

CUBIC DECOMPOSITIONS OF THE q -APPELL POLYNOMIAL SEQUENCES

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ABSTRACT. In this paper, we delve into a comprehensive examination of a specific type of lowering operator denoted as Λ_q . This operator is defined as a finite sum of lowering operators and can be formulated in various configurations. Our focus lies in characterizing polynomial sequences that adhere to an Appell relation with respect to Λ_q . Through a detailed exploration of a concrete cubic decomposition of a simple q -Appell sequence, we establish that the polynomial component sequences exhibit Λ_{q^3} -Appell properties, with Λ_{q^3} defined as previously, albeit through a three-term sum.

1. INTRODUCTION AND MOTIVATION

Let \mathcal{P} be the vector space of polynomials with complex coefficients. Let \mathcal{O} be a lowering operator, that is, a linear operator that decreases in one unit the degree of a polynomial and such that $\mathcal{O}(1) = 0$, i.e., an operator that satisfies the following conditions

$$\mathcal{O}(\mathcal{P}) = \mathcal{P}, \quad \mathcal{O}(1) = 0 \quad \text{and} \quad \deg\{\mathcal{O}(x^n)\} = n - 1, \quad n \in \mathbb{N},$$

where, and in what follows, \mathbb{N} denotes, as usual, the set of natural numbers and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \dots\}.$$

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Given a MPS $\{B_n\}_{n \in \mathbb{N}_0}$, we construct another MPS

$$\left\{ B_n^{[1]}(\cdot; \mathcal{O}) := \frac{1}{\rho_{n+1}(\mathcal{O})} \mathcal{O}(B_{n+1}(\cdot)) \right\}_{n \in \mathbb{N}_0},$$

where $\rho_{n+1}(\mathcal{O}) \neq 0$ is such that $\mathcal{O}(x^{n+1}) = \rho_{n+1}(\mathcal{O})\varpi_n(x)$ where $\varpi_n(x)$ is a monic polynomial such that $\deg \varpi_n = n$, for any integer $n \in \mathbb{N}_0$.

Definition 1.1 ([6, 7]). A MPS $\{B_n\}_{n \in \mathbb{N}_0}$ is called an Appell sequence with respect to a lowering (annihilating) operator \mathcal{O} or simply \mathcal{O} -Appell sequence if $B_n^{[1]}(\cdot, \mathcal{O}) = B_n(\cdot)$ for every $n \in \mathbb{N}_0$.

This expands upon the concept initially introduced by Appell [4]. When \mathcal{O} is a lowering operator such that $\varpi_n(x) = x^n$, a sequence $\{B_n\}_{n \in \mathbb{N}_0}$ is termed an \mathcal{O} -Appell sequence if and only if the sequence $\{B_n\}_{n \in \mathbb{N}_0}$ has as a generating function

$$A(t)B(xt) = \sum_{n \geq 0} B_n(x) \frac{t^n}{(\rho_n)!},$$

where $A(t) = \sum_{n \geq 0} \frac{1}{(\rho_n)!} a_n t^n$, $B(t) = \sum_{n \geq 0} \frac{1}{(\rho_n)!} t^n$ and $(\rho_n)! = \rho_1 \rho_2 \cdots \rho_n$ for $n \in \mathbb{N}$, with the conventions $a_0 = 1$ and $(\rho_0)! = 1$. The polynomials having this property are known as Brenke-type polynomials [12].

Hahn has delineated the assembly of orthogonal polynomial sequences $\{B_n\}_{n \in \mathbb{N}}$ that share the characteristic of the derivative sequence $\{B'_n\}_{n \in \mathbb{N}}$ is orthogonal, commonly referred to as classical sequences (see [3], [24] and [31]). However, one might inquire about the common properties among all orthogonal sequences $\{B_n\}_{n \in \mathbb{N}}$ such that the novel sequence $\{\mathcal{O}B_n(\cdot)\}_{n \in \mathbb{N}}$, where \mathcal{O} represents either a lowering operator or a (linear) isomorphism in \mathcal{P} , is also an orthogonal sequence. In other words, with the introduction of an operator \mathcal{O} mapping \mathcal{P} into itself, possessing a certain number of necessary properties, we are, in Hahn's sense, seeking all the \mathcal{O} -classical sequences. Regarding their significance, a more formal description follows.

Definition 1.2. A MOPS $\{B_n\}_{n \in \mathbb{N}}$ is said to be an \mathcal{O} -classical sequence whenever the MPS $\{B_n^{[1]}(\cdot; \mathcal{O})\}_{n \in \mathbb{N}_0}$ is also orthogonal.

The origins of this problem trace back to 1949, when Hahn [13] unveiled notable properties shared by what are now known as $\mathcal{L}_{q,\omega}$ -classical sequences. Here, $\mathcal{L}_{q,\omega}$ is defined as $\mathcal{L}_{q,\omega}f(x) := \frac{f(qx+\omega)-f(x)}{(q-1)x+\omega}$, with complex numbers q, ω , and for any $f \in \mathcal{P}$. This inquiry has given rise to various questions, particularly in the realm of classical q -analog polynomial sequences, a domain that has been extensively explored.

The \mathcal{O} -classical sequences, specifically when $\mathcal{O} = \mathcal{L}_{1,\omega}$ or $\mathcal{L}_{q,0}$, are thoroughly elucidated in the works [1] and [15], respectively. Recent research has delved into Dunkl-classical polynomial sequences, which are classical sequences with respect to the Dunkl (resp. q -Dunkl) operator defined as $T_\mu = D + \mu H_{-1}$ (resp. $T_{\theta,q} = H_q + \theta H_{-1}$). Noteworthy contributions in this area include the works [8, 21, 36]. Recently, a further

lowering operator to treat analogous questions was considered (see [22, 23, 30]). These examples underscore the importance of studying classical sequences in a broader sense, as it unveils desirable properties of certain orthogonal sequences.

On the other hand, the study of the quadratic and the cubic decomposition of an orthogonal polynomial sequence (OPS) is a subject that received special attention along the last decades (see [9–11, 16–19, 25, 29]).

This work deals with Appell sequences with respect to lowering (or annihilating) operators involving the Hahn’s operator, often called Jackson’s q -derivative operator, here denoted as H_q :

$$(H_q f)(x) := \frac{f(qx) - f(x)}{(q - 1)x}, \quad f \in \mathcal{P},$$

where q belongs to the set $\tilde{\mathbb{C}} := \mathbb{C} - \bigcup_{n \in \mathbb{N}_0} U_n$, where

$$U_n = \begin{cases} \{0\}, & n = 0, \\ \{z \in \mathbb{C} : z^n = 1\}, & n \in \mathbb{N}. \end{cases}$$

The q -Appell sequences have been extensively investigated [2, 15, 34, 37]. Notably, the existence of orthogonal q -Appell polynomial sequences is well-documented and has been approached from various perspectives. These sequences are equivalent to the Al-Salam and Carlitz polynomial sequence [2, 15], albeit under a linear transformation. For a more in-depth exploration, refer to Ismail’s book [14, Chap. 18].

Our aim is to explore the cubic decomposition (CD) of these q -Appell sequences. In fact, while investigating the cubic decomposition of a q -Appell sequence, we came across polynomial sequences that satisfy a similar identity: $\Lambda_q B_{n+1}(x) = \rho_{n,q} B_n(x)$, $n \in \mathbb{N}_0$, where Λ_q is a certain lowering operator, and $\rho_{n,q}$ are normalization constants. The specific lowering operators arising from the context of the cubic decomposition take the form of either $H_{q^3}(q[3]_q x H_{q^3} + I_{\mathcal{P}})(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})$, $(q[3]_q x H_{q^3} + I_{\mathcal{P}})H_{q^3}(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})$ or $(q[3]_q x H_{q^3} + I_{\mathcal{P}})(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})H_{q^3}$, which can be expressed as $\Lambda_{q^3} = a_{0,q^3} H_{q^3} + a_{1,q^3} H_{q^3} x H_{q^3} + a_{2,q^3} (H_{q^3} x)^2 H_{q^3}$, with constant coefficients a_{0,q^3} , a_{1,q^3} , and a_{2,q^3} .

The various configurations mentioned are easy to elucidate and can be readily identified for a straightforward verification. However, the analysis of these operators, typically expressed with k factors or terms, does not entail a substantial effort and streamlines the comparison of the presented results with subsequent contributions, especially in contexts involving highly specific q -differential operators. Consequently, we establish a concise list of three approaches to present these operators-without imposing particular restrictions-and highlight the relationships among their constant coefficients. Moreover, different versions of these operators (when $q \rightarrow 1$) appear often in the study of special functions, for instance, with regard to the monomiality principle (see, for example, [33]).

The manuscript follows this organization. Section 2 presents the fundamental definitions and necessary tools, as well as the definition and characterization of the so-called \mathcal{O} -Appell polynomial sequences (for some annihilating operator \mathcal{O}) will be recalled. Section 3 is devoted to the study of the operator $\Lambda_q = \sum_{i=0}^k a_{i,q}(H_q x)^i H_q$, where k is a positive integer and $a_{k,q} \neq 0$, and to a characterization of the sequences herein called Λ_q -Appell. To recap, the first two sections set the essential foundation for the upcoming results. In Section 4, we consider a cubic decomposition of a q -Appell sequence and it is indicated what kind of Appell behavior have the respective component sequences. Indeed, the primary objective of this section is to acquire insights into the cubic decomposition of the q -Appell sequences. This is equivalent to characterize the nine polynomial sequences linked to terms of order $3n$, $3n + 1$, and $3n + 2$ in a q -Appell sequence. This cubic decomposition leads to the emergence of Appell sequences with respect to a new operator, which is of second order in H_{q^3} .

2. NOTATIONS, DEFINITIONS AND PRELIMINARIES

2.1. Basic tools. Let \mathcal{P} denote the vector space of polynomials with coefficients in \mathbb{C} and let \mathcal{P}' be its dual. We indicate by $\langle u, f \rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ and we denote by $(u)_n := \langle u, x^n \rangle$, $n \in \mathbb{N}_0$, the moments of u . A form u is equivalent to the numerical sequence $\{(u)_n\}_{n \in \mathbb{N}_0}$.

A linear operator $T : \mathcal{P} \rightarrow \mathcal{P}$ has a transpose ${}^tT : \mathcal{P}' \rightarrow \mathcal{P}'$ defined by

$$(2.1) \quad \langle {}^tT(u), f \rangle = \langle u, T(f) \rangle, \quad u \in \mathcal{P}', f \in \mathcal{P}.$$

Given $\varpi \in \mathcal{P}$ and $u \in \mathcal{P}'$, the form ϖu , called the left-multiplication of u by the polynomial ϖ , is defined by

$$(2.2) \quad \langle \varpi u, p \rangle = \langle u, \varpi p \rangle, \quad \text{for all } p \in \mathcal{P}.$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial $(uf)(x) := \left\langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle$. This allows us to define the Cauchy product of two forms

$$\langle vu, f \rangle := \langle v, uf \rangle, \quad u \in \mathcal{P}', f \in \mathcal{P}.$$

The above product is commutative, associative and distributive with respect to the sum of forms.

We define the operator $\sigma_\varpi : \mathcal{P}' \rightarrow \mathcal{P}'$ by

$$\langle \sigma_\varpi(u), f \rangle := \langle u, \sigma_\varpi(f) \rangle, \quad u \in \mathcal{P}', f \in \mathcal{P},$$

where the linear operator $\sigma_\varpi : \mathcal{P} \rightarrow \mathcal{P}$ is defined by $\sigma_\varpi(f)(x) := f(x^3)$ for every $f \in \mathcal{P}$.

Consequently, this yields

$$\left(\sigma_\varpi(u) \right)_n = (u)_{3n}, \quad n \in \mathbb{N}_0.$$

We define the operator H_q , also known as Jackson operator, by

$$(H_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad f \in \mathcal{P},$$

where q belongs to the set $\tilde{\mathbb{C}}$. This operator is a particular case of a (p, q) -derivative operator [5].

This linear operator H_q has a transpose ${}^t H_q$, from \mathcal{P}' into \mathcal{P}' , defined by duality according to (2.1) and we have the operator H_q on \mathcal{P}' as minus the transpose of the q -derivative operator on \mathcal{P} , that is, ${}^t H_q := -H_q$, so that

$$(2.3) \quad \langle H_q u, f \rangle := -\langle u, H_q f \rangle, \quad f \in \mathcal{P}, u \in \mathcal{P}'.$$

In particular, this yields

$$(H_q u)_n = -[n]_q (u)_{n-1},$$

with the convention $(u)_{-1} = 0$ and

$$[n]_q := \frac{q^n - 1}{q - 1}, \quad \text{for } n \in \mathbb{N}_0.$$

In the sequel, we will call polynomial sequence (PS) to any sequence $\{B_n\}_{n \in \mathbb{N}_0}$ such that $\deg B_n = n$ for all $n \in \mathbb{N}_0$. We will also call monic polynomial sequence (MPS) a PS so that all polynomials have leading coefficient equal to one. Notice that if $\langle u, B_n \rangle = 0$, for all $n \in \mathbb{N}_0$, then $u = 0$. Given a MPS $\{B_n\}_{n \in \mathbb{N}_0}$, there are complex sequences, $\{\beta_n\}_{n \in \mathbb{N}_0}$ and $\{\chi_{n,v}\}_{0 \leq v \leq n, n \in \mathbb{N}_0}$, such that

$$B_0(x) = 1, \quad B_1(x) = x - \beta_0,$$

$$B_{n+2}(x) = (x - \beta_{n+1}) B_{n+1}(x) - \sum_{\nu=0}^n \chi_{n,\nu} B_\nu(x).$$

This relation is called the structure relation of $\{B_n\}_{n \in \mathbb{N}_0}$, and $\{\chi_{n,v}\}_{0 \leq v \leq n, n \in \mathbb{N}_0}$, $\{\beta_n\}_{n \in \mathbb{N}_0}$ are called the structure coefficients. Moreover, there exists a unique sequence $\{u_n\}_{n \in \mathbb{N}_0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{B_n\}_{n \in \mathbb{N}_0}$, such that

$$\langle u_n, B_m \rangle = \delta_{n,m}, \quad n, m \in \mathbb{N}_0,$$

where $\delta_{n,m}$ denotes the Kronecker symbol. Let us remark that, if p is a polynomial and $\langle u_n, p \rangle = 0$ for all $n \in \mathbb{N}_0$, then $p = 0$. Besides, it is well known that [27]

$$\beta_n = \langle u_n, x B_n(x) \rangle, \quad n \in \mathbb{N}_0,$$

$$\chi_{n,v} = \langle u_v, x B_{n+1}(x) \rangle, \quad 0 \leq v \leq n, n \in \mathbb{N}_0.$$

Lemma 2.1 ([27]). *For each $u \in \mathcal{P}'$ and each $m \in \mathbb{N}$, the two following statements are equivalent.*

- (a) $\langle u, B_{m-1} \rangle \neq 0$, $\langle u, B_n \rangle = 0$, $n \geq m$.
- (b) *Exists $\lambda_v \in \mathbb{C}$, $0 \leq v \leq m - 1$, $\lambda_{m-1} \neq 0$, such that $u = \sum_{v=0}^{m-1} \lambda_v u_v$, with $\lambda_v = \langle u, B_v \rangle$.*

Definition 2.1 ([26, 28]). A PS $\{B_n\}_{n \in \mathbb{N}_0}$ is regularly orthogonal with respect to the form u if and only if it fulfils

$$(2.4) \quad \langle u, B_n B_m \rangle = 0, \quad n \neq m, n, m \in \mathbb{N}_0,$$

$$(2.5) \quad \langle u, B_n^2 \rangle \neq 0, \quad n \in \mathbb{N}_0.$$

Then, the form u is said to be regular (or quasi-definite) and $\{B_n\}_{n \in \mathbb{N}_0}$ is an orthogonal polynomial sequence (OPS). The conditions (2.4) are called the orthogonality conditions and the conditions (2.5) are called the regularity conditions.

We can normalize $\{B_n\}_{n \in \mathbb{N}_0}$ in order that it becomes monic; then it is unique and we note it as a MOPS. If $\{u_n\}_{n \in \mathbb{N}_0}$ is the corresponding dual sequence, then $u = \lambda u_0$, with $\lambda = (u)_0 \neq 0$.

2.2. Lowering operators and q -Appell sequences. A q -Appell (or H_q -Appell) sequence $\{B_n\}_{n \in \mathbb{N}_0}$ is usually defined by the condition [15, 32]

$$B_n^{[1]}(x; H_q) := \frac{1}{[n+1]_q} (H_q B_{n+1})(x) = B_n(x), \quad n \in \mathbb{N}_0.$$

The q -differential operator H_q is an example of an usually called lowering operator, i.e., is a linear mapping $\mathcal{O} : \mathcal{P} \rightarrow \mathcal{P}$ such that

$$\mathcal{O}(1) = 0 \quad \text{and} \quad \deg(\mathcal{O}(x^n)) = n - 1, \quad n \in \mathbb{N}.$$

More generally, for any lowering operator \mathcal{O} , we can construct the polynomial sequence $\{B_n^{[1]}(\cdot; \mathcal{O})_n\}_{n \in \mathbb{N}_0}$, defined by

$$B_n^{[1]}(x; \mathcal{O}) = \rho_n^{-1} (\mathcal{O} B_{n+1})(x), \quad n \in \mathbb{N}_0,$$

where $\rho_n \in \mathbb{C} \setminus \{0\}$ is chosen in order to make $B_n^{[1]}(x; \mathcal{O})$ monic.

Definition 2.2 ([6, 7]). A MPS $\{B_n\}_{n \in \mathbb{N}_0}$ is called an \mathcal{O} -Appell sequence with respect to a lowering operator \mathcal{O} if $B_n(\cdot) = B_n^{[1]}(\cdot; \mathcal{O})$ for every $n \in \mathbb{N}_0$.

In an upcoming section, it will be crucial to comprehend the utilization of a lowering operator on a sequence of polynomials $\{g_n\}_{n \in \mathbb{N}_0}$ that may not satisfy all of the MPS attributes, for example, when $\deg g_n < n$ for certain values of n . In such cases, by imposing certain hypotheses, we can ensure a specific structure for these sequences, as indicated in the following straightforward results.

Lemma 2.2. *Let \mathcal{O} be a lowering operator and $f \in \mathcal{P}$. If $\deg(f) > 0$, then $\deg(\mathcal{O}(f)) = \deg(f) - 1$.*

Proposition 2.1. *Let \mathcal{O} be a lowering operator and let $\{f_n\}_{n \in \mathbb{N}_0}$ be a sequence in \mathcal{P} such that $\rho_n f_n = \mathcal{O}(f_{n+1})$, with $\rho_n \in \mathbb{C} \setminus \{0\}$. Then, either $f_n = 0, n \in \mathbb{N}_0$, or there is $n_0 \in \mathbb{N}_0$ such that $f_{n_0} \neq 0, f_i = 0, 0 \leq i \leq n_0 - 1$, if $n_0 \geq 1$, and $\deg(f_{n+1}) = \deg(f_n) + 1, n \geq n_0$.*

Let us now fix a non-negative integer k and some constants $a_{i,q} \in \mathbb{C}$, $i = 0, \dots, k$, and let us set

$$\Lambda_q = \sum_{i=0}^k a_{i,q} (H_q x)^i H_q.$$

Taking into account that x^n is an eigenvector of xH_q with eigenvalue equal to $[n]_q$, we get the following lemma.

Lemma 2.3. *The following identity holds, for any positive integer n ,*

$$(2.6) \quad \Lambda_q(x^n) = \left([n]_q \sum_{i=0}^k a_{i,q} ([n]_q)^i \right) x^{n-1}.$$

As a consequence, we have the following.

Proposition 2.2. (i) *For any positive integer k , $\Lambda_q = 0$ if and only if $a_{i,q} = 0$ $i = 0, \dots, k$.*

(ii) *Suppose $a_{k,q} \neq 0$, then the operator Λ_q is a lowering operator if and only if the polynomial $f(x) = \sum_{i=0}^k a_{i,q} x^i$ has no roots in the form $[n]_q$.*

Proof. (i) Suppose $\Lambda_q = 0$, that is, $\Lambda_q(f) = 0$ for any $f \in \mathcal{P}$. In particular,

$$0 = \Lambda_q(x^n) = \left([n]_q \sum_{i=0}^k a_{i,q} ([n]_q)^i \right) x^{n-1}.$$

Therefore, $\sum_{i=0}^k a_{i,q} ([n]_q)^i = 0$, for all positive integer n , which implies $a_{i,q} = 0$, $i = 0, \dots, k$.

(ii) Since $a_{k,q} \neq 0$, the polynomial $f(x) = \sum_{i=0}^k a_{i,q} x^i$ is not identically zero and $\Lambda_q \neq 0$. Moreover, from (2.6) we get the indicated result. □

3. THE Λ_q OPERATOR

3.1. Further definitions and the transpose operator. The form employed in defining the operator Λ_q may appear restrictive; however, as we will see later on, this q -differential operator can be expressed in various configurations. In particular, the next results clarify that a Λ_q operator (for all non-negative integer k) can be expressed as a product of simpler operators or as a linear combination of operators of the form $x^i H_q^{i+1}$, with $i = 0, \dots, k$. The first Lemma is a simple case of the problem of normal ordering of words in H_q and x , where the q -Stirling numbers play a significant role.

With regard to the operators x and H_q , it will be useful later on to recall that

$$(H_q x) H_q = H_q + q x H_q^2$$

or

$$(3.1) \quad x H_q^2 = q^{-1} H_q x H_q - q^{-1} H_q.$$

Lemma 3.1. *For all non-negative integer i , it holds*

$$(3.2) \quad (H_q x)^i H_q = \sum_{m=0}^i S(i+1, m+1)_q q^{\binom{m+1}{2}} x^m H_q^{m+1},$$

where $S(n, k)_q$ are the q -Stirling numbers of second kind. Conversely,

$$(3.3) \quad x^i H_q^{i+1} = q^{-\binom{i+1}{2}} \sum_{m=0}^i s(i+1, m+1)_q (H_q x)^m H_q,$$

where $s(n, k)_q$ are the q -Stirling numbers of first kind.

Proof. Identity (3.2) can be established through induction and by using relation

$$S(n, k)_q = S(n-1, k-1)_q + [k]_q S(n-1, k)_q.$$

The orthogonal relation

$$\sum S(n, k)_q s(k, m)_q = \delta_{n,m}$$

shows that each of the relations above implies the other. □

Remark 3.1. Notice that for $i = 2$, (3.2) and (3.3) read as

$$(H_q x)^2 H_q = H_q + q(2+q)xH_q^2 + q^3 x^2 H_q^3$$

and

$$x^2 H_q^3 = q^{-3}(1+q)H_q - q^{-3}(2+q)H_q x H_q + q^{-3}(H_q x)^2 H_q.$$

As a straightforward consequence, we may write the next identities

$$\begin{aligned} \sum_{i=0}^k a_{i,q} (H_q x)^i H_q &= \sum_{m=0}^k \left(\sum_{i=m}^k a_{i,q} S(i+1, m+1) \right) q^{\binom{m+1}{2}} x^m H_q^{m+1}, \\ \sum_{i=0}^k a_{i,q} x^i H_q^{i+1} &= \sum_{m=0}^k \left(\sum_{i=m}^k a_{i,q} s(i+1, m+1) q^{-\binom{i+1}{2}} \right) (H_q x)^m H_q. \end{aligned}$$

In the special case where $k = 2$, it is essential to highlight that

$$(3.4) \quad \begin{aligned} &a_{0,q} H_q + a_{1,q} (H_q x) H_q + a_{2,q} (H_q x)^2 H_q \\ &= (a_{0,q} + a_{1,q} + a_{2,q}) H_q + q(a_{1,q} + (2+q)a_{2,q}) x H_q^2 + a_{2,q} q^3 x^2 H_q^3. \end{aligned}$$

Lastly, we would like to put in evidence a product form for the operator Λ_q that appears dealing with a cubic decomposition of a q -Appell sequence. The next Lemma is self-explanatory being mainly justified by previous identities, as for example (3.1), and by the complete factorization of any element of \mathcal{P} .

Lemma 3.2. *For any two non-negative integers l and t and any set of constants $\alpha_{i,q}, \beta_{i,q} \in \mathbb{C}$, $i = -l, \dots, t$, with $\alpha_{0,q} = 1$, $\beta_{0,q} = 0$, let us consider the operator*

$$S_{l,t} = \prod_{-l}^0 (\alpha_{i,q} I_{\mathcal{P}} + \beta_{i,q} x H_q) H_q \prod_0^t (\alpha_{i,q} I_{\mathcal{P}} + \beta_{i,q} x H_q),$$

where $I_{\mathcal{P}}$ denotes the identity operator on \mathcal{P} . Then, for some constants $a_{i,q} \in \mathbb{C}$, $i = 0, \dots, l + t$, we have

$$S_q^{l,t} = \sum_{i=0}^{l+t} a_{i,q} (H_q x)^i H_q.$$

Conversely, an operator $\Lambda_q = \sum_{i=0}^k a_{i,q} (H_q x)^i H_q = \sum_{i=0}^k a_{i,q} H_q (x H_q)^i$ can be written as

$$\Lambda_q = S_q^{0,k} = H_q \prod_0^k (\alpha_{i,q} I_{\mathcal{P}} + \beta_{i,q} x H_q),$$

where coefficients $\alpha_{i,q}$ and $\beta_{i,q}$ are obtained through the factorization of the polynomial $f(x) = \sum_{i=0}^k a_{i,q} x^i$, that is, $\prod_0^k (\alpha_{i,q} + \beta_{i,q} x) = \sum_{i=0}^k a_{i,q} x^i$.

In this way, we have confirmed that the three operators:

$$\sum_{i=0}^k a_{i,q} (H_q x)^i H_q, \quad \sum_{i=0}^k \tilde{a}_{i,q} x^i H_q^{i+1}$$

and

$$S_{l,t} = \prod_{-l}^0 (\alpha_{i,q} I_{\mathcal{P}} + \beta_{i,q} x H_q) H_q \prod_0^t (\alpha_{i,q} I_{\mathcal{P}} + \beta_{i,q} x H_q),$$

represent the same category of operators. As a result, we are able to opt for the most appropriate form at each stage of our analysis. Taking into account the initially suggested configuration, we subsequently present its transposed expression.

Lemma 3.3. *The transpose operator $t_{\Lambda_q} : \mathcal{P}' \rightarrow \mathcal{P}'$ is expressed by*

$$(3.5) \quad t_{\Lambda_q} = \sum_{i=0}^k a_{i,q} (-1)^{i+1} (H_q x)^i H_q.$$

Proof. Drawing upon equations (2.2) and (2.3), it is straightforward to observe that the transpose of $(H_q x)^i H_q$ is $(-1)^{i+1}$ times itself, resulting in the indicated sum. \square

3.2. Λ_q -Appell sequences. Let us suppose that the fixed operator

$$\Lambda_q = \sum_{i=0}^k a_{i,q} (H_{q^3} x)^i H_{q^3}$$

is a lowering operator according to Proposition 2.2. Taking into account

$$(\Lambda_q B_{n+1})(x) = \Lambda_q (x^{n+1} + \dots) = [n + 1]_q \left(\sum_{i=0}^k a_{i,q} ([n + 1]_q)^i \right) x^n + \Lambda_q(\dots),$$

and recalling that $\sum_{i=0}^k a_{i,q} ([n + 1]_q)^i \neq 0$, we have

$$B_n^{[1]}(x; \Lambda_q) = [n + 1]_q^{-1} \left(\sum_{i=0}^k a_{i,q} ([n + 1]_q)^i \right)^{-1} (\Lambda_q B_{n+1})(x).$$

Denoting by $\{u_n^{[1]}(\Lambda_q)\}_{n \in \mathbb{N}_0}$ the dual sequence of $\{B_n^{[1]}(\cdot; \Lambda_q)\}_{n \in \mathbb{N}_0}$ we have the next result.

Proposition 3.1.

$$(3.6) \quad t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right) = \rho_{n,q} u_{n+1},$$

where t_{Λ_q} is detailed in (3.5) and $\rho_{n,q} = [n + 1]_q \left(\sum_{i=0}^k a_{i,q} ([n + 1]_q)^i \right)$.

Proof. We have

$$\left\langle u_n^{[1]}(\Lambda_q), B_m^{[1]}(x; \Lambda_q) \right\rangle = \delta_{n,m}, \quad n, m \in \mathbb{N}_0.$$

That is,

$$\left\langle u_n^{[1]}(\Lambda_q), \Lambda_q B_{m+1}(x) \right\rangle = \rho_{m,q} \delta_{n,m}$$

or

$$\left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_{m+1}(x) \right\rangle = \rho_{m,q} \delta_{n,m}.$$

In particular, $\left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_{m+1}(x) \right\rangle = 0, \quad m \geq n + 1, \quad n \in \mathbb{N}_0.$ So, by Lemma 2.1, we get

$$t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu} u_\nu, \quad n \in \mathbb{N}_0,$$

with $\lambda_{n,\nu} = \left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_\nu(x) \right\rangle, \quad 0 \leq \nu \leq n + 1.$ Notice that

$$\lambda_{n,0} = \left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_0(x) \right\rangle = \sum_{i=0}^k a_{i,q} (-1)^{i+1} \left\langle (H_{q^3} x)^i H_{q^3} \left(u_n^{[1]}(\Lambda_q) \right), 1 \right\rangle = 0.$$

Also, if $1 \leq \nu \leq n,$ then

$$\lambda_{n,\nu} = \left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_\nu(x) \right\rangle = [\nu]_q \left(\sum_{i=0}^k a_{i,q} ([\nu]_q)^i \right) \delta_{n,\nu-1} = 0$$

and

$$\lambda_{n,n+1} = \left\langle t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right), B_{n+1}(x) \right\rangle = \rho_{n,q} \neq 0.$$

Thus, $t_{\Lambda_q} \left(u_n^{[1]}(\Lambda_q) \right) = \lambda_{n,n+1} u_{n+1}.$ □

Proposition 3.2. *Given a MPS $\{B_n\}_{n \in \mathbb{N}_0},$ the following statements are equivalent.*

- (a) $\{B_n\}_{n \in \mathbb{N}_0}$ is Λ_q -Appell, that is, $B_n^{[1]}(x; \Lambda_q) = B_n(x).$
- (b) $t_{\Lambda_q}(u_n) = \rho_{n,q} u_{n+1},$ where $\rho_{n,q} = [n + 1]_q \left(\sum_{i=0}^k a_{i,q} ([n + 1]_q)^i \right).$
- (c) $u_n = ([n]_q!)^{-1} \left(\prod_{s=1}^n \left(\sum_{i=0}^k a_{i,q} ([s]_q)^i \right) \right)^{-1} (t_{\Lambda_q})^n(u_0),$

where the symbol $[z]_q! = [z]_q [z - 1]_q \cdots [1]_q$ represents the q -factorial of the integer $z.$

Proof. If $\{B_n\}_{n \in \mathbb{N}_0}$ is Λ_q -Appell, then $u_n = u_n^{[1]}(\Lambda_q)$ From (3.6), we obtain

$$t_{\Lambda_q}(u_n) = \rho_{n,q} u_{n+1} \quad \text{or} \quad u_{n+1} = \rho_{n,q}^{-1} t_{\Lambda_q}(u_n).$$

Recursively, we get that $u_n = \prod_{j=0}^{n-1} \rho_{j,q}^{-1} (t_{\Lambda_q})^n (u_0)$. Since

$$\prod_{j=0}^{n-1} \rho_{j,q}^{-1} = ([n]_q!)^{-1} \left(\prod_{s=1}^n \left(\sum_{i=0}^k a_{i,q} ([s]_q)^i \right) \right)^{-1}$$

we obtain (c).

Conversely, if we suppose the identity of item (b), then, from (3.6), we conclude that

$$t_{\Lambda_q} (u_n^{[1]}(\Lambda_q)) = t_{\Lambda_q}(u_n).$$

Similarly, based on the identity in item (c), we can derive the relation $u_{n+1} = \rho_{n,q}^{-1} t_{\Lambda_q}(u_n)$, as the following calculations elucidate. Furthermore, in light of (3.6), we deduce $t_{\Lambda_q}(u_n^{[1]}(\Lambda_q)) = t_{\Lambda_q}(u_n)$

$$\begin{aligned} u_{n+1} &= ([n+1]_q!)^{-1} \left(\prod_{s=1}^{n+1} \left(\sum_{i=0}^k a_{i,q} [s]_q^i \right) \right)^{-1} (t_{\Lambda_q})^{n+1} u_0 \\ &= [n+1]_q^{-1} \left(\sum_{i=0}^k a_{i,q} [n+1]_q^i \right)^{-1} (t_{\Lambda_q}) ([n]_q!)^{-1} \left(\prod_{s=1}^n \left(\sum_{i=0}^k a_{i,q} [s]_q^i \right) \right)^{-1} (t_{\Lambda_q})^n u_0 \\ &= [n+1]_q^{-1} \left(\sum_{i=0}^k a_{i,q} [n+1]_q^i \right)^{-1} \Lambda_q(u_n). \end{aligned}$$

For any lowering operator \mathcal{O} , we can assure that $t_{\mathcal{O}} : \mathcal{P}' \rightarrow \mathcal{P}'$ is a one-to-one operator, and thus, in both situations, we conclude that $u_n^{[1]}(\Lambda_q) = u_n$ proving that $\{B_n\}_{n \in \mathbb{N}_0}$ is Λ_q -Appell. \square

4. A CUBIC DECOMPOSITION OF A q -APPELL MPS

For any MPS $\{W_n\}_{n \in \mathbb{N}_0}$ there are three MPSs $\{P_n\}_{n \in \mathbb{N}_0}$, $\{Q_n\}_{n \in \mathbb{N}_0}$ and $\{R_n\}_{n \in \mathbb{N}_0}$, so that

$$(4.1) \quad W_{3n}(x) = P_n(x^3) + x a_{n-1}^1(x^3) + x^2 a_{n-1}^2(x^3),$$

$$(4.2) \quad W_{3n+1}(x) = b_n^1(x^3) + x Q_n(x^3) + x^2 b_{n-1}^2(x^3),$$

$$(4.3) \quad W_{3n+2}(x) = c_n^1(x^3) + x c_n^2(x^3) + x^2 R_n^2(x^3),$$

with $\deg a_{n-1}^1 \leq n-1$, $\deg a_{n-1}^2 \leq n-1$, $\deg b_n^1 \leq n$, $\deg b_{n-1}^2 \leq n-1$, $\deg c_n^1 \leq n$, $\deg c_n^2 \leq n$ and $a_{-1}^1(x) = a_{-1}^2(x) = b_{-1}^1(x) = 0$. This is a particular case of the general cubic decomposition of any MPS presented in [29], where all the parameters involved are considered zero. In this cubic decomposition (CD) (4.1)–(4.3) of $\{W_n\}_{n \in \mathbb{N}_0}$, the sequences:

- $\{P_n\}_{n \in \mathbb{N}_0}$, $\{Q_n\}_{n \in \mathbb{N}_0}$, $\{R_n\}_{n \in \mathbb{N}_0}$ are called the principal components;

- $\{a_{n-1}^1\}_{n \in \mathbb{N}_0}$, $\{a_{n-1}^2\}_{n \in \mathbb{N}_0}$, $\{b_n^1\}_{n \in \mathbb{N}_0}$, $\{b_{n-1}^2\}_{n \in \mathbb{N}_0}$, $\{c_n^1\}_{n \in \mathbb{N}_0}$, $\{c_n^2\}_{n \in \mathbb{N}_0}$ are called the secondary components, since they are sequences of polynomials although not necessarily bases for the vector space of polynomials \mathcal{P} .

The nine component sequences are assembled in the following matrix [29].

$$(4.4) \quad M_n(x) = \begin{pmatrix} P_n(x) & a_{n-1}^1(x) & a_{n-1}^2(x) \\ b_n^1(x) & Q_n(x) & b_{n-1}^2(x) \\ c_n^1(x) & c_n^2(x) & R_n(x) \end{pmatrix}.$$

Lemma 4.1 ([29]). *Let $P(x)$, $Q(x)$ and $R(x)$ be three polynomials. $P(x^3) + xQ(x^3) + x^2R(x^3) = 0$ implies $P(x) = Q(x) = R(x) = 0$.*

Proposition 4.1. *A q -Appell MPS $\{W_n\}_{n \in \mathbb{N}_0}$ admits the CD (4.1)–(4.3) if and only if the following relations are fulfilled for $n \in \mathbb{N}_0$,*

$$(4.5) \quad (q[3]_q x H_{q^3} + I_{\mathcal{P}})Q_n(x) = [3n+1]_q P_n(x),$$

$$(4.6) \quad (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})b_{n-1}^2(x) = [3n+1]_q a_{n-1}^1(x),$$

$$(4.7) \quad [3]_q H_{q^3} b_n^1(x) = [3n+1]_q a_{n-1}^2(x),$$

$$(4.8) \quad (q[3]_q x H_{q^3} + I_{\mathcal{P}})c_n^2(x) = [3n+2]_q b_n^1(x),$$

$$(4.9) \quad (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})R_n(x) = [3n+2]_q Q_n(x),$$

$$(4.10) \quad [3]_q H_{q^3} c_n^1(x) = [3n+2]_q b_{n-1}^2(x),$$

$$(4.11) \quad (q[3]_q x H_{q^3} + I_{\mathcal{P}})a_n^1(x) = [3n+3]_q c_n^1(x),$$

$$(4.12) \quad (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}})a_n^2(x) = [3n+3]_q c_n^2(x),$$

$$(4.13) \quad [3]_q H_{q^3} P_{n+1}(x) = [3n+3]_q R_n(x),$$

where $I_{\mathcal{P}}$ denotes the identity operator on \mathcal{P} .

The proof of this proposition will follow immediately once the following lemma is established.

Lemma 4.2. *Representing by $\sigma_{\varpi} : \mathcal{P} \rightarrow \mathcal{P}$ the linear operator defined by $(\sigma_{\varpi} f)(x) := f(x^3)$, for any $f \in \mathcal{P}$,*

$$(4.14) \quad H_q \circ \sigma_{\varpi} = [3]_q x^2 \sigma_{\varpi} \circ H_{q^3},$$

$$(4.15) \quad H_q x = qx H_q + I_{\mathcal{P}},$$

$$(4.16) \quad H_q x \circ \sigma_{\varpi} = \sigma_{\varpi} (q[3]_q x H_{q^3} + I_{\mathcal{P}}),$$

$$(4.17) \quad H_q x^2 = q^2 x^2 H_q + [2]_q x I_{\mathcal{P}},$$

$$(4.18) \quad H_q x^2 \circ \sigma_{\varpi} = x \sigma_{\varpi} (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}).$$

Proof. The proof of this lemma is straightforward from the definitions of the operators H_q and σ_{ϖ} and will be omitted. \square

Proof of Proposition 4.1. Representing by $\{W_n\}_{n \in \mathbb{N}_0}$ a q -Appell sequence, we proceed to its cubic decomposition in accordance with (4.1)–(4.3). Operating with H_q on both sides of (4.1), with n replaced by $n + 1$, and on (4.2) and (4.3), then, under the assumptions, we, respectively, obtain

$$(4.19) \quad \begin{aligned} & [3n + 3]_q \{c_n^1(x^3) + xc_n^2(x^3) + x^2R_n^2(x^3)\} \\ & = (H_q P_{n+1}(\xi^3))(x) + (H_q \xi a_n^1(\xi^3))(x) + (H_q \xi^2 a_n^2(\xi^3))(x), \quad n \in \mathbb{N}_0, \end{aligned}$$

$$(4.20) \quad \begin{aligned} & [3n + 2]_q \{b_n^1(x^3) + xQ_n(x^3) + x^2b_{n-1}^2(x^3)\} \\ & = (H_q c_n^1(\xi^3))(x) + (H_q \xi c_n^2(\xi^3))(x) + (H_q \xi^2 R_n(\xi^3))(x), \quad n \in \mathbb{N}_0, \end{aligned}$$

$$(4.21) \quad \begin{aligned} & [3n + 1]_q \{P_n(x^3) + xa_{n-1}^1(x^3) + x^2a_{n-1}^2(x^3)\} \\ & = (H_q b_n^1(\xi^3))(x) + (H_q \xi Q_n(\xi^3))(x) + (H_q \xi^2 b_{n-1}^2(\xi^3))(x), \quad n \in \mathbb{N}_0. \end{aligned}$$

Based on (4.14), (4.16) and (4.18) the relations (4.19)–(4.21) become, respectively, as follows

$$\begin{aligned} & [3n + 3]_q \{ \sigma_\varpi c_n^1(x) + x\sigma_\varpi c_n^2(x) + x^2\sigma_\varpi R_n^2(x) \} \\ & = [3]_q x^2 (\sigma_\varpi \circ H_{q^3}) [P_{n+1}](x) + \sigma_\varpi (q[3]_q x H_{q^3} + I_{\mathcal{P}}) [a_n^1](x) \\ & \quad + x\sigma_\varpi (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) [a_n^2](x), \quad n \in \mathbb{N}_0, \\ & [3n + 2]_q \{ \sigma_\varpi b_n^1(x) + x\sigma_\varpi Q_n(x) + x^2\sigma_\varpi b_{n-1}^2(x) \} \\ & = [3]_q x^2 (\sigma_\varpi \circ H_{q^3}) [c_n^1](x) + \sigma_\varpi (q[3]_q x H_{q^3} + I_{\mathcal{P}}) [c_n^2](x) \\ & \quad + x\sigma_\varpi (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) [R_n](x), \quad n \in \mathbb{N}_0, \\ & [3n + 1]_q \{ \sigma_\varpi P_n(x) + x\sigma_\varpi a_{n-1}^1(x) + x^2\sigma_\varpi a_{n-1}^2(x) \} \\ & = [3]_q x^2 (\sigma_\varpi \circ H_{q^3}) [b_n^1](x) + \sigma_\varpi (q[3]_q x H_{q^3} + I_{\mathcal{P}}) [Q_n](x) \\ & \quad + x\sigma_\varpi (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) [b_{n-1}^2](x), \quad n \in \mathbb{N}_0. \end{aligned}$$

Applying Lemma 4.1, we get (4.5)–(4.13). □

Remark 4.1. Let us remark that the nine conditions of Proposition 4.1 can be written using matrix identities, as follows

$$(4.22) \quad \left(q[3]_q x H_{q^3} + I_{\mathcal{P}} \right) \begin{pmatrix} c_{n-1}^2(x) \\ a_{n-1}^1(x) \\ Q_n(x) \end{pmatrix} = \begin{pmatrix} [3n - 1]_p & 0 & 0 \\ 0 & [3n]_p & 0 \\ 0 & 0 & [3n + 1]_p \end{pmatrix} \begin{pmatrix} b_{n-1}^1(x) \\ c_{n-1}^1(x) \\ P_n(x) \end{pmatrix},$$

(4.23)

$$\left(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}} \right) \begin{pmatrix} a_{n-1}^2(x) \\ b_{n-1}^2(x) \\ R_n(x) \end{pmatrix} = \begin{pmatrix} [3n]_p & 0 & 0 \\ 0 & [3n+1]_p & 0 \\ 0 & 0 & [3n+2]_p \end{pmatrix} \begin{pmatrix} c_{n-1}^2(x) \\ a_{n-1}^1(x) \\ Q_n(x) \end{pmatrix},$$

(4.24)

$$[3]_q H_{q^3} \begin{pmatrix} b_n^1(x) \\ c_n^1(x) \\ P_{n+1}(x) \end{pmatrix} = \begin{pmatrix} [3n+1]_p & 0 & 0 \\ 0 & [3n+2]_p & 0 \\ 0 & 0 & [3n+3]_p \end{pmatrix} \begin{pmatrix} a_{n-1}^2(x) \\ b_{n-1}^2(x) \\ R_n(x) \end{pmatrix}.$$

The assumption of the q -Appell character of the MPS $\{B_n\}_{n \in \mathbb{N}_0}$ actually implies the Appell character, but with respect to another linear operator, of the some associated sequences; because of the importance of this, we formally state the result as a lemma.

Lemma 4.3. *Let us consider a q -Appell MPS $\{W_n\}_{n \in \mathbb{N}_0}$ and the correspondent CD (4.1)–(4.3). Then, for $n \in \mathbb{N}$, we have*

$$\begin{aligned} a_{n-1}^2(x) &= [3]_q \left([3n+1]_q [3n+2]_q [3n+3]_q \right)^{-1} \Lambda_{q^3,1} a_n^2(x), \\ b_{n-1}^2(x) &= [3]_q \left([3n+2]_q [3n+3]_q [3n+4]_q \right)^{-1} \Lambda_{q^3,1} b_n^2(x), \\ R_n(x) &= [3]_q \left([3n+3]_q [3n+4]_q [3n+5]_q \right)^{-1} \Lambda_{q^3,1} R_{n+1}(x), \\ a_{n-1}^1(x) &= [3]_q \left([3n+1]_q [3n+2]_q [3n+3]_q \right)^{-1} \Lambda_{q^3,2} a_n^1(x), \\ Q_n(x) &= [3]_q \left([3n+2]_q [3n+3]_q [3n+4]_q \right)^{-1} \Lambda_{q^3,2} Q_{n+1}(x), \\ c_{n-1}^2(x) &= [3]_q \left([3n]_q [3n+1]_q [3n+2]_q \right)^{-1} \Lambda_{q^3,2} c_n^2(x), \\ P_n(x) &= [3]_q \left([3n+1]_q [3n+2]_q [3n+3]_q \right)^{-1} \Lambda_{q^3,3} P_{n+1}(x), \\ b_{n-1}^1(x) &= [3]_q \left([3n-1]_q [3n]_q [3n+1]_q \right)^{-1} \Lambda_{q^3,3} b_n^1(x), \\ c_{n-1}^1(x) &= [3]_q \left([3n]_q [3n+1]_q [3n+2]_q \right)^{-1} \Lambda_{q^3,3} c_n^1(x), \end{aligned}$$

where

$$\begin{aligned} \Lambda_{q^3,1} &= H_{q^3} \left(q[3]_q x H_{q^3} + I_{\mathcal{P}} \right) \left(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}} \right), \\ \Lambda_{q^3,2} &= \left(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}} \right) H_{q^3} \left(q[3]_q x H_{q^3} + I_{\mathcal{P}} \right), \\ \Lambda_{q^3,3} &= \left(q[3]_q x H_{q^3} + I_{\mathcal{P}} \right) \left(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}} \right) H_{q^3}. \end{aligned}$$

Proof. Let us consider (4.23), with $n \mapsto n + 1$. Applying $q[3]_q x H_{q^3} + I_{\mathcal{P}}$, we have

$$\left(q[3]_q x H_{q^3} + I_{\mathcal{P}} \right) \left(q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}} \right) \begin{pmatrix} a_{n-1}^2(x) \\ b_{n-1}^2(x) \\ R_n(x) \end{pmatrix}$$

$$= \begin{pmatrix} [3n + 3]_p & 0 & 0 \\ 0 & [3n + 4]_p & 0 \\ 0 & 0 & [3n + 5]_p \end{pmatrix} (q[3]_q x H_{q^3} + I_{\mathcal{P}}) \begin{pmatrix} c_n^2(x) \\ a_n^1(x) \\ Q_{n+1}(x) \end{pmatrix}.$$

Attending to identity (4.22), with $n \mapsto n + 1$, we get

$$\begin{aligned} & (q[3]_q x H_{q^3} + I_{\mathcal{P}}) (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) \begin{pmatrix} a_{n-1}^2(x) \\ b_{n-1}^2(x) \\ R_n(x) \end{pmatrix} \\ &= \begin{pmatrix} [3n + 3]_p & 0 & 0 \\ 0 & [3n + 4]_p & 0 \\ 0 & 0 & [3n + 5]_p \end{pmatrix} \begin{pmatrix} [3n + 2]_p & 0 & 0 \\ 0 & [3n + 3]_p & 0 \\ 0 & 0 & [3n + 3]_p \end{pmatrix} \begin{pmatrix} b_n^1(x) \\ c_n^1(x) \\ P_{n+1}(x) \end{pmatrix}. \end{aligned}$$

Applying $[3]_q H_{q^3}$ and using (4.24) we finally obtain

$$\begin{aligned} & [3]_q H_q (q[3]_q x H_{q^3} + I_{\mathcal{P}}) (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) \begin{pmatrix} a_n^2(x) \\ b_n^2(x) \\ R_{n+1}(x) \end{pmatrix} \\ &= \begin{pmatrix} [3n + 1]_q [3n + 2]_q [3n + 3]_q a_{n-1}^2(x) \\ [3n + 2]_q [3n + 3]_q [3n + 4]_q b_{n-1}^2(x) \\ [3n + 3]_q [3n + 4]_q [3n + 5]_q R_n(x) \end{pmatrix}, \end{aligned}$$

which explains the first three announced relations. The remaining six relations can be obtained in a similar way, considering, in the beginning of the process, identity (4.22) and identity (4.24), respectively. \square

Remark 4.2. In other words, the component sequences of a CD of a q -Appell sequence are also Appell with respect to certain operators. Furthermore, let us note that $\Lambda_{q^3,1}$, $\Lambda_{q^3,2}$ and $\Lambda_{q^3,3}$ define the operator related to the component sequences of the third, second and first column of matrix $M_n(x)$ (4.4), respectively.

Taking into account identity (3.1) and Lemma 3.2, the operators of Lemma 4.3 have the following alternative expression

$$\begin{aligned} \Lambda_{q^3,1} &= H_{q^3} (q[3]_q x H_{q^3} + I_{\mathcal{P}}) (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) \\ &= q^3 [3]_q^2 H_{q^3} x H_{q^3} x H_{q^3} + q[3]_q (2q + 1) H_{q^3} x H_{q^3} + [2]_q H_{q^3}, \\ \Lambda_{q^3,2} &= (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) H_{q^3} (q[3]_q x H_{q^3} + I_{\mathcal{P}}) \\ &= [3]_q^2 H_{q^3} x H_{q^3} x H_{q^3} + [3]_q (q^{-1} - 1) H_{q^3} x H_{q^3} - q^{-1} H_{q^3}, \\ \Lambda_{q^3,3} &= (q[3]_q x H_{q^3} + I_{\mathcal{P}}) (q^2[3]_q x H_{q^3} + [2]_q I_{\mathcal{P}}) H_{q^3} \\ &= q^{-3} [3]_q^2 H_{q^3} x H_{q^3} x H_{q^3} + q^{-3} [3]_q (q + 2) H_{q^3} x H_{q^3} + q^{-3} [2]_q H_{q^3}. \end{aligned}$$

Assuming that the polynomials

$$f_1(x) = q^3[3]_q^2x^2 + q[3]_q(2q + 1)x + [2]_q,$$

$$f_2(x) = [3]_q^2x^2 + [3]_q(q^{-1} - 1)x - q^{-1}$$

and

$$f_3(x) = q^{-3}[3]_q^2x^2 + q^{-3}[3]_q(q + 2)x + q^{-3}[2]_q,$$

do not have roots of the form $[n]_q$, $n \in \mathbb{N}_0$, so, by Proposition 2.2 these are lowering operators. Attending to the following identities given by Lemma 4.3, we conclude that the principal component sequence $\{R_n\}_{n \in \mathbb{N}_0}$ is a $\Lambda_{q^3,1}$ -Appell sequence, $\{Q_n\}_{n \in \mathbb{N}_0}$ is a $\Lambda_{q^3,2}$ -Appell sequence and $\{P_n\}_{n \in \mathbb{N}_0}$ is a $\Lambda_{q^3,3}$ -Appell sequence

$$R_n^{[1]}(x; \Lambda_{q^3,1}) = [3]_q \left([3n + 3]_q [3n + 4]_q [3n + 5]_q \right)^{-1} \Lambda_{q^3,1} R_{n+1}(x) = R_n(x),$$

$$Q_n^{[1]}(x; \Lambda_{q^3,2}) = [3]_q \left([3n + 2]_q [3n + 3]_q [3n + 4]_q \right)^{-1} \Lambda_{q^3,2} Q_{n+1}(x) = Q_n(x),$$

$$P_n^{[1]}(x; \Lambda_{q^3,3}) = [3]_q \left([3n + 1]_q [3n + 2]_q [3n + 3]_q \right)^{-1} \Lambda_{q^3,3} P_{n+1}(x) = P_n(x).$$

Concerning the secondary components, we arrive at the scenario described by the following result.

Theorem 4.1. *Let us consider the secondary components of the CD (4.1)–(4.3) of a q -Appell MPS.*

- *Either the three sequences $\{a_n^1\}_{n \in \mathbb{N}_0}$, $\{c_n^1\}_{n \in \mathbb{N}_0}$ and $\{b_n^2\}_{n \in \mathbb{N}_0}$ are the zero sequence, or the aforementioned sequences are nonzero, and in this case there are a positive integer κ and three numerical sequences $\mu_{1,n}$, $\mu_{2,n}$ and $\mu_{3,n}$ so that, for $n \in \mathbb{N}_0$,*

$$(4.25) \quad b_{n+\kappa}^2(x) = \mu_{1,n} \hat{b}_n^2(x), \quad a_{n+\kappa}^1(x) = \mu_{2,n} \hat{a}_n^1(x), \quad c_{n+\kappa}^1(x) = \mu_{3,n} \hat{c}_n^1(x),$$

where $\{\hat{a}_n^1\}_{n \in \mathbb{N}_0}$, $\{\hat{c}_n^1\}_{n \in \mathbb{N}_0}$ and $\{\hat{b}_n^2\}_{n \in \mathbb{N}_0}$ are monic polynomial sequences (MPSs).

- *Either the three sequences $\{a_n^2\}_{n \in \mathbb{N}_0}$, $\{c_n^2\}_{n \in \mathbb{N}_0}$ and $\{b_n^1\}_{n \in \mathbb{N}_0}$ are the zero sequence, or the aforementioned sequences are nonzero, and in this case there are a positive integer τ and three numerical sequences $\alpha_{1,n}$, $\alpha_{2,n}$ and $\alpha_{3,n}$ so that, for $n \in \mathbb{N}_0$,*

$$(4.26) \quad a_{n+\tau}^2(x) = \alpha_{1,n} \hat{a}_n^2(x), \quad c_{n+\tau}^2(x) = \alpha_{2,n} \hat{c}_n^2(x), \quad b_{n+\tau}^1(x) = \alpha_{3,n} \hat{b}_n^1(x),$$

where $\{\hat{a}_n^2\}_{n \in \mathbb{N}_0}$, $\{\hat{c}_n^2\}_{n \in \mathbb{N}_0}$ and $\{\hat{b}_n^1\}_{n \in \mathbb{N}_0}$ are monic polynomial sequences (MPSs).

Furthermore, there are two nonzero constants b_κ^2 and a_τ^2 so that

$$(4.27) \quad \mu_{j,n} = \frac{[2]_q [3n + 3\kappa + 5 - j]_q!}{[3n + 3 - j]_q! [3\kappa + 4]_q!} b_\kappa^2, \quad n \in \mathbb{N}, j = 1, 2, 3,$$

and

$$(4.28) \quad \alpha_{j,n} = \frac{[2]_q [3n + 3\tau + 4 - j]_q!}{[3n + 3 - j]_q! [3\tau + 3]_q!} a_\tau^2, \quad n \in \mathbb{N}, j = 1, 2, 3,$$

with

$$\begin{aligned} \mu_{2,0} &= \frac{[2]_q b_\kappa^2}{[3\kappa + 4]_q}, & \mu_{3,0} &= \frac{[2]_q b_\kappa^2}{[3\kappa + 3]_q [3\kappa + 4]_q}, \\ \alpha_{2,0} &= \frac{[2]_q a_\tau^2}{[3\tau + 3]_q}, & \alpha_{3,0} &= \frac{[2]_q a_\tau^2}{[3\tau + 2]_q [3\tau + 3]_q}, \end{aligned}$$

and where the symbol $[z]_q! = [z]_q [z - 1]_q \cdots [1]_q$ represents the q -factorial of the integer z .

Proof. Let us suppose that $\{b_n^2\}_{n \in \mathbb{N}_0}$ is a nonzero sequence and let κ be the smallest index such that

$$b_\kappa^2(x) \neq 0 \quad \text{and} \quad b_n^2 = 0, \quad 0 \leq n \leq \kappa - 1, \quad \text{if } \kappa \geq 1.$$

Thus, due to (4.6), $a_n^1 = 0, 0 \leq n \leq \kappa - 1$, and hence by (4.11), $c_n^1 = 0, 0 \leq n \leq \kappa - 1$. According to (4.10), $c_\kappa^1(x) = c_\kappa^1$ (constant) and in virtue of (4.11) $a_\kappa^1(x) = a_\kappa^1$ (constant). Similarly, (4.6) implies $b_\kappa^2(x) = b_\kappa^2 \neq 0$ (nonzero constant), consequently, $a_\kappa^1 \neq 0$ and $c_\kappa^1 \neq 0$. In particular, $\{a_n^1\}_{n \in \mathbb{N}_0}$ and $\{c_n^1\}_{n \in \mathbb{N}_0}$ are also nonzero sequences. Taking into consideration Proposition 2.1 and Lemma 4.3, we conclude (4.25).

Indeed, by virtue of Proposition 4.1, the relevant fact is that if exists a positive integer n_0 such that $a_{n_0}^1 = 0$, then $a_{n_0-1}^1 = 0$ and likewise for $\{b_n^2\}_{n \in \mathbb{N}_0}$ and $\{c_n^1\}_{n \in \mathbb{N}_0}$.

The relation (4.26) is explained in the same manner, by supposing the smallest index τ such that

$$a_\tau^2(x) \neq 0 \quad \text{and} \quad a_n^2 = 0, \quad 0 \leq n \leq \tau - 1, \quad \text{if } \tau \geq 1,$$

and arguing with (4.7), (4.8) and (4.12).

With regard to the initial conditions and following the indicated notation, we begin to set

$$\mu_{1,0} = b_\kappa^2 \quad \text{and} \quad \alpha_{1,0} = a_\tau^2.$$

Using (4.6) and (4.12), we get

$$\mu_{2,0} = \frac{[2]_q}{[3\kappa + 4]_q} b_\kappa^2 \quad \text{and} \quad \alpha_{2,0} = \frac{[2]_q}{[3\tau + 3]_q} a_\tau^2.$$

From (4.11) and (4.8), we have

$$\mu_{3,0} = \frac{[2]_q}{[3\kappa + 3]_q [3\kappa + 4]_q} b_\kappa^2 \quad \text{and} \quad \alpha_{3,0} = \frac{[2]_q}{[3\tau + 2]_q [3\tau + 3]_q} a_\tau^2.$$

Inserting the expressions of (4.25) in identities (4.6), (4.10) and (4.11), all three with an appropriate shift of the index n , and comparing the leading coefficients of both sides, we obtain the following system

$$\begin{aligned} \left(q^2 [3]_q [n]_{q^3} + q + 1 \right) \mu_{1,n} &= [3n + 3\kappa + 4]_q \mu_{2,n}, \\ [3]_q [n + 1]_{q^3} \mu_{3,n+1} &= [3n + 3\kappa + 5]_q \mu_{1,n}, \\ \left(q [3]_q [n]_{q^3} + 1 \right) \mu_{2,n} &= [3n + 3\kappa + 3]_q \mu_{3,n}. \end{aligned}$$

Therefore, using $[3]_q[n]_{q^3} = [3n]_q$, $q^2[3n]_q + q + 1 = [3n + 2]_q$ and $q[3n]_q + 1 = [3n + 1]_q$, respectively (which, in view of the following well-known $[x + a]_q = [x]_q + q^x[a]_q$ [35]), the last equations become

$$(4.29) \quad [3n + 2]_q \mu_{1,n} = [3n + 3\kappa + 4]_q \mu_{2,n},$$

$$(4.30) \quad [3n + 3]_q \mu_{3,n+1} = [3n + 3\kappa + 5]_q \mu_{1,n},$$

$$(4.31) \quad [3n + 1]_q \mu_{2,n} = [3n + 3\kappa + 3]_q \mu_{3,n}.$$

Its resolution yields

$$\mu_{3,n+1} = \frac{[3n + 3\kappa + 3]_q [3n + 3\kappa + 4]_q [3n + 3\kappa + 5]_q}{[3n + 1]_q [3n + 2]_q [3n + 3]_q} \mu_{3,n},$$

and we get identity

$$\mu_{3,n} = \prod_{i=1}^{3n} \frac{[3\kappa + 2 + i]_q}{[i]_q} \mu_{3,0},$$

without forgetting the initial conditions $\mu_{3,0} = \frac{[2]_q b_\kappa^2}{[3\kappa+3]_q [3\kappa+4]_q}$. Therefore,

$$(4.32) \quad \mu_{3,n} = \prod_{i=3}^{3n} \frac{[3\kappa + 2 + i]_q}{[i]_q} b_\kappa^2 = \frac{[2]_q [3n + 3\kappa + 2]_q!}{[3n]_q! [3\kappa + 4]_q!} b_\kappa^2, \quad n \in \mathbb{N},$$

which is exactly the relation (4.27) with $j = 3$. The expressions for $\mu_{1,n}$ and $\mu_{2,n}$ can be computed by revisiting the previous system, specifically considering the relation (4.31) and (4.29), respectively, and using (4.32). The equality (4.28) is obtained in an analogous way. \square

Remark 4.3. Concerning the proof of Theorem 4.1, we could have proceeded without using Proposition 2.1, strictly relying on the relations of Proposition 4.1. However, by choosing this approach, our intention is to emphasize that the underlying feature can be considered a general characteristic applicable to any sequence that satisfies the assumptions of Proposition 2.1.

REFERENCES

- [1] F. Abdelkarim and P. Maroni, *The D_w -classical orthogonal polynomials*, Results Math. **32** (1997), 1–28. <https://doi.org/10.1007/s00009-024-02638-9>
- [2] W. A. Al-Salam and L. Carlitz, *Some orthogonal q -polynomials*, Math. Nachr. **30** (1965), 47–61. <https://doi.org/10.1002/mana.19650300105>
- [3] W. A. Al-Salam, *Characterization theorems for orthogonal polynomials*, in: P. Nevai (Ed.), *Orthogonal Polynomials: Theory and Practice*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1990. https://doi.org/10.1007/978-94-009-0501-6_1
- [4] P. Appell, *Sur une classe de polynomes*, Ann. Sci. Éc. Norm. Supér. (4) **9** (1880), 119–144. <https://doi.org/10.24033/asens.186>
- [5] S. Araci, U. Duran, M. Acikgoz and H. M. Srivastava, *A certain (p, q) -derivative operator and associated divided differences*, J. Inequal. Appl. **2016**(301) (2016). <https://doi.org/10.1186/s13660-016-1240-8>
- [6] Y. Ben Cheikh, *Some results on quasi-monomiality*, Appl. Math. Comput. **141** (2003), 63–76. [https://doi.org/10.1016/S0096-3003\(02\)00321-1](https://doi.org/10.1016/S0096-3003(02)00321-1)

- [7] Y. Ben Cheikh and H. Chaggara, *Connection problems via lowering operators*, J. Comput. Appl. Math. **178** (2005), 45–61. <https://doi.org/10.1016/j.cam.2004.02.024>
- [8] Y. Ben Cheikh and M. Gaided, *Characterization of the Dunkl-classical orthogonal polynomials*, Appl. Math. Comput. **187** (2007), 105–114. <https://doi.org/10.1016/j.amc.2006.08.108>
- [9] I. Ben Salah and M. Khalfallah, *A description via second degree character of a family of quasi-symmetric forms*, Period. Math. Hungar. **85** (2022), 81–108. <https://doi.org/10.1007/s10998-021-00420-y>
- [10] I. Ben Salah and M. Khalfallah, *Third-degree semiclassical forms of class one arising from cubic decomposition*, Integral Transforms Spec. Funct. **31** (2020), 720–743. <https://doi.org/10.1080/10652469.2020.1735384>
- [11] I. Ben Salah, F. Marcellán and M. Khalfallah, *Second degree linear forms and semiclassical forms of class one: A case study*, Filomat **36** (2022), 781–800. <https://doi.org/10.2298/FIL2203781B>
- [12] T. S. Chihara, *Orthogonal polynomials with Brenke-type generating functions*, Duke Math. J. **35** (1968), 505–518. <https://doi.org/10.1215/S0012-7094-68-03551-5>
- [13] W. Hahn, *Über orthogonalpolynome, die q -differenzgleichungen genügen*, Math. Nachr. **2** (1949), 4–34. <https://doi.org/10.1002/mana.19490020103>
- [14] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications Vol. **98**, Cambridge University Press, Cambridge, 2005. <https://doi.org/10.1017/CB09781107325982>
- [15] L. Khérifi and P. Maroni, *The H_q -classical orthogonal polynomials*, Acta Appl. Math. **71** (2002), 49–115. <https://doi.org/10.1023/A:1014597619994>
- [16] M. Khalfallah, *A description of second degree semiclassical forms of class two arising via cubic decomposition*, Mediterr. J. Math. **19** (2022), Article ID 30, 22 pages. <https://doi.org/10.1007/s00009-021-01948-6>
- [17] M. Khalfallah, *A description via third degree character of quasi-symmetric semiclassical forms of class two*, Period. Math. Hungar. **86** (2023), 58–75. <https://doi.org/10.1007/s10998-022-00458-6>
- [18] M. Khalfallah, *Characterization theorem for strict third-degree semiclassical forms of class one obtained via cubic decomposition*, Z. Anal. Anwend. **42**(1/2) (2023), 17–36. <https://doi.org/10.4171/ZAA/1720>
- [19] M. Khalfallah, *Several characterizations of third degree semiclassical linear forms of class two appearing via cubic decomposition*, Integral Transforms Spec. Funct. **35**(3) (2024), 206–233. <https://doi.org/10.1080/10652469.2023.2296623>
- [20] M. Khalfallah, *A large family of third degree semiclassical forms of class three*, Afr. Mat. **36**(69) (2025). <https://doi.org/10.1007/s13370-025-01294-w>
- [21] M. Khalfallah, *On the quadratic decomposition of q -Dunkl-Appell polynomial sequences*, Period. Math. Hungar. (2026) (to appear).
- [22] A. Loureiro and P. Maroni, *Quadratic decomposition of Appell sequences*, Expo. Math. **26** (2008), 177–186. <https://doi.org/10.1016/j.exmath.2007.10.002>
- [23] A. Loureiro and P. Maroni, *Quadratic decomposition of Laguerre polynomials via lowering operators*, J. Approx. Theor. **163** (2011), 888–903. <https://doi.org/10.1016/j.jat.2010.07.009>
- [24] F. Marcellán, A. Branquinho and J. Petronilho, *Classical orthogonal polynomials: A functional approach*, Acta Appl. Math. **34** (1994), 283–303. <https://doi.org/10.1007/BF00998681>
- [25] F. Marcellán and M. Khalfallah, *On the characterizations of third-degree semiclassical forms via polynomial mappings*, Integral Transforms Spec. Funct. **34** (2023), 65–87. <https://doi.org/10.1080/10652469.2022.2089134>

- [26] P. Maroni, *An algebraic theory of orthogonal polynomials: Applications to semi-classical orthogonal polynomials*, in: C. Brezinski et al. (Eds.), *Orthogonal Polynomials and Their Applications*, Annals of Computing and Applied Mathematics Vol. **9**, Baltzer, Basel, 1991, 95–130.
- [27] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. **48** (1993), 133–155. [https://doi.org/10.1016/0377-0427\(93\)90319-7](https://doi.org/10.1016/0377-0427(93)90319-7)
- [28] P. Maroni, *Fonctions eulériennes. Polynômes orthogonaux classiques*, Mathématiques **A-154** (1994), 1–30. <https://doi.org/10.51257/a-v1-a154>
- [29] P. Maroni, T. A. Mesquita and Z. da Rocha, *On the general cubic decomposition of polynomial sequences*, J. Difference Equ. Appl. **17**(9) (2011), 1303–1332. <https://doi.org/10.1080/10236190903527236>
- [30] P. Maroni and T. A. Mesquita, *Appell polynomial sequences with respect to some differential operators*, Period. Math. Hungar. **72**(2) (2016), 200–217. <https://doi.org/10.1007/s10998-016-0142-3>
- [31] H. M. Srivastava, *An introductory overview of Bessel polynomials, the generalized Bessel polynomials and the q -Bessel polynomials*, Symmetry **15** (2023), Article ID 822, 28 pages. <https://doi.org/10.3390/sym15040822>
- [32] H. M. Srivastava, *Some characterizations of Appell and q -Appell polynomials*, Ann. Mat. Pura Appl. (Ser. 4) **130** (1982), 321–329. <https://doi.org/10.1007/BF01761501>
- [33] H. M. Srivastava and Y. Ben Cheikh, *Orthogonality of some polynomial sets via quasi-monomiality*, Appl. Math. Comput. **141** (2003), 415–425. [https://doi.org/10.1016/S0096-3003\(02\)00961-X](https://doi.org/10.1016/S0096-3003(02)00961-X)
- [34] H. M. Srivastava, S. Khan and M. Riyasat, *q -Difference equations for the 2-iterated q -Appell and mixed type q -Appell polynomials*, Arabian J. Math. **8** (2019), 63–77. <https://doi.org/10.1007/s40065-018-0211-y>
- [35] H. M. Srivastava, T. Kim and Y. Simsek, *q -Bernoulli numbers and polynomials associated with multiple q -zeta functions and basic L -series*, Russian J. Math. Phys. **12** (2005), 241–268. <https://doi.org/10.48550/arXiv.math/0502019>
- [36] H. M. Srivastava, J. Souissi and M. Khalfallah, *Some characterizations of the symmetric q -Dunkl-semiclassical orthogonal q -polynomials*, J. Difference Equ. Appl. **30**(3) (2024), 377–407. <https://doi.org/10.1080/10236198.2023.2290525>
- [37] H. M. Srivastava, G. Yasmin, A. Muhyi and S. Araci, *Certain results for the twice-iterated 2D q -Appell polynomials*, Symmetry **11** (2019), Article ID 1307, 23 pages. <https://doi.org/10.3390/sym11101307>

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