Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 95–104.

ON CONTACT CR-SUBMANIFOLD OF A KENMOTSU MANIFOLD WITH KILLING TENSOR FIELD

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Abstract. The object of this paper is to study the Contact CR-submanifold of a Kenmotsu manifold with the help of a killing tensor field and deduce some results.

1. Introduction

K. Kenmotsu [5] introduced the notion of Kenmotsu manifold and later several authors studied this manifold [2, 14, 15]. M. Kobayashi and N. Papaghuic [10, 11] investigated the geometry of semi-invariant submanifolds of a Kenmotsu manifold. The geometry of Contact CR-submanifolds, invariant and anti-invariant submanifolds of an almost contact metric structure are studied by A. Bejancu [1].

Gupta et al. [13] studied the intrinsic characterization of a slant submanifold of a Kenmotsu manifold in case of induced metric and obtained some examples of the slant submanifold of a Kenmotsu manifold. Avik De [2] studied and obtained few examples of a 3-dimensional Kenmotsu manifold with parallel Ricci tensor and obtained killing condition for a vector field in Kenmotsu manifold.

Moreover, the Contact CR-submanifolds of Kenmotsu manifolds are studied by some other authors [8, 9]. The notion of a killing tensor field was introduced by Professor D. E. Blair [4]. In [12], we have investigated and characterized a slant submanifold of a Kenmotsu manifold using killing tensor fields. In this paper, we have studied Contact CR-submanifold of a Kenmotsu manifold using the notion of a killing tensor field and obtained some results.

Key words and phrases. Contact manifold, CR-submanifold, Kenmotsu manifold, killing tensor field.

2010 *Mathematics Subject Classification.* Primary: 53C15. Secondary: 53C40.

Received: January 17, 2020.

Accepted: July 16, 2020.

2. Preliminaries

A $(2m+1)$-dimensional manifold M is said to admit an almost contact metric structure if there exist a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$\varphi\xi = 0, \quad \varphi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \eta(\varphi U) = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad g(U, \xi) = \eta(U), \tag{2.2}$$

where U and V are vector fields on M [3, 7].

Moreover, if

$$\left(\overline{\nabla}_U \varphi\right) V = -g(U, \varphi V)\xi - \eta(V)\varphi U, \quad \overline{\nabla}_U \xi = U - \eta(U)\xi, \tag{2.3}$$

where $\overline{\nabla}$ be a Levi-Civita connection on $\overline{M}$, then the structure $(M, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold [5].

Suppose M is an isometrically immersed submanifold in $\overline{M}$ and ∇, $\overline{\nabla}$ be the Riemannian connections on M, $\overline{M}$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.4}$$

and

$$\overline{\nabla}_U W = -\mathrm{A}_W U + \nabla^{\perp}_U W, \tag{2.5}$$

for any vector fields $U, V \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ be the normal connection on $T^{\perp}M$, A and h be the shape operator and second fundamental form of M in $\overline{M}$.

Both h and A are related as

$$g(A_W U, V) = g(h(U, V), W). \tag{2.6}$$

In Kenmotsu manifold, M is isometrically immersed submanifold. For any vector field U tangent to M, we put

$$\varphi U = pU + fU, \tag{2.7}$$

where pU and fU denote the tangent and normal component of φU, respectively.

The covariant derivative of p, f are given by

$$\begin{aligned}(\nabla_U p)V &= \nabla_U pV - p\nabla_U V,\\ (\nabla_U f)V &= \nabla^{\perp}_U fV - f\nabla_U V.\end{aligned}$$

Similarly, for any vector field W normal to M, we have

$$\varphi W = bW + cW, \tag{2.8}$$

where bW and cW are the tangent and normal component of φW.

The covariant derivative of b, c are given by

$$(\nabla_U b)W = \nabla_U bW - b\nabla_U^{\perp} W,$$
$$(\nabla_U c)W = \nabla_U^{\perp} cW - c\nabla_U^{\perp} W.$$

Let p be the endomorphism defined by (2.7), then we have

$$g(pU, V) + g(U, pV) = 0. \tag{2.9}$$

Definition 2.1 ([9])**.** Let M be a submanifold of a Kenmotsu manifold $\overline{M}$. Then M is said to be a contact CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $D : x \to D_x \subseteq T_x(M)$ on M satisfying the following conditions:

(i) $TM = D \oplus D^{\perp}$, $\xi \in D$;

(ii) D is invariant with respect to φ, that is, $\varphi D_x \subset T_x(M)$;

(iii) the orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subseteq T_x(M)$ satisfies $\varphi D_x^{\perp} \subseteq T_x^{\perp}(M)$ for each $x \in M$.

A contact CR-submanifold is said to be proper if neither $D_x = \{0\}$ nor $D_x^{\perp} = \{0\}$. If $D_x = \{0\}$, then M is anti-invariant submanifold and if $D_x^{\perp} = \{0\}$, then M becomes invariant submanifold.

Now, let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. For any $U, V \in \Gamma(TM)$, by (2.3), (2.7), (2.8) together with the Gauss and Weingarten formulae [9], we have

$$\left(\overline{\nabla}_U \varphi\right) V = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V \tag{2.10}$$

or

$$-g(U, \varphi V) - \eta(V)\varphi U = \overline{\nabla}_U pV + \overline{\nabla}_U fV - \varphi \nabla_U V - \varphi h(U, V).$$

By comparing the tangent and normal component of the above equation, we have

$$(\nabla_U p)V = A_{fV}U + bh(U, V) + g(pU, V)\xi - \eta(V) pU \tag{2.11}$$

and

$$(\nabla_U f) V = ch(U, V) - h(U, pV) - \eta(V) fU. \tag{2.12}$$

If ξ be the structure vector field tangent to submanifold M, then by (2.3) and (2.6), we have

$$A_W \xi = h(U, \xi) = 0, \tag{2.13}$$

for all $U \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$. Thus, (2.11) reduces to

$$(\nabla_U p)V = g(pU, V)\xi - \eta(V) pU, \tag{2.14}$$

for any $U, V \in \Gamma(D)$. This shows that, the induced structure p is a Kenmotsu structure on M [9].

Let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$, then equation (2.11) reduces to

$$(\nabla_U p)V = bh(U, V) + g(pU, V)\xi - \eta(V) pU, \tag{2.15}$$

for any $U, V \in \Gamma(D)$ [8].

If the second fundamental form h is zero, then submanifold M is totally geodesic. A submanifold M is totally umbilical if

$$h(U,V) = g(U,V)H,$$

where H is the mean curvature vector. In addition, if $H = 0$, then the submanifold M is minimal.

A tensor field φ is called killing [4], if it satisfies the following condition

$$\left(\overline{\nabla}_U \varphi\right) V + \left(\overline{\nabla}_V \varphi\right) U = 0. \tag{2.16}$$

3. Contact CR-Submanifold of a Kenmotsu Manifold $\overline{M}$ with Killing Tensor Field

In this section, we discuss some results on contact CR-submanifold of a Kenmotsu manifold with killing tensor field.

Theorem 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\overline{\nabla}_U pV + \overline{\nabla}_V pU) + \left(\overline{\nabla}_U fV + \overline{\nabla}_V fU\right) = p(\overline{\nabla}_U V + \overline{\nabla}_V U) + f(\overline{\nabla}_U V + \overline{\nabla}_V U). \tag{3.1}$$

Proof. From the equation (2.10), we have

$$(\overline{\nabla}_U \varphi)V = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V \varphi\right) U = \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U.$$

On clubbing above equations, we get

$$\left(\overline{\nabla}_U \varphi\right) V + \left(\overline{\nabla}_V \varphi\right) U = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V + \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U.$$

Using (2.16), we get

$$0 = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V + \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U. \tag{3.2}$$

Using (2.7), above equation yields

$$(\overline{\nabla}_U pV + \overline{\nabla}_V pU) + (\overline{\nabla}_U fV + \overline{\nabla}_V fU) = p(\overline{\nabla}_U V + \overline{\nabla}_V U) + f(\overline{\nabla}_U V + \overline{\nabla}_V U). \qquad \square$$

Theorem 3.2. *Suppose M denotes a contact CR-submanifold with killing tensor field φ of a Kenmotsu manifold $\overline{M}$, then*

$$\eta(V) pU + \eta(U) pV = 0 \tag{3.3}$$

and

$$\eta(V) fU + \eta(U) fV = 0. \tag{3.4}$$

Proof. From equation (2.3), we have

$$\left(\overline{\nabla}_U\varphi\right)V = g\left(\varphi U, V\right)\xi - \eta\left(V\right)\varphi U.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V\varphi\right)U = -g\left(\varphi U, V\right)\xi - \eta\left(U\right)\varphi V.$$

Clubbing above two equations, we get

$$\left(\overline{\nabla}_U\varphi\right)V + \left(\overline{\nabla}_V\varphi\right)U = -\eta\left(V\right)\varphi U - \eta\left(U\right)\varphi V.$$

By using (2.16), we get

$$-\eta\left(V\right)\varphi U - \eta\left(U\right)\varphi V = 0. \tag{3.5}$$

By using (2.7) in above equation, then comparing the tangential and normal components, we get the result. □

Theorem 3.3. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then the induced structure p satisfies*

$$(\nabla_U p)V + (\nabla_V p)U = 0. \tag{3.6}$$

Proof. From (2.14), we have

$$(\nabla_U p)V = -g\left(U, pV\right)\xi - \eta\left(V\right)pU.$$

By swapping U and V in above equation, we get

$$(\nabla_V p)U = g\left(U, pV\right)\xi - \eta\left(U\right)pV.$$

On clubbing above two equations, we have

$$(\nabla_U p)V + (\nabla_V p)U = -\eta\left(V\right)pU - \eta\left(U\right)pV.$$

By using (3.3) in above equation, we get the result. □

Theorem 3.4. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. If second fundamental form h is parallel then contact CR-submanifold M is a totally geodesic.*

Proof. By swapping U and V in (2.15), we have

$$(\nabla_V p)U = bh\left(U, V\right) - g\left(V, pU\right)\xi - \eta\left(U\right)pV. \tag{3.7}$$

Combining (2.15) and (3.7), we have

$$(\nabla_U p)V + (\nabla_V p)U = 2bh\left(U, V\right) - \eta\left(V\right)pU - \eta\left(U\right)pV.$$

Now, using (3.3) and (3.6), yields $h\left(U, V\right) = 0$ for any $U, V \in \Gamma(TM)$. □

Lemma 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$A_{fV}U + A_{fU}V + 2bh\left(U, V\right) = 0. \tag{3.8}$$

Proof. By swapping U and V in (2.11), we have

$$(\nabla_V p)U = A_{fU}V + bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV. \tag{3.9}$$

On clubbing (2.11) and (3.9), we get

$$\begin{aligned}(\nabla_U p)V + (\nabla_V p)U =& A_{fV}U + A_{fU}V + 2bh\,(U,V) + g\,(pU,V)\,\xi \\ &+ g\,(pV,U)\,\xi - \eta\,(U)\,pV - \eta\,(V)\,pU.\end{aligned}$$

By using (2.9), it follows that

$$(\nabla_U p)V + (\nabla_V p)U = A_{fV}U + A_{fU}V + 2bh\,(U,V) - \eta\,(U)\,pV - \eta\,(V)\,pU.$$

Since p satisfies (3.3) and (3.6), we get the desired result. □

Proposition 3.1. *Suppose M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. Then M is anti-invariant submanifold in $\overline{M}$ if the endomorphism p is parallel.*

Proof. By interchanging U and V in (2.15), we get

$$(\nabla_V p)\,U = bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV,$$

for any $U, V \in \Gamma(D)$.

Clubbing above equation with (2.15), we get

$$(\nabla_U p)\,V + (\nabla_V p)\,U = 2bh\,(U,V) + g\,(pU,V)\,\xi + g\,(pV,U)\,\xi - \eta\,(V)\,pU - \eta\,(U)\,pV.$$

By using (2.9) and (3.6), above equation yields

$$2bh\,(U,V) - \eta\,(V)\,pU - \eta\,(U)\,pV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), we get $pU = 0$, which establishes our assertion. □

Proposition 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then M is invariant (submanifold) in $\overline{M}$ if the endomorphism f is parallel.*

Proof. By swapping U and V in (2.12), we get

$$(\nabla_V f)\,U = ch\,(U,V) - h\,(V,pU) - \eta\,(U)\,fV, \tag{3.10}$$

for any $U, V \in \Gamma(TM)$.

Clubbing (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

If f is parallel, then above equation becomes

$$2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), it follows that $fU = 0$. □

Lemma 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 0 \tag{3.11}$$

if and only if

$$2ch\,(U,V) = h\,(U,pV) + h\,(V,pU)\,. \tag{3.12}$$

Proof. Taking into consideration (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

By using (3.4), above equation yields

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU)\,.$$

Hence, the result. □

4. Examples

In this section, we give a few examples of Kenmotsu manifolds with killing φ.

Example 4.1. Let us consider the three dimensional manifold $\overline{M} = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Suppose metric g on $\overline{M}$ is given by

$$g = \eta \otimes \eta + e^{2z}(dx \otimes dx + dy \otimes dy).$$

Now, we choose

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} = \xi.$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i,e_j) = 0$ for $i \neq j$ and $g(e_i,e_j) = 1$ for $i = j$, for $1 \leq i,j \leq 3$. The 1-form η is given by $\eta(U) = g(U,e_3)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1,1)$, defined by $\varphi(e_1) = 0$, $\varphi(e_2) = 0$, $\varphi(e_3) = 0$. Now, using the linearity property of φ and g, we get

$$\varphi^2 U = -U + \eta(U)\xi, \quad \eta(e_3) = 1, \quad g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields,

$$\begin{aligned}
\overline{\nabla}_{e_1} e_1 &= -\,e_3, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = e_1,\\
\overline{\nabla}_{e_2} e_1 &= 0, \quad \overline{\nabla}_{e_2} e_2 = -e_3, \quad \overline{\nabla}_{e_2} e_3 = e_2,\\
\overline{\nabla}_{e_3} e_1 &= e_1, \quad \overline{\nabla}_{e_3} e_2 = e_2, \quad \overline{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U \xi = U - \eta\,(U)\,\xi$ for $\xi = e_3$. Hence, the manifold is a Kenmotsu manifold. From the above relations, we obtain the following equations

$$
\text{(4.1)} \qquad \begin{cases}
(\overline{\nabla}_{e_1}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_1 = 0, & (\overline{\nabla}_{e_1}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_1 = 0,\\
(\overline{\nabla}_{e_1}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_1 = 0, & (\overline{\nabla}_{e_2}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_2 = 0,\\
(\overline{\nabla}_{e_2}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_2 = 0, & (\overline{\nabla}_{e_2}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_2 = 0,\\
(\overline{\nabla}_{e_3}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_3 = 0, & (\overline{\nabla}_{e_3}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_3 = 0,\\
(\overline{\nabla}_{e_3}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_3 = 0. &
\end{cases}
$$

From the equations (4.1), it follows that φ is the killing tensor field. Hence, the manifold $\overline{M}$ is a Kenmotsu manifold with the killing tensor field φ. Moreover, we have

$$
\text{(4.2)} \qquad \begin{cases}
\overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 + \overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 = 0,\\
\overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 + \overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 = 0,\\
\overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 + \overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 = 0,\\
\overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 + \overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 = 0,\\
\overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 + \overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 = 0,\\
\overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 + \overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 = 0,\\
\overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 + \overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 = 0,\\
\overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 + \overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 = 0,\\
\overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 + \overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 = 0,
\end{cases}
$$

and

$$
\text{(4.3)} \qquad \begin{cases}
\eta(e_1)\varphi(e_1) + \eta(e_1)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_1) + \eta(e_1)\varphi(e_2) = 0,\\
\eta(e_3)\varphi(e_1) + \eta(e_1)\varphi(e_3) = 0, & \eta(e_1)\varphi(e_2) + \eta(e_2)\varphi(e_1) = 0,\\
\eta(e_2)\varphi(e_2) + \eta(e_2)\varphi(e_2) = 0, & \eta(e_3)\varphi(e_2) + \eta(e_2)\varphi(e_3) = 0,\\
\eta(e_1)\varphi(e_3) + \eta(e_3)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_3) + \eta(e_3)\varphi(e_2) = 0,\\
\eta(e_3)\varphi(e_3) + \eta(e_3)\varphi(e_3) = 0. &
\end{cases}
$$

The equations (4.1) and (4.2) satisfy the equation (3.2) and the equations (4.1) and (4.3) satisfy the equation (3.5).

Analogous to [14], we have the following example of five-dimensional Kenmotsu manifold with the killing tensor field.

Example 4.2. Let us consider the five dimensional manifold $\overline{M} = \{(x_1, x_2, x_3, x_4, v) \in \mathbb{R}^5, v \neq 0\}$, where (x_1, x_2, x_3, x_4, v) are the standard coordinates in $\mathbb{R}^5$. Suppose metric g on $\overline{M}$ is given by

$$
g = \eta \otimes \eta + e^{2v} \sum_{i=1}^{4} dx_i \otimes dx_i.
$$

Now, we choose

$$
e_1 = e^{-v}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-v}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-v}\frac{\partial}{\partial x_3}, \quad e_4 = e^{-v}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial v} = \xi.
$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i, e_j) = 0$ for $i \neq j$ and $g(e_i, e_j) = 1$ for $i = j$, where $i, j = 1, 2, 3, 4, 5$. The 1-form η is given by $\eta(U) = g(U, e_5)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1, 1)$, defined by $\varphi(e_1) = 0$, $\varphi(e_2) = 0$, $\varphi(e_3) = 0$, $\varphi(e_4) = 0$, $\varphi(e_5) = 0$.

Now, using the linearity property of φ and g, we have

$$\varphi^2 U = -U + \eta(U)\xi, \quad \eta(e_5) = 1, \quad g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields

$$\begin{aligned}
&\overline{\nabla}_{e_1} e_1 = - e_5, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = 0, \quad \overline{\nabla}_{e_1} e_4 = 0, \quad \overline{\nabla}_{e_1} e_5 = e_1,\\
&\overline{\nabla}_{e_2} e_1 = 0, \quad \overline{\nabla}_{e_2} e_2 = -e_5, \quad \overline{\nabla}_{e_2} e_3 = 0, \quad \overline{\nabla}_{e_2} e_4 = 0, \quad \overline{\nabla}_{e_2} e_5 = e_2,\\
&\overline{\nabla}_{e_3} e_1 = 0, \quad \overline{\nabla}_{e_3} e_2 = 0, \quad \overline{\nabla}_{e_3} e_3 = -e_5, \quad \overline{\nabla}_{e_3} e_4 = 0, \quad \overline{\nabla}_{e_3} e_5 = e_3,\\
&\overline{\nabla}_{e_4} e_1 = 0, \quad \overline{\nabla}_{e_4} e_2 = 0, \quad \overline{\nabla}_{e_4} e_3 = 0, \quad \overline{\nabla}_{e_4} e_4 = -e_5, \quad \overline{\nabla}_{e_4} e_5 = e_4,\\
&\overline{\nabla}_{e_5} e_1 = e_1, \quad \overline{\nabla}_{e_5} e_2 = e_2, \quad \overline{\nabla}_{e_5} e_3 = e_3, \quad \overline{\nabla}_{e_5} e_4 = e_4, \quad \overline{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U \xi = U - \eta(U)\xi$ for $\xi = e_5$. Moreover, on the similar pattern of Example 4.1, it follows that φ is a killing tensor field. Hence $\overline{M}$ is a five-dimensional Kenmotsu manifold with the killing tensor field. Also, analogous to Example 4.1, it can be seen that the equations (3.2) and (3.5) are satisfied.

Acknowledgments. We express our profound gratitude towards the anonymous referees for their valuable inputs to improve the quality of the paper.

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