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INEQUALITIES FOR MAXIMUM MODULUS OF RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract. In this paper we prove some results concerning the rational functions with prescribed poles and restricted zeros. These results in fact generalize or strengthen some known inequalities for rational functions with prescribed poles and in turn produce new results besides the refinements of some known polynomial inequalities. Our method of proof may be useful for proving other inequalities for polynomials and rational functions.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most n and $P'(z)$ be the derivative of $P(z)$. Let $D_k^- := \{z : |z| < k\}$, $D_k^+ := \{z : |z| > k\}$ and $T_k := \{z : \ |z| = k\}$. For a function f defined on the circle T_1 in the complex plane $\mathcal{C}$, we write

$$\|f\| := \sup_{z \in T_1} |f(z)|, \quad w(z) := \prod_{j=1}^{n} (z - a_j)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \dots, a_n) := \left\{ \frac{p(z)}{w(z)} : p \in \mathcal{P}_n \right\},$$

where $a_j \in D_1^+,\ j = 1, 2, \dots n$.

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Thus, $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \dots, a_n$ at most and with finite limit at ∞. We observe that the Blashke product $B(z) \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right).$$

For every $P \in \mathcal{P}_n$, the following inequality is due to Bernstein [5]:

$$\|P'\| \leq n\|P\|,$$

where as by an application of maximum modulus principle

$$\|P(R, \cdot)\| \leq R^n \|P\|,$$

where $\|P(R, \cdot)\| = \sup_{z \in T_R} |P(z)|$. Both these inequalities are sharp and equality holds for polynomials having all zeros at the origin. In case $P(z) \neq 0$ for $z \in D_1^-$, then we have for $z \in T_1$

$$\|P'\| \leq \frac{n}{2}\|P\| \tag{1.1}$$

and

$$\|P(R, \cdot)\| \leq \frac{R^n + 1}{2}\|P\|, \tag{1.2}$$

whereas if $P(z) \neq 0$ for $z \in D_1^+$, then

$$\|P'\| \geq \frac{n}{2}\|P\|. \tag{1.3}$$

Inequality (1.1) was conjectured by Erdös and proved by Lax [9], whereas inequality (1.2) is due to Ankeny and Rivilin [1]. Inequality (1.3) is due to Turán [14]. In all the inequalities (1.1), (1.2), and (1.3) equality holds for polynomials having all zeros on the unit disk.

Li, Mohapatra and Rodriguez [10] extended inequalities (1.1) and (1.3) to rational functions $r \in \mathcal{R}_n$ and proved the following results.

Theorem 1.1. *Suppose $r \in \mathcal{R}_n$ and all the zeroes of r lie in $T_1 \cup D_1^+$. Then for $z \in T_1$*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|\|r\|.$$

Theorem 1.2. *Suppose $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ and all the zeros of r lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\{|B'(z)| - (n - m)\}|r(z)|,$$

where m is the number of zeros of r.

The inequality (1.2) was extended to rational functions by Govil and Mohapatra [7] (see also Aziz and Rather [3]) to read as follows.

Theorem 1.3. *Suppose $r \in \mathcal{R}_n$ and all the zeroes of r lie in $T_1 \cup D_1^+$. Then for $z \in T_1$*

$$|r'(z)| \leq \frac{|B(Rz)| + 1}{2}|r(z)|.$$

2. Main Results

Theorem 2.1. *Suppose $r \in \mathcal{R}_n$, where r has n poles at $a_1, a_2, \dots, a_n$ and all the zeros of r lie in $T_1 \cup D_1^-$ with a zero of multiplicity s at origin, then the following inequality holds for each point $z \in T_1$, such that $r(z) \neq 0$*

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{1}{2}\left\{|B'(z)| + (s+m-n) + \frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}, \tag{2.1}$$

where m is the number of zeros of r. Inequality (2.1) *is sharp and equality holds for*

$$r(z) = \frac{z^s(z^{m-s}-1)}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad z \in T_1,\ a \geq 1.$$

Since $\left|\frac{zr'(z)}{r(z)}\right| \geq \operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\}$, from Theorem 2.1, we immediately have the following.

Corollary 2.1. *Suppose $r \in \mathcal{R}_n$, where r has n poles at $a_1, a_2, \dots, a_n$ and all its zeros lie in $T_1 \cup D_1^-$, with s-fold zeros at origin, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + (s+m-n) + \frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}|r(z)|, \tag{2.2}$$

where m is the number of zeros of r. The result is sharp and equality holds for

$$r(z) = \frac{z^s(z^{m-s}-1)}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n,$$

at $z = 1$ and $a \geq 1$.

Note. Inequality (2.2) is trivally true in case $r(z) = 0$ for $z \in T_1$.

If we take $s = 0$ in Corollary 2.1, we get the following result, which is an improvement of Theorem 1.2, earlier proved by Li, Mohapatra and Rodriguez [10, Theorem 4].

Corollary 2.2. *Suppose $r \in \mathcal{R}_n$, where r has n poles and all the zeros of r lie in $T_1 \cup D_1^-$. Then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - (n-m) + \frac{|c_m|-|c_0|}{|c_m|+|c_0|}\right\}|r(z)|,$$

where m is the number of zeros of r. The result is sharp and equality holds for

$$r(z) = \frac{(z+1)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n$$

at $z = 1$ and $a \geq 1$.

Since $|c_m| \geq |c_0|$, therefore as mentioned above, Corollary 2.2 is an improvement of Theorem 1.2. In case number of poles of r is same as its zeros, that is, when $m = n$ then Corollary 2.2 gives an improvement of [4, inequality (12)].

Taking $a_i = \alpha,\ i = 1, 2, \dots, n$, in Corollary 1, we have the following.

Corollary 2.3. *If $P(z)$ is a polynomial of degree m having all zeros in $T_1 \cup D_1^-$ with s-fold zero at origin, then*

$$|D_\alpha P(z)| \geq \frac{|\alpha|-1}{2}\left\{m+s+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}|P(z)|, \tag{2.3}$$

where $D_\alpha P(z) := nP(z)+(\alpha-z)P'(z)$ is called the polar derivative of the polynomial $P(z)$ with respect to the point α and it generalizes the ordinary derivative of $P(z)$ of degree n in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z).$$

By taking $s = 0$ in Corollary 2.3, we get the following sharp result which is also an extension of a result of Dubinin [6] to the polar derivative of $P(z)$.

Corollary 2.4. *If $P(z) := \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all zeros in $T_1 \cup D_1^-$, then*

$$|D_\alpha P(z)| \geq \frac{|\alpha|-1}{2}\left\{n+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}|P(z)|. \tag{2.4}$$

Equality in (2.4) holds for a polynomial $P(z) = (z-1)^n$ with $\alpha > 1$. Since $|c_n| \geq |c_0|$, it follows that Corollary 2.4 is a refinement of a result of Shah [13].

Dividing both sides of (2.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 2.5. *If $P \in \mathcal{P}_n$ is such that $P(z)$ has all its zeros in $T_1 \cup D_1^-$ with s-fold zero at origin, then*

$$|P'(z)| \geq \frac{1}{2}\left(n+s+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right)|P(z)|. \tag{2.5}$$

For $s = 0$ (2.5) reduces to the result of Dubinin [6] and is an improvement of a classical result of Turán [14].

Next we prove the following refinement of a result of Aziz and Shah [4, Theorem 1].

Theorem 2.2. *Suppose $r \in \mathcal{R}_n$, where r has exactly n poles $a_1, a_2, \ldots, a_n$ and all the zeros of r lie in $T_k \cup D_k^-$, $k \leq 1$, with a zero of order s at origin. Then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)|-n+\frac{2(m+sk)}{1+k}\right\}|r(z)|, \tag{2.6}$$

where m is the number of zeros of r. The result is sharp and equality holds for

$$r(z) = \frac{z^s(z+k)^{m-s}}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n$$

at $z = 1$ and $a \geq 1$.

The result of Aziz and Shah [4, Theorem 1] is a special case of Theorem 2.2, if we take $s = 0$.

As in previous case, if we take $a_i = \alpha$, $i = 1, 2, \ldots, n$, in Theorem 2.2, we get the following result on the polar derivatives of a polynomial.

Corollary 2.6. *If $P \in P_n$ is such that $P(z) \neq 0$ in D_k^+, $k \leq 1$ with s-fold zero at origin, then for every α with $|\alpha| \geq 1$*

$$|D_\alpha P(z)| \geq \frac{n(|\alpha|-1)(1+ks)}{1+k}|P(z)|. \tag{2.7}$$

Remark 2.1. In Corollary 2.6 if we take $s = 0$, we have the following generalization of a result of Shah [13].

Corollary 2.7. *If $P \in \mathcal{P}_n$ is such that $P(z)$ has all zeros in $T_k \cup D_k^-$, then for $|\alpha| \geq 1$ and $z \in T_1$*

$$|D_\alpha P(z)| \geq \frac{n(|\alpha|-1)}{1+k}|P(z)|.$$

Remark 2.2. Dividing both sides of (2.7) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following sharp result.

Corollary 2.8. *If $P \in \mathcal{P}_n$ is such that $P(z) \neq 0$ for $z \in D_k^+$ with s-fold zero at origin, then for $z \in T_1$*

$$|P'(z)| \geq \frac{n+ks}{1+k}|P(z)|.$$

Equality holds for $P(z) = z^s(z-k)^{n-s}$.

For $s = 0$, this gives result of Malik [11], whereas for $k = 1$, $s = 0$, it reduces to the classical theorem of Turán [14].

Theorem 2.3. *Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $T_k \cup D_k^+$. Then for $z \in T_1$*

$$|r(Rz)| \leq \frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\Big\{|B(Rz)|+1\Big\}\|r\|. \tag{2.8}$$

Remark 2.3. Theorem 1.3 is a special case of Theorem 2.3, when $k = 1$.

Remark 2.4. Let $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, so that

$$r(z) = \frac{P(z)}{(z-\alpha)^n} \text{ and } B(z) = \prod_1^n \frac{1-\overline{\alpha}z}{z-\alpha} = \Big(\frac{1-\overline{\alpha}z}{z-\alpha}\Big)^n.$$

Using this in Theorem 2.3, it can be easily verified that for $|\alpha| \geq R > 1$ and $z \in T_1$

$$|P(Rz)| \leq \frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\Big\{\Big|\frac{1-\overline{\alpha}Rz}{Rz-\alpha}\Big|^n+1\Big\}\|P\|.$$

Letting $|\alpha| \to \infty$, we get the following result.

If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k \geq 1$, then for $R > 1$ and $z \in T_1$

$$\|P(R,\cdot)\| \leq \frac{(R+k)^n(R^n+1)}{(R+k)^n+(1+Rk)^n}\|P\|.$$

This result was earlier proved by Aziz and Mohammad [2].

3. Lemmas and Proofs

For the proofs of these theorems we need the following lemmas.

Lemma 3.1 ([8])**.** *Let $f : D \to D$ be holomorphic. Assume that $f(0) = 0$. Further assume that there is a $b \in \partial D$, the boundary of D, so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

The next lemma is due to Aziz and Rather.

Lemma 3.2 ([3])**.** *If $r \in \mathcal{R}_n$ and $z \in T_1$, then for every $R \geq 0$,*

$$|r(Rz)| + |r^*(Rz)| \leq \{|B(Rz)| + 1\}\|r\|,$$

where $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$.

Proof of Theorem 2.1. Suppose that $r(z) \neq 0$ for $z \in T_1$ and all the poles of $r(z)$ lie in D_1^+. Since $r(z)$ has a zero at origin of multiplicity s. Therefore,

$$r(z) = \frac{P(z)}{w(z)} = \frac{z^s h(z)}{w(z)},$$

where

$$h(z) := \sum_{j=0}^{m-s} c_{s+j} z^j = c_m \prod_{j=1}^{m-s} (z - z_j), \quad z_j \in D_1^-,\ j = 1, 2, \ldots, m - s,$$

and

$$w(z) = \prod_{j=1}^{n} (z - a_j).$$

This gives

$$\frac{r'(z)}{r(z)} = \frac{s}{z} + \frac{h'(z)}{h(z)} - \frac{w'(z)}{w(z)}.$$

Equivalently,

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) = s + \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) - \operatorname{Re}\left(\frac{zw'(z)}{w(z)}\right). \tag{3.1}$$

Since $h(z)$ has all zeros in D_1^-, therefore

$$h^*(z) = z^{m-s}\overline{h\left(\frac{1}{\bar{z}}\right)}$$

has all zeros in D_1^+, and hence

$$G(z) = \frac{zh(z)}{h^*(z)} = z\frac{c_m}{\overline{c_m}} \prod_{j=1}^{m-s} \left(\frac{z - z_j}{1 - z\overline{z_j}}\right) \tag{3.2}$$

is analytic in $T_1 \cup D_1^-$, with $G(0) = 0$ and $|G(z)| = 1$ for $z \in T_1$.

Applying Lemma 3.1 to $G(z)$, we get for $z \in T_1$

$$|G'(z)| \geq \frac{2}{1+|G'(0)|}. \tag{3.3}$$

Since

$$|G'(0)| = \left|\prod_{j=1}^{m-s} z_j\right| = \frac{|c_s|}{|c_m|},$$

it can be easily verified (see [15, proof of Lemma 1]) that for every $z \in T_1$

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \geq \frac{m-s-1}{2} + \frac{|c_m|}{|c_m|+|c_s|}. \tag{3.4}$$

Again we have

$$B(z) = \frac{w^*(z)}{w(z)},$$

where

$$w^*(z) = z^n \overline{w\left(\frac{1}{\overline{z}}\right)}.$$

This gives (see [15, Lemma 1])

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \frac{n-|B'(z)|}{2}. \tag{3.5}$$

Now using (3.4) and (3.5) in (3.1), we conclude that

$$\operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left\{s + |B'(z)| - (n-m) + \frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}.$$

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. Suppose that for each point $z \in T_1$, $r(z) \neq 0$ and all the poles of $r(z)$ lie in D_1^+. Since $r(z)$ has a zero of order s at origin, therefore

$$r(z) = \frac{P(z)}{w(z)} = \frac{z^s Q(z)}{w(z)},$$

where

$$Q(z) = \sum_{j=0}^{m-s} c_{s+j} z^j$$

is a polynomial of degree $m-s$ having all zeros in $|z| \leq k$. This gives

$$\operatorname{Re}\frac{zr'(z)}{r(z)} = s + \operatorname{Re}\frac{zQ'(z)}{Q(z)} - \operatorname{Re}\frac{zw'(z)}{w(z)}. \tag{3.6}$$

We write $Q(z) = c_m \prod_{j=1}^{m-s}(z-z_j)$, $|z_j| \leq k \leq 1$, and it can be easily verified that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geq \frac{m-s}{1+k}.$$

Also, as in (3.5)

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \frac{n - |B'(z)|}{2}.$$

Using this in (3.6), we get

$$\operatorname{Re}\frac{zr'(z)}{r(z)} \geq s + \frac{m-s}{1+k} - \frac{n-|B'(z)|}{2}.$$

Now for $z \in T_1$ such that $r(z) \neq 0$, we have

$$\left|\frac{zr'(z)}{r(z)}\right| \geq \operatorname{Re}\frac{zr'(z)}{r(z)} \geq \frac{1}{2}\left\{|B'(z)| + \frac{2(m+sk) - n(1+k)}{1+k}\right\}.$$

This gives for $z \in T_1$ such that $r(z) \neq 0$,

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - n + \frac{2(m+sk)}{1+k}\right\}|r(z)|.$$

Since the result is trivially true if $r(z) = 0$ for $z \in T_1$, it follows that

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - n + \frac{2(m+sk)}{(1+k)}\right\}|r(z)|,$$

for all $z \in T_1$. The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3. Since zeros of $r(z)$ lie in $T_k \cup D_k^+$, therefore all zeros of $P(z)$ lie in $T_k \cup D_k^+$, $k \geq 1$ and $w(z) = \prod_{j=1}^{n}(z - \alpha_j)$, $|\alpha_j| > 1$ for all $j = 1, 2, \ldots, n$.

Also

$$\begin{aligned} r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)} &= \prod_{j=1}^{n}\left(\frac{1-\overline{\alpha_j}z}{z-\alpha_j}\right)\overline{\left(\frac{P(\frac{1}{\bar{z}})}{w(\frac{1}{\bar{z}})}\right)} \\ &= \prod_{j=1}^{n}\left(\frac{1-\overline{\alpha_j}z}{z-\alpha_j}\right)\frac{z^n\overline{P(\frac{1}{\bar{z}})}}{\prod_{j=1}^{n}(1-\overline{\alpha_j}z)} = \frac{z^n\overline{P(\frac{1}{\bar{z}})}}{\prod_{j=1}^{n}(z-\alpha_j)} = \frac{P^*(z)}{w(z)}. \end{aligned}$$

Therefore,

$$\frac{r(z)}{r^*(z)} = \frac{P(z)}{P^*(z)}. \tag{3.7}$$

We write

$$P(z) = \prod_{j=1}^{n}(z - r_j e^{i\theta_j}), \quad \text{where } r_j \geq k \geq 1,\ j = 1, 2, \ldots, n,$$

so that $P^*(z) = z^n\overline{P(\frac{1}{\bar{z}})} = \prod_{j=1}^{n}(1 - zr_j e^{i\theta_j})$.

For $0 \leq \theta < 2\pi$ and $R > 1$ we have

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{1 - r_j Re^{i(\theta-\theta_j)}}\right|. \tag{3.8}$$

Now

$$\begin{aligned}
\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{1-r_jRe^{i(\theta-\theta_j)}}\right|^2 &= \prod_{j=1}^{n}\left|\frac{Re^{i(\theta-\theta_j)}-r_j}{1-r_jRe^{i(\theta-\theta_j)}}\right|^2\\
&= \prod_{j=1}^{n}\left(\frac{Re^{i(\theta-\theta_j)}-r_j}{1-r_jRe^{i(\theta-\theta_j)}}\cdot\frac{Re^{-i(\theta-\theta_j)}-r_j}{1-r_jRe^{-i(\theta-\theta_j)}}\right)\\
&= \prod_{j=1}^{n}\frac{R^2-2Rr_j\cos(\theta-\theta_j)+r_j^2}{1-2Rr_j\cos(\theta-\theta_j)+r_j^2R^2}\le\prod_{j=1}^{n}\left(\frac{R+r_j}{1+Rr_j}\right)^2.
\end{aligned}$$

Therefore, from (3.8), we have

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right|^2\le\prod_{j=1}^{n}\left(\frac{R+r_j}{1+Rr_j}\right)^2. \tag{3.9}$$

Since $|r_j|\ge k,\ k\ge 1$, it can be easily verified that

$$\frac{R+r_j}{1+Rr_j}\le\frac{R+k}{1+Rk}.$$

Using this in (3.9), we get

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right|\le\prod_{j=1}^{n}\left(\frac{R+k}{1+Rk}\right)=\left(\frac{R+k}{1+Rk}\right)^n. \tag{3.10}$$

Combining (3.7) and (3.10), we get

$$\left|\frac{r(Rz)}{r^*(Rz)}\right|\le\left(\frac{R+k}{1+Rk}\right)^n,\quad \text{for } z\in T_1,\ k\ge 1,\ R>1.$$

Equivalently,

$$\left(\frac{R+k}{1+Rk}\right)^{-n}|r(Rz)|\le|r^*(Rz)|. \tag{3.11}$$

Now using Lemma 3.2, we get from (3.11)

$$\left\{\left(\frac{R+k}{1+Rk}\right)^{-n}+1\right\}|r(Rz)|\le|r(Rz)|+|r^*(Rz)|.$$

That is,

$$|r(Rz)|\le\frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\{|B(Rz)|+1\}|r|.$$

The proof of Theorem 2.3 is completed. □

Proof of Corollary 2.3. Since $r(z)$ has a pole of order n at $z=\alpha,\ |\alpha|>1$, we have

$$r(z)=\frac{P(z)}{(z-\alpha)^n}.$$

Therefore,

$$r'(z) = \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}} = \frac{-[nP(z) + (\alpha - z)P'(z)]}{(z-\alpha)^{n+1}} = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}.$$

Also,

$$B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$$

gives

$$B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using these facts, we immediately get for $z \in T_1$ from inequality (2.2)

$$\left|\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}\right| \geq \frac{1}{2}\left\{\frac{n|z-\alpha|^{n-1}(|\alpha|^2-1)}{|z-\alpha|^{n+1}} + \left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)\right\}\frac{|P(z)|}{|z-\alpha|^n}.$$

This gives

$$\begin{aligned}
|D_\alpha P(z)| &\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|} + \left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)|z-\alpha|\right\}|P(z)| \\
&\geq \frac{(|\alpha|-1)}{2}\left\{n + \left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)\right\}|P(z)| \\
&= \frac{(|\alpha|-1)}{2}\left\{s+m+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}|P(z)|, \quad z \in T_1,\ |\alpha| \geq 1.
\end{aligned}$$

Using the argument of continuity in case of poles the proof of Corollary 2.3 completes. □

Proof of Corollary 2.6. Since we have

$$r(z) = \frac{P(z)}{(z-\alpha)^n} \quad \text{and} \quad B(z) = \left(\frac{1-\alpha z}{z-\alpha}\right)^n,$$

therefore

$$r'(z) = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}$$

and

$$B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using in inequality (2.6), with $m = n$, we get

$$\left|\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}\right| \geq \frac{1}{2}\left\{\frac{n|z-\alpha|^{n-1}(|\alpha|^2-1)}{|z-\alpha|^{n+1}} + \frac{n(1-k)+2ks}{1+k}\right\}\frac{|P(z)|}{|z-\alpha|^n}.$$

This gives

$$|D_\alpha P(z)| \geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|} + \frac{n(1-k)+2ks}{1+k}|z-\alpha|\right\}|P(z)|$$

$$
\begin{aligned}
&\geq \frac{1}{2}\left\{n(|\alpha|-1)+\frac{n(1-k)+2ks}{1+k}(|\alpha|-1)\right\}|P(z)| \\
&= \frac{n(|\alpha|-1)(1+ks)}{1+k}|P(z)|, \quad z\in T_1,\ |\alpha|\geq 1.
\end{aligned}
$$

Using the argument of continuity in case of poles the proof of Corollary 2.6 completes.

□

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