Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1035–1047.

GENERALIZED CAPUTO PROPORTIONAL BOUNDARY VALUE LANGEVIN FRACTIONAL DIFFERENTIAL EQUATIONS VIA KURATOWSKI MEASURE OF NONCOMPACTNESS

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Abstract. This manuscript aims to discuss the existence of solutions for nonlinear boundary value Langevin fractional differential equations involving the generalized Caputo proportional fractional derivative via Kuratowski measure of noncompactness in an arbitrary Banach space. Using the measure of noncompactness approach and Mönch's fixed point theorem, we demonstrate the existence result. An illustrative example is provided as an application to illustrate our main results.

1. Introduction

The concept of fractional calculus has enormous potential to change the way we see and model the nature around us. With a particular attention of physicists as well as engineers, remarkable research activities has been devoted to fractional calculus. They claimed that the use of fractional derivative and integration operators are desirable for describing the properties of several materials. Indeed, we discovered that a number of theoretical and experimental investigations demonstrate that differential equations with fractional derivatives govern specific thermal systems (heat diffusion) [1], physical systems (electricity) [2], and reological systems (viscoelasticity) [3]. For additional information regarding fractional differential equations (see [4, 5, 22–25]). Recently, particular attention has been focused on the study of Langevin fractional differential equations. The Langevin equation, which describes the evolution of physical events in fluctuating environments, was first proposed by Langevin in 1908 and is a key theory

Key words and phrases. Differential equations, generalized Caputo proportional fractional derivative, measure of noncompactness, fixed point theorem.
2020 *Mathematics Subject Classification.* Primary: 34A12. Secondary: 34K37.
DOI
Received: November 13, 2023.
Accepted: September 04, 2024.

of Brownian motion [6, 7]. A more flexible fractional model for simulating fractal processes is the Langevin equation's fractional model, which is a generalization of the classical one and yields a fractional Gaussian process parametrized by two indices [8,9]. In [10], the authors established the existence of solutions for nonlinear Langevin equations with multistrip boundary and nonlocal multipoint conditions. The authors of [11], studied an antiperiodic boundary value problem for the Langevin equation with two fractional orders. For more details on Langevin fractional differential equations, we recommend reading [15–17].

Inspired by the previously stated research, we address the existence of solutions for the following nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative

(1.1)
$$\begin{cases} {}^{C}_{\delta}D^{\alpha,g}_{a^+}\left({}^{C}_{\delta}D^{\beta,g}_{a^+}+\chi\right)(x(t)-K(t,x(t))=L(t,x(t)), \quad t\in\Delta=[a,b],\\ (x(t)-K(t,x(t))_{t=a}=(x(t)-K(t,x(t))_{t=b}=0,\\ {}^{C}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t))_{t=a}=\mu, \quad {}^{C}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t))_{t=b}=\nu, \quad \mu,\nu\in\mathbb{R}, \end{cases}$$

where $\Delta=[a,b]$ be a finite interval of $\mathbb{R}$ with $(a>0)$, $1<\alpha<2$, $0<\beta<1$, ${}^{C}_{\delta}D^{\alpha,g}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order α, $\chi\in\mathbb{R}^*$, and $K,L\in C(\Delta\times E,E)$ are given functions checking certain assumptions that will be defined later and E is a Banach space with the norm $\|\cdot\|$.

To the best of our knowledge, this is the first time that the nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative (1.1) is being studied.

2. Preliminaires

Some definitions and properties of the generalized Caputo proportional fractional derivative and the Kuratowski measure of noncompactness are the focus of this section.

- The Banach space of continuous functions with the norm $\|h\|_\infty=\sup\{\|h(t)\|: t\in\Delta\}$ is denoted as $C(\Delta,E)$.
- We say that the function $f:\Delta\times E\to E$ satisfies the Caratheodory conditions if:

(i) $f(t,x)$ is measurable with respect to t for every $x\in E$;
(ii) $f(t,x)$ is continuous with respect to $x\in E$ for every $t\in\Delta$.

- Throughout this paper we consider that the function $g:\Delta\to\mathbb{R}$ is a strictly positive increasing differentiable function.

Definition 2.1 ([12, 13]). Let $0<\delta<1$, $\alpha>0$, $\Phi\in L^1([a,b],E)$. The left-sided generalized proportional fractional integral with respect to g of order α of the function Φ is given by:

$${}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)=\frac{1}{\delta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\alpha-1}\Phi(s)ds,$$

where $\Gamma(\alpha)=\int_a^{+\infty}e^{-\tau}\tau^{\alpha-1}d\tau$, $\alpha>0$, is the Euler gamma function.

Definition 2.2 ([12, 13])**.** Let $0 < \delta < 1$, $\zeta, \rho : [0,1] \times \mathbb{R} \to [0, +\infty)$ be continuous functions such that $\lim_{\delta \to 0^+} \zeta(\delta, t) = 0$, $\lim_{\delta \to 1^-} \zeta(\delta, t) = 1$, $\lim_{\delta \to 0^+} \rho(\delta, t) = 1$, $\lim_{\delta \to 1^-} \rho(\delta, t) = 0$, and $\zeta(\delta, t) \neq 0$, $\rho(\delta, t) \neq 0$ for each $\delta \in [0, 1]$, $t \in \mathbb{R}$. Then, the proportional derivative of order δ with respect to g of the function Φ is given by

$$ {}_{\delta}D^{g}\Phi(t) = \rho(\delta, t)\Phi(t) + \zeta(\delta, t)\frac{\Phi'(t)}{g'(t)}. $$

In particular, if $\zeta(\delta, t) = \delta$ and $\rho(\delta, t) = 1 - \delta$, then we have

$$ {}_{\delta}D^{g}\Phi(t) = (1 - \delta)\Phi(t) + \delta\frac{\Phi'(t)}{g'(t)}. $$

Definition 2.3 ([12, 13])**.** Let $\delta \in (0, 1]$. The left-sided generalized Caputo proportional fractional derivative of order $n - 1 < \alpha < n$ is defined by

$$ {}^{C}_{\delta}D^{\alpha;g}_{a^+}\Phi(t) = \frac{1}{\delta^{n-\alpha}\Gamma(n - \alpha)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t) - g(s))^{n-\alpha-1}({}_{\delta}D^{n,g}\Phi)(s)ds, $$

where $n = [\alpha] + 1$ and ${}_{\delta}D^{n,g} = \underbrace{{}_{\delta}D^{g}{}_{\delta}D^{g} \cdots {}_{\delta} D^{g}}_{n-\text{times}}$.

Lemma 2.1 ([12, 13])**.** *Let $t \in \Delta$, $\delta \in (0, 1]$, $\alpha, \theta > 0$, and $\Phi \in L^1(\Delta, E)$ then we have*

$$ {}_{\delta}I^{\alpha;g}_{a^+}({}_{\delta}I^{\theta;g}_{a^+}\Phi(t)) ={}_{\delta} I^{\theta;g}_{a^+}({}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)) ={}_{\delta} I^{\alpha+\theta;g}_{a^+}\Phi(t). $$

Throughout this paper, as a simplification, we set

$$ \Omega^{\theta}_{g}(t, a) = e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\theta}. \tag{2.1} $$

Lemma 2.2 ([12, 13])**.** *Let $\alpha > 0$, $\theta > 0$, and $\delta \in (0, 1]$. Then, we have*

(*i*) $({}_{\delta}I^{\alpha;g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(a))}(g(\tau) - g(a))^{\theta-1})(t) = \frac{\Gamma(\theta)}{\delta^{\alpha}\Gamma(\alpha+\theta)}e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\alpha+\theta-1}$;

(*ii*) ${}^{C}_{\delta}D^{\alpha;g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(a))}(g(\tau) - g(a))^{\theta-1})(t) = \frac{\delta^{\alpha}\Gamma(\theta)}{\Gamma(\theta-\alpha)}e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\theta-\alpha-1}$.

Lemma 2.3 ([12, 13])**.** *Let $\alpha > 0$, $\delta \in (0, 1]$, and $\Phi \in L^1(\Delta, E)$. Then, we have* $\lim_{t \to a}({}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)) = 0$.

Lemma 2.4 ([14])**.** *Let $\delta \in (0, 1]$, $n - 1 < \alpha < n$, $n = [\alpha] + 1$. Then, we have*

$$ {}_{\delta}I^{\alpha;g}_{a^+}({}^{C}_{\delta}D^{\alpha,g}_{a^+}\Phi(t)) = \Phi(t) - \sum_{k=0}^{n-1}\frac{({}_{\delta}D^{k,g}\Phi)(0)}{\delta^{k}\Gamma(k + 1)}\Omega^{k}_{g}(t, a). $$

Next, we give some definitions and properties concerning the Kuratowski measure of noncompactness.

Definition 2.4 ([18])**.** Let E be a Banach space and F be a bounded subset of E. We define the Kuratowski measure of noncompactness by the mapping $\Lambda : F \to [0, +\infty)$ defined as follows

$$\Lambda(A) = \inf \left\{ \varrho > 0/A \subseteq \cup_{i=1}^{n} A_i \text{ and } \operatorname{diam}(A_i) \leq \varrho \right\}.$$

Lemma 2.5 ([18])**.** *Let N and G be two bounded subset of the Banach space E, then we have the following properties:*

1. $N \subseteq G \Rightarrow \Lambda(N) \leq \Lambda(G)$;
2. $\Lambda(N + G) \leq \Lambda(N) + \Lambda(G)$;
3. $\Lambda(\alpha N) = |\alpha| \Lambda(N)$, $\alpha \in \mathbb{R}$*;*
4. $\Lambda(N) = 0 \Leftrightarrow N$ *is relatively compact in E;*
5. $\Lambda(N \cup G) = \max\{\Lambda(N), \Lambda(G)\}$*;*
6. $\Lambda(N) = \Lambda(\overline{N}) = \Lambda(\overline{\operatorname{conv}} N)$*, where $\overline{N}$ and* $\operatorname{conv} N$ *denote the closure and the convex hull of N, respectively.*

Definition 2.5 ([21])**.** Let $\Upsilon : F \to E$ be a continuous bounded operator. Then, Υ is Λ-Lipschitz if there exists a constant $\theta > 0$, such that for all $N \subset F$ we have

$$\Lambda(\Upsilon(N)) \leq \theta \Lambda(N).$$

Lemma 2.6 ([19])**.** *Consider that $N \in C(\Delta, E)$ is a bounded and equicontinuous subset. Then, the function $t \mapsto \Lambda(N(t))$ is continuous on Δ and*

$$\Lambda \left(\int_a^b x(s) ds \right) \leq \int_a^b \Lambda(N(s)) ds,$$

where $N(s) = \{x(s) : x \in N\}$, $s \in \Delta$.

Theorem 2.1 (Mönch's fixed point theorem [20])**.** *Let W be a closed, bounded and convex subset in a Banach space E such that $0 \in E$ and let $J : W \to W$ be a continuous mapping satisfying:*

$$U = \overline{\operatorname{conv}} J(U) \quad \text{or} \quad U = J(U) \cup \{0\} \Rightarrow \Lambda(U) = 0, \quad \text{for all } U \subset W. \tag{2.2}$$

Then, the mapping J has a fixed point.

3. Auxiliary Results

In this section, we prove the equivalence between an integral equation and our problem (1.1), and we give an existence theorem of the solution to the nonlinear boundary value Langevin fractional differential equation (1.1).

Lemma 3.1. *Let $\Delta = [a, b]$ and $K, L \in C(\Delta \times E, E)$. Then, x is a solution of problem* (1.1) *if and only if*

$$\begin{aligned} x(t) =& {}_\delta I_{a^+}^{\alpha+\beta,g} L(t, x(t)) - \chi {}_\delta I_{a^+}^{\beta,g} (x(t) - K(t, x(t)) + \mu ({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ & + \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))} (g(t) - g(a))^{\beta+1} + K(t, x(t)), \end{aligned} \tag{3.1}$$

where

$$\Pi = \left(\frac{\nu - {}_{\delta}I_{a^+}^{\alpha;g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta^{\beta}\Gamma(\beta+2)e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right). \tag{3.2}$$

Proof. Consider that $x(t)$ is a solution of the nonlinear boundary value Langevin fractional differential equation (1.1). Applying the operator ${}_{\delta}I_{0^+}^{\alpha;g}(\cdot)$ on both sides of the problem (1.1) and using Lemma 2.4, we get

$$\begin{aligned} {}_{\delta}^{C}D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) &= {}_{\delta}I_{a^+}^{\alpha;g} L(t,x(t)) - \chi\left(x(t) - K(t,x(t))\right) + C_0 e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &\quad + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta}(g(t)-g(a)). \end{aligned} \tag{3.3}$$

Now, applying the operator ${}_{\delta}I_{0^+}^{\beta;g}(\cdot)$ on both sides of the integral equation (3.3) and using Lemma 2.4, we get

$$\begin{aligned} & x(t) - K(t,x(t)) \\ &= {}_{\delta}I_{a^+}^{\alpha+\beta;g} L(t,x(t)) - \chi {}_{\delta}I_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) + C_0({}_{\delta}I_{a^+}^{\beta;g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &\quad + C_1({}_{\delta}I_{a^+}^{\beta;g}) \left(\frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta}(g(t)-g(a)) \right) + C_2 e^{\frac{\delta-1}{\delta}(g(t)-g(a))}, \end{aligned} \tag{3.4}$$

with C_0, C_1 and $C_2 \in \mathbb{R}$.

Using Lemma 2.2 (i), the integral equation (3.4) becomes

$$\begin{aligned} & x(t) - K(t,x(t)) \\ &= {}_{\delta}I_{a^+}^{\alpha+\beta;g} L(t,x(t)) - \chi {}_{\delta}I_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) + C_0({}_{\delta}I_{a^+}^{\beta;g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &\quad + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta^{\beta+1}\Gamma(\beta+2)}(g(t)-g(a))^{\beta+1} + C_2 e^{\frac{\delta-1}{\delta}(g(t)-g(a))}. \end{aligned} \tag{3.5}$$

Next, let's determine the constants C_0, C_1, and C_2. In the integral equation (3.5), putting $t = a$, using the Lemma 2.3, and the initial condition $\left(x(t) - K(t,x(t))\right)_{t=a} = 0$, then we have

$$C_2 = \left(x(t) - K(t,x(t))\right)_{t=a} = 0.$$

Now we proceed to determine C_0, by applying the operator ${}_{\delta}^{C}D_{a^+}^{\beta;g}(\cdot)$ on both sides of the integral equation (3.5) with $C_2 = 0$ and using Lemma 2.2 (ii), we get the integral equation (3.3), putting $t = a$ in this equation, using Lemma 2.3, and the initial condition ${}_{\delta}^{C}D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=a} = \mu$, we get

$$C_0 = {}_{\delta}^{C}D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=a} = \mu.$$

This time, putting $t = b$ in the integral equation (3.3) with $(C_0 = \mu)$ and using the initial condition ${}_{\delta}^{C}D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=b} = \nu$, we get

$${}_{\delta}^{C}D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=b} = {}_{\delta}I_{a^+}^{\alpha;g} L(t,x(t))|_{t=b} - \chi\left(x(t) - K(t,x(t))\right)_{t=b}$$

$$+ \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))} + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta}(g(b)-g(a)),$$

this implies that

$$\nu =_\delta I_{a^+}^{\alpha,g} L(b,x(b)) + \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))} + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta}(g(b)-g(a)),$$

therefore

$$C_1 = \delta \left(\frac{\nu -_\delta I_{a^+}^{\alpha,g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right).$$

Substituting C_0, C_1, and C_2 in (3.5) we obtain

$$\begin{aligned} &x(t) - K(t,x(t)) \\ =&_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi _\delta I_{a^+}^{\beta,g} (x(t) - K(t,x(t)) + \mu ({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ \left(\frac{\nu -_\delta I_{a^+}^{\alpha,g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta^\beta \Gamma(\beta+2) e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right) e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1}. \end{aligned}$$

This implies that

$$\begin{aligned} x(t) =&_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi _\delta I_{a^+}^{\beta,g} (x(t) - K(t,x(t)) + \mu ({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + K(t,x(t)), \end{aligned}$$

where Π is given by (3.2).

The opposite follows by direct computation. □

Next, we introduce the following conditions.

(C_1) The functions $L, K \in C(\Delta \times E, E)$ satisfy the Carathéodory conditions.

(C_2) The function $K : \Delta \times E \to E$ is Λ-Lipschitz, i.e., there exists a constant $\theta > 0$, such that for all $N \subset E$ we have

$$\Lambda(K(t,N)) \leq \theta \Lambda(N).$$

(C_3) There exists a constant ξ such that for each $x \in C(\Delta, E)$ and $t \in \Delta$, we have

$$\|K(t,x)\| \leq \xi \|x\|.$$

(C_4) There exist $\omega \in C(\Delta, \mathbb{R}^+)$ and $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that ψ being nondecreasing. Then, for all $t \in \Delta$ and $x \in E$, we have:

$$\|L(t,x)\| \leq \omega(t)\psi(\|x\|).$$

(C_5) For every bounded subset $N \subseteq E$ and for each $t \in \Delta$, we have:

$$\Lambda(L(t,N)) \leq \omega(t)\Lambda(N).$$

Let $x \in C(\Delta, E)$ and $t \in \Delta$, then we define the operator Σ as follows:

$$\begin{aligned} (\Sigma x)(t) =&_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi _\delta I_{a^+}^{\beta,g} (x(t) - K(t,x(t)) + \mu ({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + K(t,x(t)), \end{aligned}$$

where Π is given by (3.2).

Note that if the operator Σ has a fixed point, then this implies that the nonlinear Langevin fractional differential equation (1.1) has a solution in $C(\Delta, E)$.

It is now appropriate for us to provide the main result of this paper, which is the existence theorem of problem (1.1).

Theorem 3.1. *Assume that all conditions* (C_1)–(C_5) *hold. Then, the nonlinear boundary value Langevin fractional differential equation* (1.1) *has at least a solution on* Δ *provided that the following inequality is satisfied:*

$$\varpi = \frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|\chi|(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)} + \theta < 1. \tag{3.6}$$

Proof. Let $\Sigma : C(\Delta, E) \to C(\Delta, E)$. Then we consider the set B_ρ defined as:

$$B_\rho = \{x \in C(\Delta, E) : \|x\| \leq \rho\},$$

where $\rho \geq \frac{\Theta_1}{1-\Theta_2}$, with

$$\begin{aligned}\Theta_1 =& \frac{\omega^*\psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta} \\ &+ \frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \Pi(g(b)-g(a))^{\beta+1},\end{aligned} \tag{3.7}$$

where $\omega^* = \sup_{t\in\Delta}\{\omega(t)\}$, and

$$\Theta_2 = \frac{|\chi|\,(1+\xi)}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \xi. \tag{3.8}$$

It is easy to see that B_ρ is a convex, closed, and bounded subset of the Banach space $C(\Delta, E)$.

The proof is given in the several steps.

Step 1. $\Sigma(B_\rho) \subset B_\rho$.

Let $x \in B_\rho$ and $t \in \Delta$, then by using the fact that $e^{\frac{\delta-1}{\delta}(g(\cdot)-g(\cdot))} < 1$ we have

$$\begin{aligned}&\|(\Sigma x)(t)\| \\ \leq& \frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\alpha+\beta-1}\|L(s,x(s))\|ds \\ &+ \frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\|x(s)\|ds \\ &+ \frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\|K(s,x(s))\|ds \\ &+ \frac{|\mu|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds \\ &+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + \|K(t,x(t))\|\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{\omega^* \psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta} + \frac{|\chi|\rho}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} \\
& + \frac{|\chi|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}\xi\rho \\
& + \frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \Pi(g(b)-g(a))^{\beta+1} + \xi\rho \\
\leq & \frac{\omega^* \psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta} \\
& + \frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \Pi(g(b)-g(a))^{\beta+1} \\
& + \rho\left(\frac{|\chi|(1+\xi)}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \xi\right) \\
= & \Theta_1 + \rho\Theta_2 \leq \rho,
\end{aligned}
$$

where Θ_1 and Θ_2 are given by (3.7) and (3.8), respectively.

Therefore, Σ maps B_ρ into itself.

Step 2. The operator Σ is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of B_ρ such that $x_n \to x$ as $n \to +\infty$ in B_ρ. Then, we have

$$
\begin{aligned}
& \|(\Sigma x_n)(t) - (\Sigma x)(t)\| \\
\leq & {}_\delta I_{a^+}^{\alpha+\beta,g} \|L(t,x_n(t)) - L(t,x(t))\| + \chi_\delta I_{a^+}^{\beta,g} \|x_n(t) - x(t)\| \\
& + \chi_\delta I_{a^+}^{\beta,g} \|K(t,x_n(t) - K(t,x(t))\| + \|K(t,x_n(t) - K(t,x(t))\|,
\end{aligned}
$$

by using the fact that the functions L and K satisfy the Carathéodory conditions, and by Lebesgue dominated convergence theorem, from the above inequality, we get:

$$\|(\Sigma x_n)(t) - (\Sigma x)(t)\| \to 0 \quad \text{as } n \to +\infty.$$

This implies that the operator Σ is continuous.

Step 3. The operator Σ is equicontinuous.

Let $t_1, t_2 \in \Delta$, $t_1 < t_2$, and $x \in B_\rho$. Then, we have

$$
\begin{aligned}
& \|(\Sigma x)(t_2) - (\Sigma x)(t_1)\| \\
\leq & \frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{t_1} \left(\Omega_g^{\alpha+\beta-1}(t_2,a) - \Omega_g^{\alpha+\beta-1}(t_1,a)\right) g'(s)\|L(s,x(s))\|ds \\
& + \frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \Omega_g^{\alpha+\beta-1}(t_2,a) g'(s)\|L(s,x(s))\|ds \\
& + |\chi|\left[\frac{1}{\delta^{\beta}\Gamma(\beta)} \int_a^{t_1} \left(\Omega_g^{\beta-1}(t_2,a) - \Omega_g^{\beta-1}(t_1,a)\right) g'(s)\|(x(s) - K(s,x(s)))\|ds\right. \\
& \left. + \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_{t_1}^{t_2} \Omega_g^{\beta-1}(t_2,a) g'(s)\|(x(s) - K(s,x(s)))\|ds\right]
\end{aligned}
$$

$$+|\mu|\left[\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_a^{t_1}\left(\Omega_g^{\beta-1}(t_2,a)-\Omega_g^{\beta-1}(t_1,a)\right)g^{'}(s)e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds\right.$$
$$\left.+\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_{t_1}^{t_2}\Omega_g^{\beta-1}(t_2,a)g^{'}(s)e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds\right]$$
$$+\Pi\left(\Omega_g^{\beta+1}(t_2,a)-\Omega_g^{\beta+1}(t_1,a)\right)+\left\|K(t_2,x(t_2))-K(t_1,x(t_1))\right\|,$$

where $\Omega_g^{(\cdot)}(t,a)$ is given by (2.1).

By using the continuity of the functions $\Omega_g^{(\cdot)}(t,a)$, K, and by Lebesgue dominated convergence theorem, from the above inequality, we get $|(\Sigma x)(t_2)-(\Sigma x)(t_1)|\to 0$ as $t_1\to t_2$.

Therefore, the operator Σ is equicontinuous.

Step 4. Checking for condition (2.2) of Theorem 2.1.

Let $W\subseteq\overline{\text{conv}}(\Sigma(W)\cup\{0\})$ be a bounded and equicontinuous subset. Then, the function $J(t)=\Lambda(W(t))$ is continuous on Δ. Next, by using Lemma 2.6, the conditions (C_2) and (C_5), we obtain:

$$\begin{aligned}
J(t)=&\Lambda(W(t))\leq\Lambda\left(\text{conv}((\Sigma W)(t)\cup\{0\})\right)\leq\Lambda((\Sigma W)(t))\\
\leq&\Lambda\left\{\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)\right.\\
&\left.\times\ (g(t)-g(s))^{\alpha+\beta-1}L(s,x(s))ds:x\in W\right\}\\
&+\Lambda\left\{\frac{\chi}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\beta-1}x(s)ds:x\in W\right\}\\
&+\Lambda\left\{\frac{\chi}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\beta-1}K(s,x(s))ds:x\in W\right\}\\
&+\Lambda\left\{\mu({}_{\delta}I_{a^+}^{\beta;g})e^{\frac{\delta-1}{\delta}(g(t)-g(a))}:t\in\Delta\right\}\\
&+\Lambda\left\{\Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1}:t\in\Delta\right\}\\
&+\Lambda\left\{K(t,x(t)):x\in W\right\}\\
\leq&\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\alpha+\beta-1}\Lambda\left(L(s,W(s))\right)ds\\
&+\frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\beta-1}\Lambda\left(W(s)\right)ds\\
&+\frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\beta-1}\Lambda\left(K(s,W(s))\right)ds\\
&+\Lambda\left(K(t,W(t))\right)\\
\leq&\frac{\omega^*\|J\|}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t g^{'}(s)(g(t)-g(s))^{\alpha+\beta-1}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\chi|\|J\|}{\delta^{\beta}\Gamma(\beta)}\int_a^t g^{'}(s)(g(t)-g(s))^{\beta-1}ds\\
&+\frac{\theta|\chi|\|J\|}{\delta^{\beta}\Gamma(\beta)}\int_a^t g^{'}(s)(g(t)-g(s))^{\beta-1}ds+\theta\|J\|\\
\leq&\|J\|\left\{\frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{|\chi|(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)}+\theta\right\}.
\end{aligned}
$$

This implies that

$$\|J\|_\infty \leq \varpi \|J\|_\infty,$$

where ϖ is given by (3.6).

According to condition (3.6), we conclude that $\|J\| = 0$. Therefore, for all $t \in \Delta$, we have $J(t) = 0$ which implies that $\Lambda(W(t)) = 0$. Then, $W(t)$ is relatively compact in E. Hence, by using Arzelá-Ascoli theorem, we conclude that W is relatively compact in B_ρ. We observe that, all conditions of Theorem 3.1 are satisfied. As a result, there is a fixed point for the operator Σ on B_ρ which solves problem (1.1). The proof is then finished. □

4. Example

An exemplary example to demonstrate the applicability of our results will be covered in this section.

Let $\Delta = [0, \pi/2]$ and $E = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \to 0\}$ be the Banach space of real sequences $(x_n)_{n\in\mathbb{N}^*}$, such that $x_n \to 0$, with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}^*} |x_n|$.

We consider the following nonlinear boundary value Langevin fractional differential equation:

$$
(4.1)\qquad
\begin{cases}
{}^{C}_{\frac{1}{2}}D^{\frac{3}{2},t}_{0^+}\left({}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+}+\frac{2}{25}\right)\left(x(t)-\frac{1}{19}x(t)\cos^2(t)\right)\\
=\frac{1}{11+t^2}\left(x^4+\frac{1}{5^n}\sin(|x(t)|)\right), \quad t\in\Delta=[0,\frac{\pi}{2}],\\
x(0)=x\left(\frac{\pi}{2}\right)=0,\\
{}^{C}_{\delta}D^{\beta,g}_{0^+}\left(x(t)\left(1-\frac{1}{19}\cos^2(t)\right)\right)_{t=0}=\mu,\\
{}^{C}_{\delta}D^{\beta,g}_{a^+}\left(x(t)\left(1-\frac{1}{19}\cos^2(t)\right)\right)_{t=\frac{\pi}{2}}=\nu, \quad \mu,\nu\in\mathbb{R},
\end{cases}
$$

Comparing System (4.1) with Problem (1.1). Then, $\alpha = \frac{3}{2}$, $\beta = \delta = \frac{1}{2}$, $g(t) = t$, $\chi = \frac{2}{25}$, and $L, K : \Delta \times E \to E$, such that:

$$L(t,x)=\frac{1}{11+t^2}\left(x_n^4+\frac{1}{5^n}\sin(|x_n|)\right), \quad K(t,x)=\frac{1}{19}x_n\cos^2(t),$$

with $x = \{x_n\} \in E$.

Now we check for conditions (C_1)-(C_5).

It is easy to see that the functions L and K satisfy the Caratheodory conditions. We have:

$$\|K(t,x)\| \leq \frac{1}{19}\|x\| \quad \text{and} \quad \|L(t,x)\| \leq \frac{1}{11+t^2}\left(\|x\|^4+1\right).$$

Then, the conditions (C_3) and (C_4) hold with $\omega(t)=\frac{1}{11+t^2}$, $\omega^*(t)=\frac{1}{11}$, $\psi(x)=x^4+1$, and $\xi=\frac{1}{19}$. For each bounded set $N\subset E$, we have

$$\Lambda\left(L(t,N)\right)\leq\omega(t)\Lambda\left(N\right)\quad\text{and}\quad\Lambda\left(K(t,N)\right)\leq\frac{1}{19}\Lambda\left(N\right).$$

Then, the conditions (C_2) and (C_5) are satisfied with $\theta=\frac{1}{19}$.

Next, checking for the condition (3.6). Then, we have:

$$\begin{aligned}&\frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{\frac{2}{25}(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)}+\theta\\
=&\frac{\frac{1}{11}\times\left(\frac{\pi}{2}\right)^{\frac{3}{2}+\frac{1}{2}}}{\left(\frac{1}{2}\right)^{\frac{3}{2}+\frac{1}{2}}\Gamma(\frac{3}{2}+\frac{1}{2}+1)}+\frac{\frac{2}{25}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(1+\frac{1}{19})}{\left(\frac{1}{2}\right)^{\frac{1}{2}}\Gamma(\frac{1}{2}+1)}+\frac{1}{19}\simeq 0,6684<1.\end{aligned}$$

We note that all conditions of Theorem 3.1 hold. Then, the nonlinear boundary value Langevin fractional differential equation (4.1) has at least one solution on $[0,\frac{\pi}{2}]$.

5. Conclusion

The theory of nonlinear boundary value Langevin fractional differential equations has been developed in this manuscript. We have established the existence of solutions for a given nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative in arbitrary Banach space. Using the measure of noncompactness approach and Mönch's fixed point theorem, we were able to demonstrate the existence of solutions. Finally, a suitable example effectively illustrates our results.

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