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SOME ESTIMATES FOR HOLOMORPHIC FUNCTIONS AT THE BOUNDARY OF THE UNIT DISC

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Abstract. In this paper, for holomorphic function $f(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ belong to the class of $\mathcal{N}(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

1. Introduction

Let f be a holomorphic function in the unit disc $E = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc E, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = ze^{i\theta}$, where θ is a real number ([8], p. 329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2, 7]).

The basic tool in proving our results is the following lemma due to Jack.

Lemma 1.1 (Jack's lemma). *Let $f(z)$ be holomorphic function in the unit disc E with $f(0) = 0$. Then if $|f(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then there exists a real number $k \geq 1$ such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + c_2 z^2 + c_3 z^3 + \cdots ,$$

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that are holomorphic in the unit disc E. Also, $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left|\frac{zf'(z)}{f(z)}\right|^{\alpha}\left|z\left(\frac{zf'(z)}{f(z)}\right)'\right|^{\beta}<\left(\frac{1}{2}\lambda\right)^{\beta}, \tag{1.1}$$

for some real $\alpha \geq 0$, $\beta > 0$ and $\lambda = \frac{\beta}{\beta+\alpha}$.

Let $f(z) \in \mathcal{N}(\lambda)$ and define $\phi(z)$ in E by

$$\phi(z)=\frac{(h(z))^{\frac{1}{\lambda}}-1}{(h(z))^{\frac{1}{\lambda}}+1}, \tag{1.2}$$

where $h(z) = \frac{zf'(z)}{f(z)}$.

Obviously, $\phi(z)$ is holomorphic function in the unit disc E and $\phi(0) = 0$. We want to prove $|\phi(z)| < 1$ for $|z| < 1$. Differentiating (1.2) and simplifiying, we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)'=\frac{2\lambda\phi'(z)}{(1-\phi(z))^2}\left(\frac{1+\phi(z)}{1-\phi(z)}\right)^{\lambda-1}$$

and, so

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}\right|^{\alpha}\left|z\left(\frac{zf'(z)}{f(z)}\right)'\right|^{\beta}&=\left|\frac{1+\phi(z)}{1-\phi(z)}\right|^{\alpha\beta+\beta(\lambda-1)}\left|\frac{2\lambda z\phi'(z)}{(1-\phi(z))^2}\right|^{\beta}\\&=\left|\frac{2\lambda z\phi'(z)}{(1-\phi(z))^2}\right|^{\beta}<\left(\frac{\lambda}{2}\right)^{\beta}.\end{aligned}$$

If there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|}|\phi(z)|=|\phi(z_0)|=1,$$

then Jack's lemma gives us that $\phi(z_0) = e^{i\theta}$ and $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.

Thus we have

$$\begin{aligned}\left|\frac{z_0f'(z_0)}{f(z_0)}\right|^{\alpha}\left|z_0\left(\frac{z_0f'(z_0)}{f(z_0)}\right)'\right|^{\beta}&=\left|\frac{2\lambda z_0\phi'(z_0)}{(1-\phi(z_0))^2}\right|^{\beta}=\left|\frac{2\lambda ke^{i\theta}}{(1-e^{i\theta})^2}\right|^{\beta}\\&=\frac{(2\lambda k)^{\beta}}{|1-e^{i\theta}|^{2\beta}}\geq\frac{(2\lambda)^{\beta}}{2^{2\beta}}=\left(\frac{\lambda}{2}\right)^{\beta}.\end{aligned}$$

This contradict (1.1). So, there is no point $z_0 \in E$ such that $\phi(z_0) = 1$. This means that $|\phi(z)| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain

$$|c_2|\leq\frac{2\beta}{\beta+\alpha}.$$

Moreover, the equality $|c_2| = \frac{2\beta}{\beta+\alpha}$ occurs for the function

$$f(z)=e^{\int_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt}.$$

That proves the following lemma.

Lemma 1.2. *If $f(z) \in \mathcal{N}(\lambda)$, then we have*

$$|c_2| \le \frac{2\beta}{\beta+\alpha}. \tag{1.3}$$

The equality in (1.3) *occurs for the function*

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda} dt}.$$

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [21] and then rediscovered and partially improved by Osserman in [17].

Lemma 1.3. *Let $f(z)$ be a holomorphic function self-mapping of $E = \{z : |z| < 1\}$, that is $|f(z)| < 1$ for all $z \in E$. Assume that there is a $b \in \partial E$ so that f extend continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$|f'(b)| \ge \frac{2}{1+|f'(0)|}. \tag{1.4}$$

The equality in (1.4) *holds if and only if f is of the form*

$$f(z) = -z\frac{a-z}{1-az}, \quad \textit{for all } z \in E,$$

for some constant $a \in (-1, 0]$.

Corollary 1.1. *Under the hypotheses lemma, we have*

$$|f'(b)| \ge 1, \tag{1.5}$$

with equality only if f is of the form

$$f(z) = ze^{i\theta},$$

where θ is a real number.

The following Lemma 1.4 and Corollary 1.2, known as the Julia-Wolff lemma, is needed in the sequel [15].

Lemma 1.4 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in E, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial E$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \le |f'(b)| \le \infty$.*

Corollary 1.2. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \partial E$.*

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [8, 18]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1, 2, 5–7, 15–17, 19, 20] and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwar lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{z_k\}$ (see [5]).

S. G. Krantz and D. M. Burns [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. According to M. Mateljević's studies, some other types of results which are related to the subject can be found in ([13,14] and [12]). In addition, [11] was posed on ResearchGate where is discussed concerning results in more general aspects.

Also, M. Jeong [10] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [9] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, for holomorphic function $f(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ belong to the class of $\mathcal{N}(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\lambda)$. Assume that, for some $b \in \partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)} = i^{\lambda}$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{\beta}{\beta + \alpha}. \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$f(z) = e^{\int\limits_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda} dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Consider the function

$$\phi(z) = \frac{(h(z))^{\frac{1}{\lambda}} - 1}{(h(z))^{\frac{1}{\lambda}} + 1},$$

where $h(z) = \frac{zf'(z)}{f(z)}$ and $\lambda = \frac{\beta}{\beta+\alpha}$. $\phi(z)$ is a holomorphic function in the unit disc E and $\phi(0) = 0$. From the Jack's lemma and since $f(z) \in \mathcal{N}(\lambda)$, we obtain $|\phi(z)| < 1$ for $|z| < 1$. Also, we have $|\phi(b)| = 1$ for $b \in \partial E$.

From (1.5), we obtain

$$1 \leq |\phi'(b)| = \frac{2}{\lambda} \left| \frac{(h(b))^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (h(b))^{\frac{1}{\lambda}}\right)^2} \right| = \frac{2}{\lambda} \left| \frac{\left(i^{\lambda}\right)^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (i^{\lambda})^{\frac{1}{\lambda}}\right)^2} \right| = \frac{2}{\lambda} \left| \frac{\left(i^{\lambda}\right)^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (i^{\lambda})^{\frac{1}{\lambda}}\right)^2} \right|$$

and

$$1 \leq \frac{2}{\lambda}\frac{|h'(b)|}{|1+i|^2} = \frac{|h'(b)|}{\lambda}.$$

So, we take the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

Then, we have

$$\begin{aligned}
\ln f(z) =& \ln e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt} = \int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt, \\
\frac{f'(z)}{f(z)} =& \frac{1}{z}\left(\frac{1+z}{1-z}\right)^\lambda, \\
h(z) =& z\frac{f'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^\lambda
\end{aligned}$$

and

$$h'(z) = \lambda\left(\frac{1+z}{1-z}\right)^{\lambda-1}\frac{2}{(1-z)^2}.$$

Therefore, we obtain

$$h'(i) = \lambda\left(\frac{1+i}{1-i}\right)^{\lambda-1}\frac{2}{(1-i)^2}$$

and

$$|h'(i)| = \lambda = \frac{\beta}{\beta+\alpha}.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{4\beta^2}{(\beta+\alpha)\left(2\beta+(\beta+\alpha)|c_2|\right)}. \tag{2.2}$$

The inequality (2.2) *is sharp with equality for the function*

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. Using the inequality (1.4) for the function $\phi(z)$, we obtain

$$\frac{2}{1+|\phi'(0)|} \leq |\phi'(b)| = \frac{2}{\lambda}\left|\frac{(h(b))^{\frac{1}{\lambda}-1}h'(b)}{\left(1+(h(b))^{\frac{1}{\lambda}}\right)^2}\right| = \frac{2}{\lambda}\frac{|h'(b)|}{|1+i|^2} = \frac{|h'(b)|}{\lambda}.$$

Since
$$\phi'(z)=\frac{2}{\lambda}\frac{(h(z))^{\frac{1}{\lambda}-1}\,h'(z)}{\left(1+(h(z))^{\frac{1}{\lambda}}\right)^{2}}$$
and
$$|\phi'(0)|=\frac{2}{\lambda}\left|\frac{(h(0))^{\frac{1}{\lambda}-1}\,h'(0)}{\left(1+(h(0))^{\frac{1}{\lambda}}\right)^{2}}\right|=\frac{2}{\lambda}\frac{|c_2|}{4}=\frac{|c_2|}{2\lambda},$$
we have
$$\frac{2}{1+\frac{|c_2|}{2\lambda}}\leq\frac{|h'(b)|}{\lambda}$$
and
$$|h'(b)|\geq\frac{4\lambda^2}{2\lambda+|c_2|}.$$
So, we obtain the inequality (2.2).

To show that the inequality (2.2) is sharp, take the holomorphic function
$$f(z)=e^{\int\limits_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt}.$$
Then
$$h(z)=z\frac{f'(z)}{f(z)}=\left(\frac{1+z}{1-z}\right)^{\lambda}$$
and
$$|h'(i)|=\lambda.$$
Since $|c_2|=2\lambda$ is satisfied with equality. That is;
$$\frac{4\lambda^2}{2\lambda+|c_2|}=\frac{4\lambda^2}{2\lambda+2\lambda}=\lambda.$$
□

Theorem 2.3. *Let $f(z)\in\mathcal{N}(\lambda)$. Assume that, for some $b\in\partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)}=i^{\lambda}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right|\geq\lambda\left(1+\frac{2\left(2\lambda-|c_2|\right)^2}{4\lambda^2-|c_2|^2+\left|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2\right|}\right),\tag{2.3}$$

where $\lambda=\frac{\beta}{\beta+\alpha}$. The inequality (2.3) *is sharp with equality for the function*
$$f(z)=e^{\int\limits_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt}.$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. By the maximum principle for each $z\in E$, we have $|\phi(z)|\leq|z|$. So,
$$\psi(z)=\frac{\phi(z)}{z}$$

is a holomorphic function in E and $|\psi(z)|<1$ for $|z|<1$. For any real number $\mu=\frac{1}{\lambda}$ that is not a non-negative integer

$$k^{\mu}=\sum_{n=0}^{\infty}\binom{\mu}{n}(k-1)^{n},$$

where $k=\frac{zf'(z)}{f(z)}=1+c_2z+\left(2c_3-c_2^2\right)z^2+\cdots$.

From equality of $\psi(z)$, we have

$$\psi(z)=\frac{\phi(z)}{z}=\frac{1}{z}\frac{(h(z))^{\frac{1}{\lambda}}-1}{(h(z))^{\frac{1}{\lambda}}+1}=\frac{1}{z}\frac{(k)^{\mu}-1}{(k)^{\mu}+1}.$$

Thus, we take

$$|\psi(0)|=\frac{|c_2|}{2\lambda}\leq 1 \tag{2.4}$$

and

$$|\psi'(0)|=\frac{|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}{4\lambda^2}.$$

Moreover, it can be seen that

$$\frac{b\phi'(b)}{\phi(b)}=|\phi'(b)|\geq\left|(b^p)'\right|=\frac{b\left(b^p\right)'}{b^p}.$$

The function

$$\Phi(z)=\frac{\psi(z)-\psi(0)}{1-\overline{\psi(0)}\psi(z)}$$

is a holomorphic in the unit disc E, $|\Phi(z)|<1$ for $|z|<1$, $\Phi(0)=0$ and $|\Phi(b)|=1$ for $b\in\partial E$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|}\leq|\Phi'(b)|&=\frac{1-|\psi(0)|^2}{\left|1-\overline{\psi(0)}\psi(b)\right|^2}|\psi'(b)|\leq\frac{1+|\psi(0)|}{1-|\psi(0)|}|\psi'(b)|\\&=\frac{1+|\psi(0)|}{1-|\psi(0)|}\left\{|\phi'(b)|-1\right\}.\end{aligned}$$

Since

$$\Phi'(z)=\frac{1-|\psi(0)|^2}{\left(1-\overline{\psi(0)}\psi(z)\right)^2}\psi'(z),$$

$$|\Phi'(0)|=\frac{|\psi'(0)|}{1-|\psi(0)|^2}=\frac{\frac{\left|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2\right|}{4\lambda^2}}{1-\left(\frac{|c_2|}{2\lambda}\right)^2}=\frac{|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}{4\lambda^2-|c_2|^2},$$

we take

$$\frac{2}{1+\frac{\left|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2\right|}{4\lambda^2-|c_2|^2}} \le \frac{1+\frac{|c_2|}{2\lambda}}{1-\frac{|c_2|}{2\lambda}}\left\{\frac{|h'(b)|}{\lambda}-1\right\}$$
$$= \frac{2\lambda+|c_2|}{2\lambda-|c_2|}\left\{\frac{|h'(b)|}{\lambda}-1\right\}.$$

Therefore, we obtain

$$1+\frac{2\left(4\lambda^2-|c_2|^2\right)}{4\lambda^2-|c_2|^2+|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\frac{2\lambda-|c_2|}{2\lambda+|c_2|} \le \frac{|h'(b)|}{\lambda}$$

and

$$|h'(b)| \ge \lambda\left(1+\frac{2\left(2\lambda-|c_2|\right)^2}{4\lambda^2-|c_2|^2+|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\right).$$

So, we obtain the inequality (2.3).

To show that the inequality (2.3) is sharp, take the holomorphic function

$$f(z)=e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

Then

$$h(z)=z\frac{f'(z)}{f(z)}=\left(\frac{1+z}{1-z}\right)^\lambda$$

and

$$|h'(i)|=\lambda.$$

Since $|c_2|=2\lambda$, (2.3) is satisfied with equality. $\square$

If $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}-1$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{N}(\lambda)$ and $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}-1$ has no zeros in E except $z=0$ and $c_2>0$. Assume that, for some $b\in\partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)}=i^\lambda$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \ge \lambda\left(1-\frac{2\lambda|c_2|\ln^2\left(\frac{|c_2|}{2\lambda}\right)}{2\lambda|c_2|\ln\left(\frac{|c_2|}{2\lambda}\right)-|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\right), \tag{2.5}$$

where $\lambda=\frac{\beta}{\beta+\alpha}$. In addition, the equality in (2.5) occurs for the function

$$f(z)=e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt},$$

where $\lambda=\frac{\beta}{\beta+\alpha}$.

Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}} - 1$ has no zeros in E except $E - \{0\}$, we denote by $\ln \psi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln \psi(0) = \ln \left(\frac{|c_2|}{2\lambda}\right) < 0.$$

The auxiliary function

$$\Delta(z) = \frac{\ln \psi(z) - \ln \psi(0)}{\ln \psi(z) + \ln \psi(0)}$$

is a holomorphic in the unit disc E, $|\Delta(z)| < 1$, $\Delta(0) = 0$ and $|\Delta(b)| = 1$ for $b \in \partial E$.

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |\Delta'(0)|} \le |\Delta'(b)| &= \frac{|2 \ln \psi(0)|}{|\ln \psi(b) + \ln \psi(0)|^2} \left| \frac{\psi'(b)}{\psi(b)} \right| \\ &= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \left\{ |\phi'(b)| - 1 \right\}. \end{aligned}$$

Since

$$\begin{aligned} |\Delta'(0)| &= \frac{-1}{\ln \left(\frac{|c_2|}{2\lambda}\right)} \frac{\frac{\left|4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2\right|}{4\lambda^2}}{\frac{|c_2|}{2\lambda}} \\ &= \frac{-1}{\ln \left(\frac{|c_2|}{2\lambda}\right)} \frac{\left|4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2\right|}{2\lambda\,|c_2|} \end{aligned}$$

and replacing $\arg^2 \psi(b)$ by zero, then we have

$$\frac{1}{1 - \frac{\left|4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2\right|}{2\lambda |c_2| \ln\left(\frac{|c_2|}{2\lambda}\right)}} \le \frac{-1}{\ln \left(\frac{|c_2|}{2\lambda}\right)} \left\{ \frac{|h'(b)|}{\lambda} - 1 \right\}$$

and

$$1 - \frac{2\lambda\,|c_2| \ln^2 \left(\frac{|c_2|}{2\lambda}\right)}{2\lambda\,|c_2| \ln \left(\frac{|c_2|}{2\lambda}\right) - \left|4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2\right|} \le \frac{|h'(b)|}{\lambda}.$$

Thus, we obtain the inequality (2.5) with an obvious equality case. □

The following inequality (2.6) is weaker, but is simpler than (2.5) and does not contain the coeffient c_3.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \ge \frac{\beta}{\beta + \alpha} \left[1 - \ln \left((\beta + \alpha) \frac{|c_2|}{2\beta} \right) \right]. \tag{2.6}$$

Moreover, the result is sharp and the extremal function is

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda} dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Let $c_2 > 0$. Using the inequality (1.5) for the function $\Phi(z)$, we obtain

$$1 \leq |\Delta'(b)| = \frac{|2\ln\psi(0)|}{|\ln\psi(b)+\ln\psi(0)|^2}\left|\frac{\psi'(b)}{\psi(b)}\right| = \frac{-2\ln\psi(0)}{\ln^2\psi(0)+\arg^2\psi(b)}\left\{|\phi'(b)|-1\right\}.$$

Replacing $\arg^2\varphi(b)$ by zero, then we have

$$1 \leq \frac{-1}{\ln\left(\frac{|c_2|}{2\lambda}\right)}\left\{\frac{|h'(b)|}{\lambda}-1\right\}$$

and

$$|h'(b)| \geq \lambda\left[1-\ln\left(\frac{|c_2|}{2\lambda}\right)\right].$$

Thus, we obtain the inequality (2.6) with an obvious equality case. □

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