

ON THE NON-NEGATIVE RADIAL SOLUTIONS OF THE TWO DIMENSIONAL BRATU EQUATION

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ABSTRACT. In this paper, we study the boundary value problem on the unit circle for the Bratu's equation depending on the real parameter μ . From the parameter estimate, the existence of non-negative solution is set. A numerical method is suggested to justify the theoretical result. It is a combination of the adaptation of finite difference and Gauss-Seidel method allowing us to obtain a good approximation of μ_c , with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

The vast majority of phenomena that occur in nature are described by a non-linear differential equation or by a system of non-linear equations. Among these equations, the Bratu's equation, given by

$$\nabla^2 u + \mu e^u = 0,$$

is a classical example of equation with a strong nonlinear exponential term and a real parameter μ . This equation arises originally as a simplified model for the description of the combustion of solid fuels. Also it is often appears in science and engineering as a model in various physical applications, from chemical reactions, thermal combustion theory, heat transfer radiation until the Chandrasekhar's model of the universe expansion and even nanotechnology [2, 3, 9, 13]. In [5], the dynamics of the Bratu equation were analyzed and the existence of bifurcations was shown. They are also devoted to describe the Gaussian curvature problem in Riemannian geometry [15], the mean field limit of vortices in Euler flows [8], the Onsager formulation in statistical mechanics [6], the Keller-Siegel system of chemotaxis [19] and the Chern-Simon-Higgs

Key words and phrases. Non-linear eigenvalue problem, finite difference method, Gauss-Seidel method.

2010 *Mathematics Subject Classification.* Primary: 35J25. Secondary: 35J60.

Received: May 14, 2018.

Accepted: December 10, 2018.

gauge theory [7, 21].

Recently, most of the research has focused on better and more efficient solution methods for determining solutions, approximate or exact, analytical or numerical to this non-linear Bratu model [1, 4, 11, 12, 17, 18, 20].

In this paper we study the two-dimensional Bratu’s equation depending on a real parameter μ on the unit circle with the Dirichlet homogeneous boundary condition. We prove the existence of non-negative radial solutions for a certain range of the real parameter μ . A numerical method is suggested to justify the theoretical result.

1. THEORETICAL RESULT: EXISTENCE OF THE SOLUTION

We study the two dimensional Bratu’s equation on the unit circle with the homogeneous boundary condition,

$$(1.1) \quad \begin{cases} -\nabla^2 u(x, y) = \mu e^{u(x,y)}, & x^2 + y^2 < 1, \\ u(x, y) = 0, & x^2 + y^2 = 1, \end{cases}$$

where μ is a real parameter. The existence of the solution for the problem (1.1) beyond a certain limit of the parameter μ is based on a general theory of the non-linear eigenvalue problem

$$(1.2) \quad \begin{cases} -\nabla^2 u(x) = \mu f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is an open bounded region of \mathbb{R}^n and $f(x, u)$ is a non-negative and continuous function on $(x, u) \in \Omega \times \mathbb{R}$. We have the next result [17]:

Theorem 1.1. *Assume that*

$$(1.3) \quad f(x, u) \geq h(x) + r(x)u, \quad (x, u) \in \Omega \times [0, \infty),$$

where h and r are non-negative and continuous functions in Ω . Then the non-linear eigenvalue problem (1.2) has no non-negative solutions for any $\mu \geq \lambda$, where λ is the principal eigenvalue of the linear eigenvalue problem

$$(1.4) \quad \begin{cases} -\nabla^2 u(x) = \lambda r(x)u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

In particular, due to the estimate

$$f(x, u) = e^u \geq 1 + u,$$

the linear eigenvalue problem (1.4) corresponding to the nonlinear problem (1.1) is given by $h(x) = r(x) = 1$, i.e.,

$$(1.5) \quad \begin{cases} -\nabla^2 u(x, y) = \lambda u(x, y), & x^2 + y^2 < 1, \\ u(x, y) = 0, & x^2 + y^2 = 1. \end{cases}$$

Introducing the polar coordinates on the plane

$$x(r, \theta) = r \cos \theta, \quad y(r, \theta) = r \sin \theta, \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi,$$

we obtain the equivalent, to the (1.5), problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0, \quad u(1, \theta) = 0.$$

The standard method of separation of variables

$$u(r, \theta) = R(r) \Theta(\theta),$$

with the boundary values

$$(1.6) \quad R(1) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi),$$

leads to the ordinary differential equations

$$(1.7) \quad \Theta''(\theta) + K \Theta(\theta) = 0,$$

and

$$(1.8) \quad r^2 R''(r) + rR'(r) + (\lambda r^2 - K)R(r) = 0$$

where K is a constant. In order to obtain a periodic solution, according to the second relation of (1.6) for the equation (1.7), the condition $K = n^2$ is necessarily required, where $n \in \mathbb{N}$. Therefore, the equation (1.8) becomes

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0,$$

which is the Bessel's equation with the general solution

$$R(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r),$$

where c_1 and c_2 are arbitrary constants. Requiring a bounded solution, when $r = 0$, we set $c_2 = 0$. Furthermore, using the first relation of (1.6) we obtain

$$J_n(\sqrt{\lambda}) = 0,$$

which implies that

$$\lambda_{(m),n} = j_{(m),n}^2.$$

Thus, the boundary value problem (1.1) has no non-negative solution for $\mu \geq j_{(m),n}^2$. Since, the first root of the Bessel's function $J_0(x)$ is $j_{1,0} = 2.40482555769577$, then the threshold is $\mu_c = \lambda = 5.7831859629467$.

In the next section we applied this result for the corresponding radial solution.

2. RADIAL POSITIVE SOLUTION OF THE BRATU'S EQUATION

The positive solution on the unit disc B_1 has a radial symmetry i.e. depends only on $r = \sqrt{x^2 + y^2}$. In order to prove this we follow the technique developed in [10]. First, we observe that the homogeneous boundary condition implies, that ∂B_1 is a level set of the positive solution $u \in C^2(\bar{B}_1)$ and therefore for the outer unit normal vector to ∂B_1 , we have

$$\vec{\nu}(x, y) \equiv (x, y) = \pm \frac{\nabla u}{|\nabla u|} \quad \text{or} \quad (x, y) \cdot \nabla u = \pm |\nabla u|.$$

Furthermore,

$$e^{u(x,y)} = 1 + \int_0^1 \frac{d}{dt} e^{tu(x,y)} dt = 1 + u(x,y) \int_0^1 e^{tu(x,y)} dt.$$

Next, we write the Bratu’s equation in the equivalent form

$$\nabla^2 u + c(x,y)u = -\mu < 0,$$

where

$$c(x,y) = \mu \int_0^1 e^{tu(x,y)} dt > 0.$$

The Serrin’s maximum principle implies, that

$$(2.1) \quad \frac{\partial u}{\partial \bar{\nu}} = (x,y) \cdot \nabla u = -|\nabla u| < 0, \quad \text{on } \partial B_1.$$

Denote by $B_1^+ = B_1 \cap \{(x,y) \in \mathbb{R}^2, y > 0\}$ the upper unit half disc and $B_1^- = B_1 \cap \{(x,y) \in \mathbb{R}^2, y < 0\}$ the lower unit half disc. Thus, (2.1) implies

$$y \frac{\partial u}{\partial y}(x,y) = -|\nabla u(x,y)| - x \frac{\partial u}{\partial x}(x,y) < 0, \quad \text{on } \partial B_1^+,$$

which means that $\partial u / \partial y < 0$ on ∂B_1^+ . The smoothness of u implies that $\partial u / \partial y < 0$ in B_1^+ close to ∂B_1^+ . Thus, the solution u is a decreasing function on the y - direction close to ∂B_1^+ . Furthermore, define the sets $l_a = \{(x,a), x \in \mathbb{R}\}$, for $0 \leq a \leq 1$ and $E_a = \{(x,y) \in B_1^+, a < y < 1\}$. To any $(x,y) \neq (x,a)$, we assign its reflection with respect to the line l_a , the point $(x, 2a - y)$.

Theorem 2.1. *If $u \in C^2(\bar{B}_1)$ is a positive solution of the Bratu’s equation, then u is a function of $r = \sqrt{x^2 + y^2}$.*

Proof. It is sufficient to show that $u(x,y) = u(x, 2a - y)$ whenever $a = 0$ i.e. the line l_a coincides with the axis x . To this end, define

$$a_0 = \inf\{a \in [0, 1] : u(x,y) < u(x, 2\beta - y), (x,y) \in E_\beta, a \leq \beta \leq 1\}.$$

The above infimum is well defined, since the solution u is a decreasing function on the y - direction close to ∂B_1^+ . We will prove that $a_0 = 0$. Suppose that $a_0 > 0$ and define the function

$$v(x,y) = u(x, 2a_0 - y) - u(x,y), \quad (x,y) \in E_{a_0}.$$

Then $v(x,y) > 0$ and

$$\nabla^2 v(x,y) - C(x,y)v(x,y) = 0,$$

where

$$C(x,y) = [u(x, 2a_0 - y) - u(x,y)] \int_0^1 e^{[tu(x, 2a_0 - y) + (1-t)u(x,y)]} dt > 0.$$

The Serrin’s maximum principle and the above discussion implies that

$$v(x,y) > 0, (x,y) \in E_{a_0}, \quad \frac{\partial v}{\partial y}(x,y) < 0, (x,y) \in l_{a_0} \cap B_1^+$$

and, equivalently

$$u(x, y) < u(x, 2a_0 - y), (x, y) \in E_{a_0}, \quad \frac{\partial u}{\partial y}(x, y) < 0, (x, y) \in l_{a_0} \cap B_1^+,$$

with the partial derivatives with respect to y , always taken close to ∂B_1^+ . Thus, we have, that the positive solution u is also decreasing function on $l_{a_0} \cap B_1^+$. Choosing any $\varepsilon > 0$, sufficiently small for $0 < \beta = a_0 - \varepsilon < a_0$, we have

$$u(x, y) < u(x, 2a_0 - y) < u(x, 2\beta - y), \quad (x, y) \in l_\beta \cap B_1^+,$$

and by the smoothness of u

$$u(x, y) < u(x, 2\beta - y), \quad (x, y) \in E_\beta, \beta < a_0,$$

which contradicts to the definition of a_0 . Thus, necessarily $a_0 = 0$, and

$$u(x, y) \leq u(x, -y), \quad (x, y) \in B_1^+.$$

In the same way, we can obtain

$$u(x, y) \geq u(x, -y), \quad (x, y) \in B_1^-,$$

which implies $u(x, y) = u(x, -y)$ in the unit disc. Finally, the axis x can be any diameter of the unit disc, thus we have the radial symmetry of the solution. \square

3. NUMERICAL METHOD

To find the numerical solution of (1.1), we have used an adaptation of the second-order Finite Difference Method (FDM). First, we consider a rectangular region (\mathcal{R}) defined by

$$\begin{cases} -1 \leq x \leq 1, \\ -1 \leq y \leq 1, \end{cases}$$

in the cartesian system (OXY), and we insert into (\mathcal{R}) the circle (\mathcal{C}) defined by $x^2 + y^2 = 1$. Next, the region (\mathcal{R}) is subdivided into the grid $n \times n$ equal subregions: $h \times h$ where

$$h = \frac{2}{n},$$

i.e, the axis (OX) and (OY) are partitioned in n equal part each one. So, each point or node (x_i, y_j) of the grid is the intersection of the $x = x_i$ vertical line and the $y = y_j$ horizontal line, where

$$x_i = -1 + ih, \quad i = 0, \dots, n,$$

and

$$y_j = -1 + jh, \quad j = 0, \dots, n.$$

Then, it is not difficult to see the following.

(a) For an exterior point or endpoint $P_{i,j} = (x_i, y_j)$ of the circle (\mathcal{C}), i.e., $x_i^2 + y_j^2 \geq 1$. See the Figure 1.

We have

$$(3.1) \quad P_{i,j} : u(x_i, y_j) = w_{i,j} = 0.$$

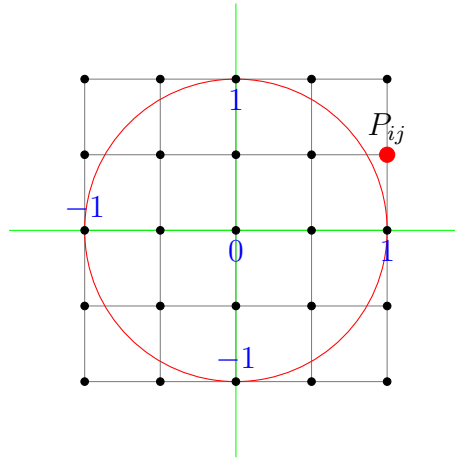


FIGURE 1.

(b) For each interior point $P_{i,j}$ of the circle (\mathcal{C}) , i.e., $x_i^2 + y_j^2 < 1$, we apply the Finite Difference Method (FDM) using the Taylor series with the variable x around x_i , and with the variable y around y_j [16], i.e., without loss of generality, around of this point $P_{i,j}$ we suppose the next four points $P_{i+1,j}, P_{i,j+1}, P_{i-1,j}, P_{i,j-1}$ which are known respectively as East(E), North(N), South(S), West(W) point with respect to $P_{i,j}$, see Figure 2.

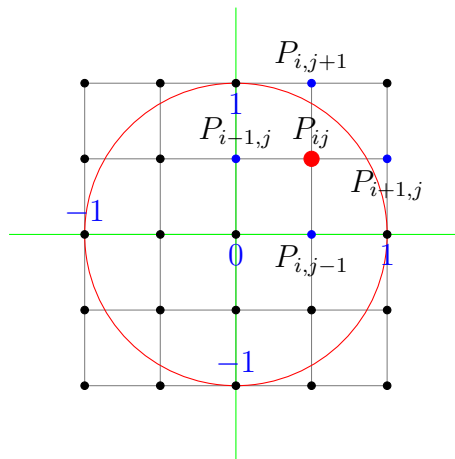


FIGURE 2.

So, we define:

$$\begin{aligned}
 P_{i+1,j} &: u(x + h, y) = u(x_{i+1}, y_j) = w_{i+1,j}, \\
 P_{i,j+1} &: u(x, y + h) = u(x_i, y_{j+1}) = w_{i,j+1}, \\
 P_{i,j-1} &: u(x, y - h) = u(x_i, y_{j-1}) = w_{i,j-1}, \\
 P_{i-1,j} &: u(x - h, y_j) = u(x_{i-1}, y_j) = w_{i-1,j}.
 \end{aligned}$$

Next, for every interior point (x, y) of the circle, we have

$$(3.2) \quad \begin{aligned} \frac{\partial^2 u}{\partial x^2} &\approx \frac{1}{h^2}[u(x+h, y) - 2u(x, y) + u(x-h, y)], \\ \frac{\partial^2 u}{\partial y^2} &\approx \frac{1}{h^2}[u(x, y+h) - 2u(x, y) + u(x, y-h)]. \end{aligned}$$

By adding these two equations (3.2), the equation (1.1) for all interior point of the circle can be replaced by the difference equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &\approx \frac{1}{h^2}[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)] \\ &= (-\mu)(1 + u(x, y)). \end{aligned}$$

Then, it is easy to see that

$$(3.3) \quad 4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = (-\mu)(-h^2)(1 + w_{i,j}),$$

where $u(x_i, y_j) = w_{i,j}$ and $e^{w_{i,j}} \cong 1 + w_{i,j} + \dots$.

So, (3.3) implies that for each interior point $P_{i,j}$ of the circle (\mathcal{C}) , we have

$$(3.4) \quad (4 - \mu h^2)w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = \mu h^2,$$

where $i = 0, \dots, n$, and $j = 0, \dots, n$.

The reader may find an illustrative example in the Appendix.

To find the value $u(x_i, y_j) = w_{i,j}$ of the point $P_{i,j}$ on the region (\mathcal{R}) , the system of linear equations (3.1) and (3.4) is established, moreover, the solution of (1.1) is reduced to the solution $w_{i,j}$ of the system of linear equations (3.1) and (3.4), depending on the parameter μ . Finally, to solve the system of linear equations (3.1) and (3.4), the Gauss-Seidel's Method is used [4].

Since (1.1) and (3.4) depends on the parameter μ , we determine the threshold μ_c , giving the grid $n \times n$ of (\mathcal{R}) , and also giving a value the parameter μ for the grid $n \times n$ of (\mathcal{R}) the norm $|w_{i,j}| = u_{max}$ is found.

Algorithm

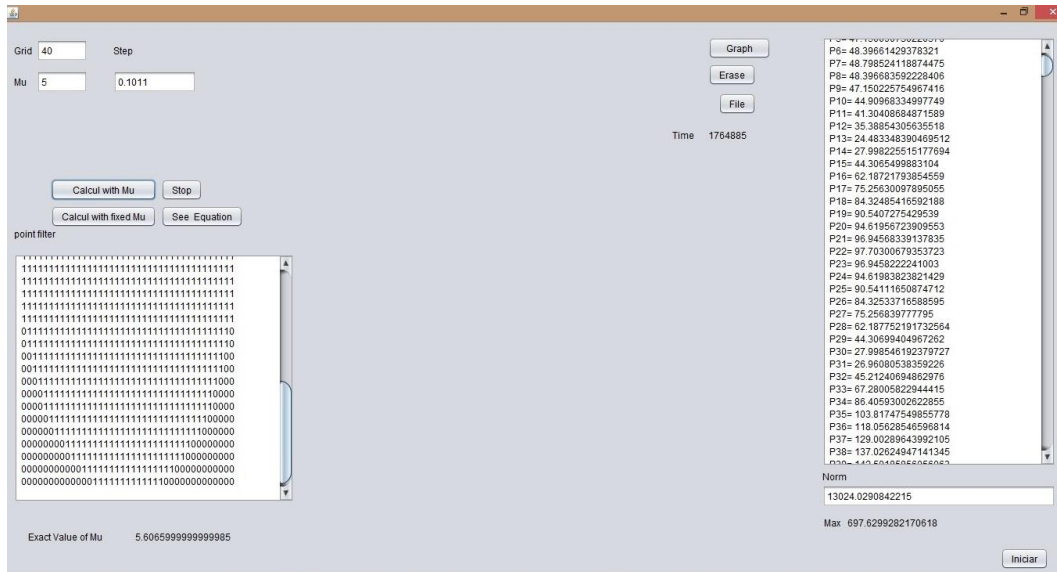
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1 Input: Value  $n$  of the subdivision of the region  $(\mathcal{R})$ ; Initial value of  $\mu = \mu_0$ ;
    $k = \text{step of } \mu$ .
2 Output: Value of the threshold  $\mu_c$ ; Approximate solution  $u(x_i, y_j) \geq 0$  of the
   interior point of the circle for certain range of  $\mu$ ; Maximum value  $u_{max}$  of the
   solution  $u(x_i, y_j)$ .
3 for  $\forall(x_i, y_j) \in (\mathcal{R})$  do
4   if  $(x_i, y_j)$  satisfy  $x_i^2 + y_j^2 \geq 1$  then
5     Output:  $u(x_i, y_j) = 0$ .
6   else
7     for  $\forall(x_i, y_j)$  that satisfy  $x_i^2 + y_j^2 < 1$  do
8       (1) Using FDM, establish the system of linear equations  $AX = B$ 
9         which depends on parameter  $\mu$ .
10      (2) Solve the system of linear equations  $AX = B$  by the
11        Gauss-Seidel's Method for given initial value  $\mu = \mu_0$ .
12      if  $\exists u(x_i, y_j) < 0$  then
13        Change Initial value of  $\mu = \mu_0$ ; goto (2)
14      else
15        (3) while  $\forall u(x_i, y_j) \geq 0$  do
16          (3.1)  $\mu = \mu + k$ .
17          (3.2) Solve the system of linear equations  $AX = B$  by the
18            Gauss-Seidel's Method.
19          (3.3) if  $\exists u(x_i, y_j) < 0$  then
20            Output:  $\mu_c = \mu - k$ ;  $u(x_i, y_j) \geq 0$ ;  $u_{max} = \max(u(x_i, y_j))$ .
21            Stop.
22          else
23            Output:  $\mu$ ;  $u(x_i, y_j) \geq 0$ ;  $u_{max} = \max(u(x_i, y_j))$ .
24          end
25        end
26      end
27    end
28  end

```

4. RESULTS

We developed a software in high-level programming language (in this case, Java) based on the algorithm mentioned above. The following tables show the result of (1.1) for the respective partition 40×40 and 70×70 (see Figure 3).

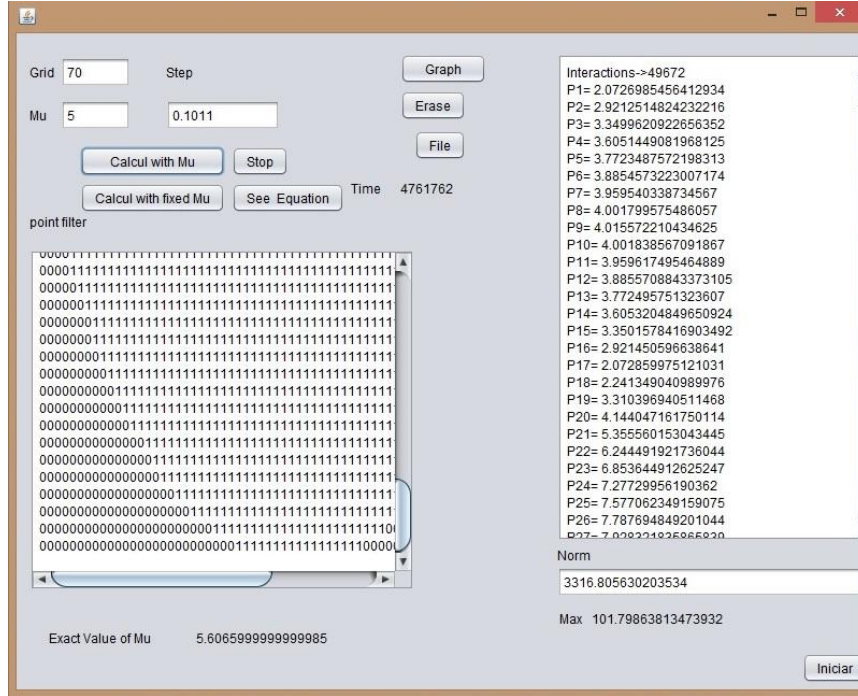


(a) Result for 40×40

4.1. Finding of μ_c . For each grid, we find the respective μ_c and the norm u_{max} given in Table 1.

TABLE 1.

$n \times n$	μ_c	u_{max}
30×30	5.505499999999999	126.26207775999859
40×40	5.606599999999985	697.6299282170618
50×50	5.606599999999985	114.5670432582935
60×60	5.606599999999985	102.23255929486518
70×70	5.606599999999985	101.79863813473932
75×75	5.584800000000021	60.012385961525645



(b) Result for 70×70

FIGURE 3. Result for 40×40 and 70×70

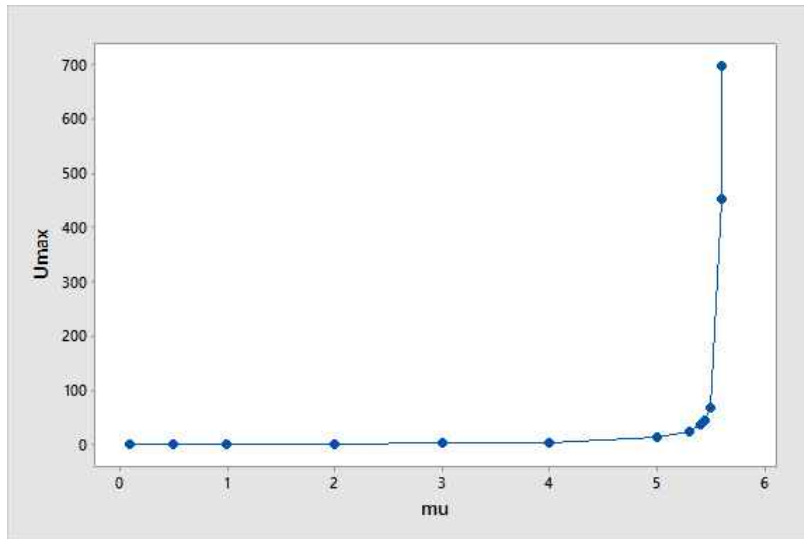


FIGURE 4. Graph of u_{max} vs μ , for 40×40

4.2. **Graph of u_{max} vs μ , for 40×40 .** We have drawn the graph of u_{max} vs μ , for 40×40 , given in Figure 4.

4.3. **The graph of the solution $u(x, y)$, for 50×50 and 75×75 with $\mu = 4$.** The graphs of the solution $u(x, y)$ of the equation (1.1) are drawn with the parameter $\mu = 4$, for the respective partition 50×50 and 75×75 , given in Figure 5.

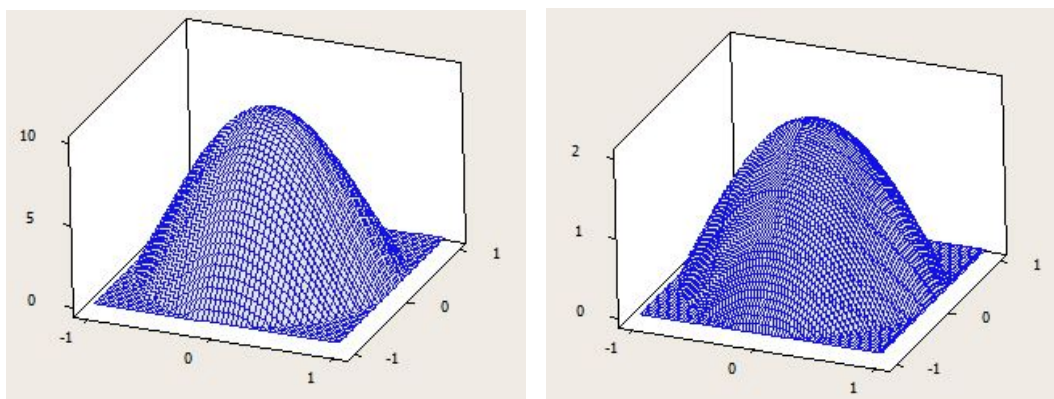


FIGURE 5. Graphs of the solution $u(x, y)$, for 50×50 and 75×75

So, from the Figure 5, we can see that, as $u \in C^2(\bar{B}_1 = \{(x, y) : x^2 + y^2 \leq 1\})$ is a positive solution of the Bratu's equation for $\mu \leq \mu_c$, then u is a function of $r = \sqrt{x^2 + y^2}$.

5. CONCLUSION

In this paper, we have studied the boundary value problem on the unit circle for the Bratu's equation which depends on a real parameter μ , we show that the boundary equation has the no non-negative solutions when $\mu \geq \mu_c = 5.7831859629467$, where we have implemented the numerical method, that is, the combination of the adaptation of finite difference and Gauss-Seidel method, which allows us to obtain a good approach of μ_c with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

A possible application of these results could be to the simplified stationary model for energy functional related to thermo-electro-hydrodynamics description of electro-spinning [14].

6. APPENDIX

Example 6.1. Let $n = 4$, so $h = \frac{2}{n} = \frac{1}{2}$. The grid has $5 \times 5 = 25$ points, in which 9 are interior points of the circle, i.e., $P_{1,1}, P_{1,2}, P_{1,3}, P_{2,1}, P_{2,2}, P_{2,3}, P_{3,1}, P_{3,2}, P_{3,3}$ (see Figure 6).

The points which satisfy $x_i^2 + y_j^2 \geq 1$, i.e.,

$$P_{0,0} : u(x_0, y_0) = u(-1, -1) = w_{0,0} = 0,$$

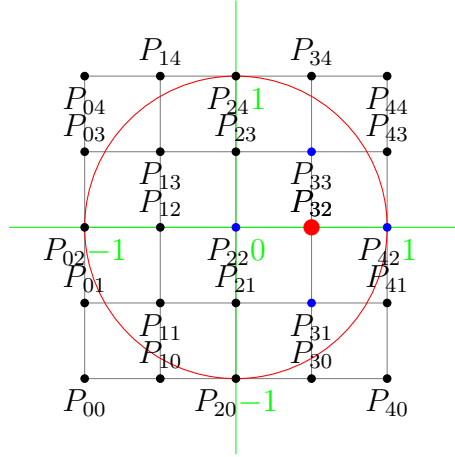


FIGURE 6.

$$\begin{aligned}
 P_{0,1} : u(x_0, y_1) &= u\left(-1, -\frac{1}{2}\right) = w_{0,1} = 0, \\
 P_{0,2} : u(x_0, y_2) &= u(-1, 0) = w_{0,2} = 0, \\
 P_{0,3} : u(x_0, y_3) &= u\left(-1, \frac{1}{2}\right) = w_{0,3} = 0, \\
 P_{0,4} : u(x_0, y_4) &= u(-1, 1) = w_{0,4} = 0, \\
 &\vdots \\
 P_{4,0} : u(x_4, y_0) &= u(1, -1) = w_{4,0} = 0, \\
 P_{4,1} : u(x_4, y_1) &= u\left(1, -\frac{1}{2}\right) = w_{4,1} = 0, \\
 P_{4,2} : u(x_4, y_2) &= u(1, 0) = w_{4,2} = 0, \\
 P_{4,3} : u(x_4, y_3) &= u\left(1, \frac{1}{2}\right) = w_{4,3} = 0, \\
 P_{4,4} : u(x_4, y_4) &= u(1, 1) = w_{4,4} = 0.
 \end{aligned}$$

So, for the interior points of the circle which satisfy $x_i^2 + y_j^2 < 1$:

$$\begin{aligned}
 P_{1,1} : \left(4 - \frac{\mu}{4}\right) w_{1,1} - w_{2,1} - w_{0,1} - w_{1,2} - w_{1,0} &= \frac{\mu}{4}, \\
 P_{1,2} : \left(4 - \frac{\mu}{4}\right) w_{1,2} - w_{2,2} - w_{0,2} - w_{1,3} - w_{1,1} &= \frac{\mu}{4}, \\
 P_{1,3} : \left(4 - \frac{\mu}{4}\right) w_{1,3} - w_{2,3} - w_{0,3} - w_{1,4} - w_{1,2} &= \frac{\mu}{4}, \\
 P_{2,1} : \left(4 - \frac{\mu}{4}\right) w_{2,1} - w_{3,1} - w_{1,1} - w_{2,2} - w_{2,0} &= \frac{\mu}{4}, \\
 P_{2,2} : \left(4 - \frac{\mu}{4}\right) w_{2,2} - w_{3,2} - w_{1,2} - w_{2,3} - w_{2,1} &= \frac{\mu}{4},
 \end{aligned}$$

$$\begin{aligned}
 P_{2,3} &: \left(4 - \frac{\mu}{4}\right) w_{2,3} - w_{3,3} - w_{1,3} - w_{2,4} - w_{2,2} = \frac{\mu}{4}, \\
 P_{3,1} &: \left(4 - \frac{\mu}{4}\right) w_{3,1} - w_{4,1} - w_{2,1} - w_{3,2} - w_{3,0} = \frac{\mu}{4}, \\
 P_{3,2} &: \left(4 - \frac{\mu}{4}\right) w_{3,2} - w_{4,2} - w_{2,2} - w_{3,3} - w_{3,1} = \frac{\mu}{4}, \\
 P_{3,3} &: \left(4 - \frac{\mu}{4}\right) w_{3,3} - w_{4,3} - w_{2,3} - w_{3,4} - w_{3,2} = \frac{\mu}{4}.
 \end{aligned}$$

It is not defficult to establish the system of linear equations $AX = B$, where

$$A = \begin{bmatrix}
 (4 - \frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & (4 - \frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & -1 & (4 - \frac{\mu}{4}) & 0 & 0 & -1 & 0 & 0 & 0 \\
 -1 & 0 & 0 & (4 - \frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 \\
 0 & -1 & 0 & 0 & (4 - \frac{\mu}{4}) & -1 & 0 & -1 & 0 \\
 0 & 0 & -1 & 0 & -1 & (4 - \frac{\mu}{4}) & 0 & 0 & -1 \\
 0 & 0 & 0 & -1 & 0 & 0 & (4 - \frac{\mu}{4}) & -1 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & -1 & (4 - \frac{\mu}{4}) & -1 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & (4 - \frac{\mu}{4})
 \end{bmatrix},$$

$$X = [w_{1,1} \ w_{1,2} \ w_{1,3} \ w_{2,1} \ w_{2,2} \ w_{2,3} \ w_{3,1} \ w_{3,2} \ w_{3,3}]^T,$$

$$B = \left[\frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4} \ \frac{\mu}{4}\right]^T.$$

Acknowledgements. This work was partially supported by Instituto Politécnico Nacional, UPIIZ, and by DICIS-University of Guanajuato.

We would like to thank the reviewers for their useful comments and corrections.

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