,\infty)$, we have

(5.1)

$$\begin{aligned} f(w) = \left(f\circ\rho^{-1}\right)(\rho(w)) &= \left(f\circ\rho^{-1}\right)(\rho(u)) + \left(f\circ\rho^{-1}\right)'(\rho(u))\,(\rho(w)-\rho(u)) \\ &\quad + \frac{(f\circ\rho^{-1})''(\rho(u))\,(\rho(w)-\rho(u))^2}{2} + \lambda_u(w)\,(\rho(w)-\rho(u))^2, \end{aligned}$$

where

$$\lambda_u(w) = \frac{(f\circ\rho^{-1})''(\rho(w)) - (f\circ\rho^{-1})''(\rho(u))}{2}. \tag{5.2}$$

Therefore, by (5.2) together with the assumption on f ensures that

$$|\lambda_u(w)| \leq \mathcal{K}, \quad \text{for all } w\in[0,\infty),$$

and is convergent to zero as $w\to u$. Now, applying the operators (2.1) to the equality (5.1), we obtain

$$\begin{aligned} \left[\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) - f(u)\right] &= \left(f\circ\rho^{-1}\right)'(\rho(u))\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(u));u\right) \\ &\quad + \frac{(f\circ\rho^{-1})''(\rho(u))\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(y))^2;u\right)}{2} \\ &\quad + \mathcal{T}^{\gamma,\delta}_{m,\rho}\left(\lambda^u(w)\left((\rho(w)-\rho(u))^2;u\right)\right). \end{aligned} \tag{5.3}$$

By Lemma 2.1 and Corollary 2.1, we get

$$\lim_{m\to\infty} m\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(u));u\right) = \gamma \tag{5.4}$$

and

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^2;u\right)=2\rho(u). \tag{5.5}$$

By estimating the last term on the right hand side of equality (5.3), we will get the proof.

Since from (5.2) for every $\epsilon>0$, $\lim_{w\to u}\lambda_u(w)=0$. Let $\sigma>0$ such that $|\lambda_u(w)|<\epsilon$ for every $w\geq 0$. By Cauchy-Schwartz inequality, we get

$$\begin{aligned}\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\lambda_u(w)|\left(\rho(w)-\rho(u)\right)^2;u\right)\leq&\epsilon\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^2;u\right)\\&+\frac{\mathcal{K}}{\sigma^2}\lim_{m\to\infty}\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^4;u\right).\end{aligned}$$

Since

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^4;u\right)=0, \tag{5.6}$$

we obtain

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\lambda_u(w)|\left(\rho(w)-\rho(y)\right)^2;y\right)=0. \tag{5.7}$$

Thus, by taking into account the equations (5.4), (5.5) and (5.7) to (5.3) the proof is completed. □

6. Local Approximation

In this section, we present local approximation theorems for the operators $\mathcal{T}_{m,\rho}^{\gamma,\delta}$. By $\mathcal{C}_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $\mathcal{C}_B[0,\infty)$ is given by

$$\|f\|=\sup_{0\leq u<\infty}|f(x)|.$$

Further let us consider the following $\mathcal{K}$-functional:

$$\mathcal{K}_2(f,\sigma)=\inf_{s\in W^2}\{\|f-s\|+\sigma\|g''\|\},$$

where $\sigma>0$ and $W^2=\{s\in\mathcal{C}_B[0,\infty):s',s''\in\mathcal{C}_B[0,\infty)\}$. By Devore and Lorentz [6, Theorem 2.4, p. 177], there exists an absolute constant $\mathcal{C}>0$ such that

$$\mathcal{K}(f,\sigma)\leq\mathcal{C}\omega_2(f,\sqrt{\sigma}). \tag{6.1}$$

Second order modulus of smoothness is as follows

$$\omega_2(f,\sqrt{\sigma})=\sup_{0<h\leq\sqrt{\sigma}}\sup_{u\in[0,\infty)}|f(u+2h)-2f(u+h)+f(u)|,$$

where $f\in C_B[0,\infty)$. The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega(f,\sigma)=\sup_{0<h\leq\sigma}\sup_{u\in[0,\infty)}|f(u+h)-f(u)|.$$

Theorem 6.1. *Let $f \in \mathcal{C}_B[0,\infty)$, with $0 \le \gamma \le \delta$. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2) and $||\rho''||$ is finite. Then, there exists an absolute constant $\mathcal{C} > 0$ such that*

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\right| \le \mathcal{C}\mathcal{K}\left(f, \sigma_m(u)\right),$$

where

$$\sigma_m(u) = \left\{\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. Let $s \in W^2$ and $u, w \in [0,\infty)$. By using Taylor's formula we have

$$s(w) = s(u) + \left(s\circ\rho^{-1}\right)'(\rho(u))(\rho(w)-\rho(u)) + \int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\left(s\circ\rho^{-1}\right)''(v)dv. \tag{6.2}$$

Now, put $v = \rho(y)$ in the last term of (6.2) and by using the equality

$$\left(s\circ\rho^{-1}\right)''(\rho(u)) = \frac{s''(u)}{(\rho'(u))^2} - s''(u)\frac{\rho''(u)}{(\rho'(u))^3}. \tag{6.3}$$

we get

$$\begin{aligned}\int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\left(s\circ\rho^{-1}\right)''(v)dv &= \int_u^w (\rho(w)-\rho(y))\left[\frac{s''(y)\rho'(y) - s'(y)\rho''(v)}{(\rho'(y))^2}\right]dy \\ &= \int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv \\ &\quad - \int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv.\end{aligned} \tag{6.4}$$

By using Lemma 2.1 and (6.4) and applying the operator (2.1) to the both sides of equality (6.2), we deduce

$$\begin{aligned}\mathcal{T}_{m,\rho}^{\gamma,\delta}(s;u) =& s(u) + \mathcal{T}_{m,\rho}^{\gamma,\delta}\left(\int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv; u\right) \\ &- \mathcal{T}_{m,\rho}^{\gamma,\delta}\left(\int_{\rho(u)}^{\rho(w)} (\rho(w)-v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv; u\right).\end{aligned}$$

As we know ρ is strictly increasing on $[0,\infty)$ and with condition (ρ_2), we get

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(s;u) - s(u)\right| \le \mathcal{M}_{m,2}^{\rho}(u)\Big(\|s''\| + \|s'\|\|\rho''\|\Big),$$

where

$$\mathcal{M}_{m,2}^{\rho}(u) = \mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(t)-\rho(u))^2; u).$$

For $f \in \mathcal{C}_B[0,\infty)$, we have

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(s;u)\right| \le \|f\circ\rho^{-1}\| 2^{-m\rho(u)} \sum_{\ell=0}^{\infty} \frac{(m\rho(u))_\ell}{2^\ell \ell!} \le \|f\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(1;u) = \|f\|. \tag{6.5}$$

Hence, we have

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\right| \le \left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f-s;u)\right| + \left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(s;u) - s(u)\right| + \left|s(u) - f(u)\right|$$

$$\leq 2\|f-g\| + \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) \right.$$
$$\left. + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\} \Big(\|s''\| + \|s'\| \|\rho''\| \Big).$$

If we choose $\mathcal{C} = \max\{2, \|\rho''\|\}$, then

$$\Big| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \Big| \leq \mathcal{C} \Bigg(2\|f-g\| + \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) \right.$$
$$\left. + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\} \|s''\|_{W^2} \Bigg).$$

Taking infimum over all $s \in W^2$ we obtain

$$\Big| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \Big| \leq \mathcal{C}\mathcal{K}\left(f, \sigma_m(u)\right). \qquad \square$$

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given in [10]. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2), $0 < \alpha \leq 1$, and $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $\mathcal{M} \geq 0$, is the set of functions f satisfying the inequality

$$\Big| f(w) - f(u) \Big| \leq \mathcal{M} \Big| \rho(w) - \rho(u) \Big|^{\alpha}, \quad u, w \geq 0.$$

Moreover, for a bounded subset $\mathcal{E} \subset [0,\infty)$, we say that the function $f \in \mathcal{C}_B[0,\infty)$ belongs to $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$, on $\mathcal{E}$ if

$$\Big| f(w) - f(u) \Big| \leq \mathcal{M}_{\alpha,f} \Big| \rho(w) - \rho(u) \Big|^{\alpha}, \quad u \in \mathcal{E} \text{ and } w \geq 0,$$

where $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f.

Theorem 6.2. *Let ρ be a function satisfying the conditions (ρ_1), (ρ_2). Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$, with $0 \leq \gamma \leq \delta$ and for every $u \in (0,\infty)$, $m \in \mathbb{N}$, we have*

$$\Big| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \Big| \leq \mathcal{M}\left(\sigma_m(u)\right)^{\frac{\alpha}{2}}, \tag{6.6}$$

where

$$\sigma_m(u) = \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\}.$$

Proof. Assume that $\alpha = 1$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned} |\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| &\leq \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(w) - f(u)|;u) \\ &\leq \mathcal{M}\mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w) - f(u)|;u). \end{aligned}$$

By applying Cauchy-Schwartz inequality, we get

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{M}\Big[\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(t) - \rho(u))^2;u)\Big]^{\frac{1}{2}} \leq \mathcal{M}\sqrt{\sigma_m(u)}.$$

Let us assume that $\alpha \in (0,1)$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)| \leq \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(w)-f(u)|;u) \leq \mathcal{M}\mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w)-f(u)|^{\alpha};u).$$

From Hölder's inequality with $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$, for $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)| \leq \mathcal{M}\Big[\mathcal{T}_{m,\rho}^{\gamma,\delta}(|(\rho(t)-\rho(u)|;u)\Big]^{\alpha}.$$

Finally by Cauchy-Schwartz inequality, we get

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)\Big| \leq \mathcal{M}\left(\sigma_m(u)\right)^{\frac{\alpha}{2}}. \qquad \square$$

A relationship between local smoothness of functions and the local approximation was given by Agratini in [2]. Here we will prove the similar result for operators $\mathcal{T}_{m,\rho}^{\gamma,\delta}$, $m \in \mathbb{N}$, for functions from $Lip_{\mathcal{M}}(\rho(u))$ on a bounded subset.

Theorem 6.3. *Let $\mathcal{E}$ be a bounded subset of $[0,\infty)$ and ρ be a function satisfying the conditions (ρ_1), (ρ_2). Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0<\alpha\leq 1$ on $\mathcal{E}$ $\alpha \in (0,1]$, we have*

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)\Big| \leq \mathcal{M}_{\alpha,f}\left\{(\sigma_m(u))^{\frac{\alpha}{2}}+2[\rho'(u)]^{\alpha}d^{\alpha}(u,\mathcal{E})\right\}, \quad u \in [0,\infty), m \in \mathbb{N},$$

where $d(u,\mathcal{E})=\inf\{\|u-y\| : y \in \mathcal{E}\}$ and $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f,

$$\sigma_m(u)=\left\{\left(\frac{m^2}{(m+\delta)^2}-\frac{2m}{m+\delta}+1\right)\rho^2(u)+\left(\frac{2\gamma m+2m}{(m+\delta)^2}-\frac{2\gamma}{m+\delta}\right)\rho(u)+\frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. Let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in $[0,\infty)$. Then there exists a point $u_0 \in \overline{\mathcal{E}}$ such that $d(u,\mathcal{E})=|u-u_0|$.

Using the monotonicity of $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ and the hypothesis of f, we obtain

$$\begin{aligned}|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)| \leq& \mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|f(w)-f(u_0)|;u\right)+\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|f(u)-f(u_0)|;u\right)\\ \leq& \mathcal{M}_{\alpha,f}\left\{\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\rho(w)-\rho(u_0)|^{\alpha};u\right)+|\rho(u)-\rho(u_0)|^{\alpha}\right\}\\ \leq& \mathcal{M}_{\alpha,f}\left\{\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\rho(w)-\rho(u)|^{\alpha};u\right)+2|\rho(u)-\rho(u_0)|^{\alpha}\right\},\end{aligned}$$

by choosing $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, as well as the fact $|\rho(u)-\rho(u_0)|=\rho'(u)|\rho(u)-\rho(u_0)|$ in the last inequality. Then by using Hölder's inequality we easily conclude

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f(u)\Big| \leq \mathcal{M}_{\alpha,f}\left\{\left[\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^2;u)\right]^{\frac{1}{2}}+2[\rho'(u)|\rho(u)-\rho(u_0)|]^{\alpha}\right\}.$$

Hence, by Corollary 2.1 we get the proof. $\square$

Now, for $f \in \mathcal{C}_B[0,\infty)$, we recall local approximation in terms of α order generalized Lipschitz-type maximal function given by Lenze [20] as

$$\tilde{\omega}_{\alpha}^{\rho}(f;u)=\sup_{w\neq u, w\in(0,\infty)}\frac{|f(w)-f(u)|}{|w-u|^{\alpha}}, \quad u \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{6.7}$$

Then we get the following result.

Theorem 6.4. *Let $f \in \mathcal{C}_B[0,\infty)$ and $\alpha \in (0,1]$, with $0 \leq \gamma \leq \delta$. Then for all $u \in [0,\infty)$ we have*

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\right| \leq \widetilde{\omega}_{\alpha}^{\rho}(f;u)\,(\sigma_m(u))^{\frac{\alpha}{2}},$$

where

$$\sigma_m(u) = \left\{\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. We know that

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(t) - f(u)|;u).$$

From (6.7), we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \widetilde{\omega}_{\alpha}^{\rho}(f;u)\mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w) - \rho(u)|^{\alpha};u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| &\leq \widetilde{\omega}_{\alpha}^{\rho}(f;u)\Big[\mathcal{T}_{m,q}^{\rho}((\rho(t) - \rho(u))^2;u)\Big]^{\frac{\alpha}{2}}\\ &\leq \widetilde{\omega}_{\alpha}^{\rho}(f;u)\,(\sigma_m(u))^{\frac{\alpha}{2}},\end{aligned}$$

which proves the desired result ☐

7. Statistical Approximation

In this section we obtain the Korovkin type weighted statistical approximation by the operators defined in (2.1). Let us recall the concept of statistical convergence which was given by Fast [8] and further studied by many authors.

Let $\mathcal{K} \subseteq N$ and $\mathcal{K}_m = \{i \leq m : i \in \mathcal{K}\}$. Then the natural density or we can say asymptotic density of $\mathcal{K}$ is defined by $\sigma(\mathcal{K}) = \lim_m \frac{1}{m}|\mathcal{K}_m|$ whenever the limit exists, where $|\mathcal{K}_m|$ denotes the cardinality of the set $\mathcal{K}_m$.

A sequence $u = (u_i)$ of real numbers is said to be statistically convergent to $\mathcal{L}$ if for every $\epsilon > 0$ the set $\{i \in N : |u_i - \mathcal{L}| \geq \epsilon\}$ has natural density zero; that is, for each $\epsilon > 0$,

$$\lim_m \frac{1}{m}|\{i \leq m : |u_i - \mathcal{L}| \geq \epsilon\}| = 0.$$

In this case, we write $st - \lim_m u_m = \mathcal{L}$. Note that convergent sequences are statistically convergent since all finite subset of of natural no have density zero. However, its converse is not true. This is demonstrated by the following example.

Example 7.1. Let us consider the sequences,

$$u = (u_m) := \begin{cases} \dfrac{1}{2m} + 1, & \text{otherwise,}\\ 0, & m = i^2 \text{ for some } i,\end{cases}$$

and

$$v = (v_m) := \begin{cases} 1, & m = i^2 \text{ for some } i, \\ 0, & m = i^2 + 1 \text{ for some } i, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that the sequence u and v are not convergent in the ordinary sense, but $st - \lim_m u_m = 1$ and $st - \lim_m v_m = 2$. All properties of convergent sequences are not true for statistical convergence. For instance, it is known that a subsequence of a convergent sequence is convergent. However, for statistical convergence this is not true. Indeed, the sequence $l = \{i : i = 1, 2, 3, \dots\}$ is a subsequence of the statistically convergent sequence u from Example 7.1. At the same time, l is statistically divergent.

Gadjiev and Orhan [11] introduced the concept of statistical convergence in approximation theory and prove the following Bohman-Korovkintype approximation theorem for statistical convergence.

Theorem 7.1 ([11])**.** *If the sequence of positive linear operators $\mathcal{A}_n : \mathcal{C}_{\mathcal{M}}[a, b] \to \mathcal{C}[a, b]$ satisfies the conditions $st - \lim_{n\to\infty} ||\mathcal{A}_n(e_v; \cdot) - e_v||_{\mathcal{C}[a,b]} = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$, then for any function $f \in \mathcal{C}_{\mathcal{M}}[a, b]$, we have*

$$st - \lim_{n\to\infty} ||\mathcal{A}_n(f; \cdot) - f||_{\mathcal{C}[a,b]} = 0,$$

where $\mathcal{C}_{\mathcal{M}}[a, b]$ denotes the space of all functions f which are continuous in $[a, b]$ and bounded on the all positive axis.

Now our first result is as follows.

Theorem 7.2. *Let $\mathcal{T}^{\gamma,\delta}_{m,\rho}(f; u)$ be the sequence of operators* (2.1)*, then for any function $f \in \mathcal{C}_B[0, \infty)$ we have*

$$st - \lim_m \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f; u) - f\|_\phi = 0. \tag{7.1}$$

Proof. Clearly for $\nu = 0$, $\mathcal{T}^{\gamma,\delta}_{m,\rho}(f; u) = 1$, which implies

$$st - \lim_m \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(1; u) \ - 1 \ \|_\phi = 0.$$

For $\nu = 1$

$$\begin{aligned} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho; u) - \rho\|_\phi &\leq \left| \frac{m}{m+\delta}\rho(u) + \frac{\gamma}{m+\delta} - \rho(u) \right| \\ &= \left| \left(\frac{m}{m+\delta} - 1 \right) \rho(u) - \frac{\gamma}{m+\delta} \right| \\ &\leq \left| \frac{m}{m+\delta} - 1 \right| + \left| \frac{\gamma}{m+\delta} \right|. \end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets:

$$\mathcal{W} = \{m : \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho; u) - \rho\|_\phi \geq \epsilon\},$$

$$\mathcal{W}' = \left\{m : 1 - \frac{m}{m+\delta} \geq \epsilon\right\},$$
$$\mathcal{W}'' = \left\{m : \frac{\gamma}{m+\delta} \geq \epsilon\right\}.$$

It is obvious that $\mathcal{W} \subseteq \mathcal{W}'' \cup \mathcal{W}'$. Then it can be written as:

$$\sigma\{i \leq m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho; u) - \rho\|_\phi \geq \epsilon\} \leq \sigma\left\{i \leq m : 1 - \frac{m}{m+\delta} \geq \epsilon\right\} + \sigma\left\{i \leq m : \frac{\gamma}{m+\delta}\| \geq \epsilon\right\}.$$

Then we have

$$st - \lim_m \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f; u) - f\|_\phi = 0.$$

Lastly for $\nu = 2$, we have

$$\begin{aligned}
\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi &\leq \left|\frac{m^2}{(m+\delta)^2}\rho^2(u) + \frac{2\gamma m + 2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2} - \rho^2(u)\right| \\
&\leq \left|\left(\frac{m^2}{(m+\delta)^2} - 1\right)\rho^2(u) + \frac{2\gamma m + 2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right| \\
&\leq \left|\frac{m^2}{(m+\delta)^2} - 1\right| + \left|\frac{2\gamma m + 2m}{(m+\delta)^2}\right| + \left|\frac{\gamma^2}{(m+\delta)^2}\right|.
\end{aligned}$$

If we choose

$$\alpha_m = \frac{m^2}{(m+\delta)^2} - 1, \quad \beta_m = \frac{2\gamma m + 2m}{(m+\delta)^2}, \quad \gamma_m = \frac{\gamma^2}{(m+\delta)^2},$$

then

$$st - \lim_m \alpha_m = st - \lim_m \beta_m = st - \lim_m \gamma_m = 0. \tag{7.2}$$

Now given $\epsilon > 0$, we define the following four sets:

$$\mathcal{W} = \{m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi \geq \epsilon\},$$
$$\mathcal{W}_1 = \left\{m : \alpha_m \geq \frac{\epsilon}{3}\right\},$$
$$\mathcal{W}_2 = \left\{m : \beta_m \geq \frac{\epsilon}{3}\right\},$$
$$\mathcal{W}_3 = \left\{m : \gamma_m \geq \frac{\epsilon}{3}\right\}.$$

It is obvious that $\mathcal{W} \subseteq \mathcal{W}_1 \bigcup \mathcal{W}_2 \bigcup \mathcal{W}_3$. Thus, we obtain

$$\delta\{i \leq m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi \geq \epsilon\} \leq \delta\left\{i \leq m : \alpha_m \geq \frac{\epsilon}{3}\right\} + \delta\left\{i \leq m : \beta_m \geq \frac{\epsilon}{3}\right\} + \delta\left\{i \leq m : \gamma_m \geq \frac{\epsilon}{3}\right\}.$$

Using (7.2), we get

$$st-\lim_{m}\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2;u)-\rho^2\|_\phi=0$$

and thus the proof is completed.

Since

$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f\|_\phi \leq \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2;u)-\rho^2\|_\phi+\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho;u)-\rho\|_\phi+\|S_{m,q_m}^{(\gamma,\delta)}(1;u)-1\|_\phi,$$

we get

$$st-\lim_{m}\| \ \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f\|_\phi$$
$$\leq st-\lim_{m}\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2;u)-\rho^2\|_\phi+st-\lim_{m}\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho;u)-\rho\|_\phi+st-\lim_{m}\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(1;u)-1\|_\phi,$$

which implies that

$$st-\lim_{m}\| \ \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u)-f\|_\phi=0.$$

This completes the proof. □

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