Kragujevac Journal of Mathematics
Volume 51(8) (2027), Pages 1321–1335.

TOPOLOGICAL ANALYSIS OF TERNARY CYCLOTOMIC POLYNOMIALS

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Abstract. A valuable way of studying the cyclotomic polynomial Φ_n is by studying topology of its associated simplicial complex. In [7], Musiker and Rainer introduced this type of simplicial complex and showed that, in most cases, its homotopy type gives us complete information about the coefficients of the cyclotomic polynomial. The only exceptions are cyclotomic polynomials Φ_n when n is a product of three different prime numbers (ternary cyclotomic polynomials), which we study in this paper.

1. Introduction

Cyclotomic polynomials are very important and extensively studied classes of polynomials that have an important role in many branches of mathematics. For a positive integer n, the n-th cyclotomic polynomial is defined as the minimal polynomial over $\mathbb{Q}$ for any primitive n^{th} root of unity ζ in $\mathbb{C}$. It is monic and irreducible with the degree given by the Euler phi function ϕ, and is given by

$$\Phi_n(x) = \prod_{j\in(\mathbb{Z}/n\mathbb{Z})^\times} (x-\zeta^j).$$

Additionally, the coefficients of the cyclotomic polynomial Φ_n lie in $\mathbb{Z}$. Although these coefficients are well-studied, there are still a lot of open questions about them.

Inspired by the works of Bolker [2], Kalai [3] and Adin [1], in [7] Musiker and Reiner give a topological interpretation of the coefficients of the cyclotomic polynomial by associating the corresponding simplicial complex to each coefficient. It turns out that

Key words and phrases. cyclotomic polynomial, simplicial complex, simplicial collapse, homotopy type.
2020 *Mathematics Subject Classification.* Primary: 05E45. Secondary: 11C08, 55P15.
DOI
Received: September 12, 2025.
Accepted: June 19, 2026.

such complexes associated to the cyclotomic polynomial Φ_n have very nice properties for all natural numbers $n = p_1p_2 \cdots p_d$, where $p_1, p_2, \dots, p_d$ are distinct prime numbers and $d \neq 3$. Namely, in cases when $d \neq 3$ the homotopy type of the associated simplicial complex gives complete information about the magnitude of the relevant coefficient of the polynomial Φ_n. However, this is not always the case when $d = 3$.

If $n = p_1^{e_1} \cdots p_d^{e_d}$, where p_i are primes, $e_i \geq 1$, it follows that $\Phi_n = \Phi_{p_1 \cdots p_d}(x^{n/p_1 \cdots p_d})$, thus it is sufficient to interpret the coefficients of the cyclotomic polynomial for squarefree n. Fix such a squarefree $n = p_1 \cdots p_d$ and let

$$K_{p_1,\dots,p_d} := K_{p_1} * \cdots * K_{p_d}$$

denote the simplicial join of $K_{p_1}, \dots, K_{p_d}$, where K_{p_i} is a 0-dimensional abstract simplicial complex with p_i vertices labeled by the residues

$$\{0 \bmod p_i, 1 \bmod p_i, \dots, (p_i - 1) \bmod p_i\}.$$

The facets of $K_{p_1,\dots,p_d}$ are labeled by sequence of residues $[j_1 \bmod p_1, \dots, j_d \bmod p_d]$, and by the Chinese Remainder Theorem, they can be denoted by residue $j \bmod n$ (we call this facets $F_{j \bmod n}$). For $A \subseteq \{0, 1, \dots, \phi(n)\}$ let K_A denote the subcomplex of $K_{p_1,\dots,p_d}$ which is generated by facets $F_{j \bmod n}$ where j runs through the set of following residues

$$A \cup \{\phi(n) + 1, \phi(n) + 2, \dots, n - 2, n - 1\}.$$

Example 1.1. Consider the previously defined complexes for $n = 5 \cdot 7$. Complex $K_{5,7}$ whose geometric realization is given in Figure 1, is a complete bipartite graph. The partitions of this graph are $\{i \bmod 5\}_{i \in \{0,\dots,4\}}$ and $\{i \bmod 7\}_{i \in \{0,\dots,6\}}$.

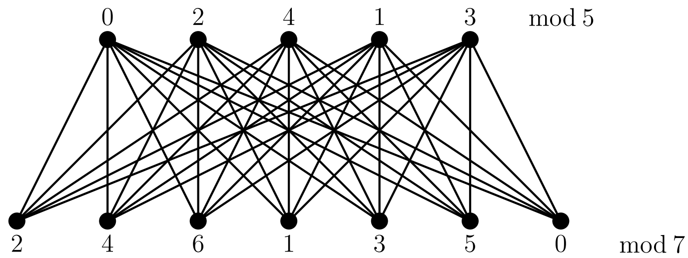

FIGURE 1. Simplicial complex $K_{5,7}$

The subcomplex $K_\emptyset$ is generated by facets $F_{j \bmod 35}$, where $j \in \{\phi(n) + 1, \phi(n) + 2, \dots, n - 1\} = \{25, 26, \dots, 34\}$. That is, $K_\emptyset$ is a bipartite graph with a set of edges $\{F_{j \bmod 35}\}_{j \in \{25,\dots,34\}}$. The geometric realization of the subcomplex $K_\emptyset$ can be seen in Figure 2.

The subcomplexes K_A, $A \subseteq \{0, 1, \dots, 24\}$, are obtained by adding the facets (edges) $\{F_{j \bmod n}\}_{j \in A}$ to subcomplex $K_\emptyset$. To illustrate this, geometric realizations of the complex K_A for some specific sets A are shown in Figure 3.

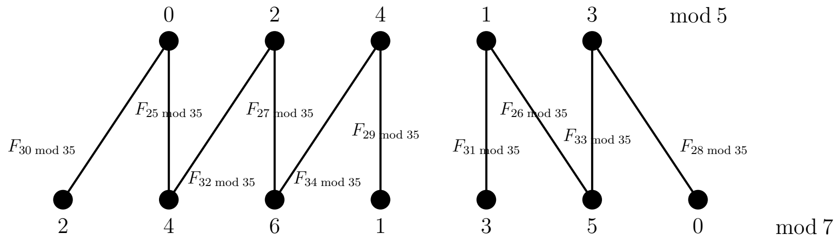

FIGURE 2. Subcomplex $K_\emptyset$ associated to polynomial Φ_{35}

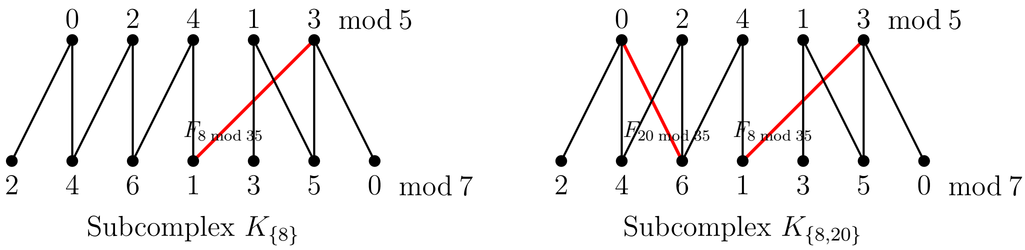

FIGURE 3. Examples of subcomplexes K_A associated to the polynomial Φ_{35}

In [7] Musiker and Reiner proved that the subcomplexes $K_\emptyset$ and $K_{\{j\}}$, where $j \in \{0, \dots, \phi(n)\}$, have an interesting topology, which we restate below.

Let $[z_{j \bmod n}] := \partial[F_{j \bmod n}]$ denote the $(d-2)$-cycle which is its image under the simplical boundary map ∂.

Theorem 1.1 (Theorem 7.1. in [7]). *Let $n = p_1 \cdots p_d$ be squarefree.*

(i) *One has a homology isomorphism $\tilde{H}_*(K_\emptyset) \cong \tilde{H}_*(\mathbb{S}^{d-2})$, with $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$ generated by the cycle $[z_{\phi(n) \bmod n}]$.*

(ii) *If $\Phi_n = \sum_{j=0}^{\phi(n)} c_j x^j$, then for $j = 0, 1, \dots, \phi(n)$, one has*

$$[z_{j \bmod n}] = c_j[z_{\phi(n) \bmod n}], \quad \text{in } \tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$$

and a homology isomorphism $\tilde{H}_*(K_{\{j\}}) \cong \tilde{H}_*(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2})$, where $\deg(f_j) = c_j$.

Stronger version of this theorem (using the same notation and assumptions) is the following.

Theorem 1.2 (Theorem 7.5. in [7]). *For $d \geq 4$ and every $A \subseteq \{0, 1, \dots, \phi(n)\}$, the complex K_A is simply-connected. Consequently, for $d \neq 3$, one has the following.*

(i) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^{d-2}$, and contains $[z_{\phi(n) \bmod n}]$ as a fundamental $(d-2)$-cycle.*

(ii) *For $j = 0, 1, \dots, \phi(n)$, the cyclotomic polynomial coefficient c_j gives the degree of the attaching map from the oriented boundary $[z_{j \bmod n}]$ of the facet $F_{j \bmod n}$ into the homotopy $(d-2)$-sphere $K_\emptyset$, with respect to the choice of $[z_{\phi(n) \bmod n}]$ as the fundamental cycle.*

(iii) *In particular, the complex* $K_{\{j\}}$ *is homotopy equivalent to* $\mathbb{S}^{d-2} \cup_{f_j} \mathbb{B}^{d-1}$, *where* $\deg(f_j) = c_j$.

However, the homotopy type of $K_\emptyset$ and $K_{\{j\}}$ when $d = 3$ remains an open question. In particular, in Question 7.6. Musiker and Rainer ask the following.

1) Is $K_\emptyset$ homotopy equivalent to the circle $\mathbb{S}^1$?
2) Is $K_{\{j\}}$ homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$, where $\deg(f_j) = c_j$, $j = 0, 1, \dots, \phi(n)$?

By giving a counter-example, the authors show in [4] that Theorem 1.2 does not hold for $d = 3$. More precisely, in case when $n = 3 \cdot 5 \cdot 7$, the authors show that the complex $K_\emptyset$ is not homotopy equivalent to the circle $\mathbb{S}^1$ and $K_{\{j\}}$ when $j = 7$ is not homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$. Additionally, in paper [5], Theorem 1.4, the author shows that $K_\emptyset$ is not homotopy equivalent to the circle $\mathbb{S}^1$ in all cases when $n = 3 \cdot 5 \cdot p$ and $p \equiv k (\mod 15)$ where $k \in \{4, 7, 8, 11\}$. Furthermore, the author shows that Theorem 1.2 holds for certain classes of prime p modulo 15.

In this paper, we analyze in detail the structure of the simplicial complexes $K_\emptyset$ and $K_{\{j\}}$ associated to Φ_n for wide classes of $n = pqr$. In Theorem 5.1 we show that Theorem 1.2 holds for all $n = pqr$, where $p < q < r$, such that $r \equiv \pm 1 (\mod pq)$.

2. Definitions and Notations

The purpose of this section is to restate some definitions and notation of the terms concerning simplicial complexes. For more details, we refer the reader to [6].

An *abstract simplicial complex* K is a finite collection of finite nonempty sets such that if $\alpha \in K$ and $\beta \subset \alpha$, then $\beta \in K$. If $\alpha \in K$ contains $n+1$ elements, then we say that α has dimension n, or alternatively, that α is an *n-simplex*. The *dimension* of the simplicial complex K is the dimension of its largest simplex.

A nonempty subset β of α in K is called a face of α, and it is a *proper face* if $\beta \subsetneq \alpha$. The *boundary* of α, denoted as $\mathrm{bd}(\alpha)$, is the set of all proper faces of α. Those simplices that are not faces of any other simplex in K are called *maximal faces* or *facets*. A face of K is called *free* if it is a face of only one facet. We can notice that every abstract simplicial complex is completely determined by its facets.

If L is a subcollection of a simplicial complex K that contains all faces of its elements, then L is another abstract simplicial complex called a *simplicial subcomplex*. A particular subcomplex $K^{(d)} = \{\alpha \in K \mid \dim \alpha \leq d\}$ is the *d-skeleton* consisting of all simplices of dimension d or less. The 0-skeleton $K^{(0)}$ is also referred to as the vertex set, denoted as $\mathrm{Vert}\, K$.

Let K and L be simplicial complexes, and let $\pi\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a map such that $\alpha \in K$ implies $\pi(\alpha) \in L$. The map π is then called the *simplicial map*. Additionally, if π is bijection and π^{-1} is a simplicial map, then π is *simplicial isomorphism*. Note that π^{-1} is not necessarily a simplicial map. For example, if $K = \{\{0\}, \{1\}\}$, $L = \{\{a\}, \{b\}, \{a, b\}\}$ and $\pi\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ given by $\pi(0) = a$, $\pi(1) = b$, then clearly π is a simplicial map, while π^{-1} is not, since the image of the simplex $\{a, b\}$ of L under π^{-1} is not a simplex in K.

Let K be an abstract simplicial complex such that $|\operatorname{Vert} K| = n$ and consider an injective mapping $f\colon \operatorname{Vert} K \to \mathbb{R}^n$. Let us denote by $\operatorname{conv}(P)$ the convex hull of the point set $P \subset \mathbb{R}^n$. We say that $|K|$ is a *geometric realization* of K in $\mathbb{R}^n$ if $|K|$ is a geometric simplicial complex, such that for any face $\alpha \in K$, the set $f(\alpha)$ is a set of affinely independent points of $\mathbb{R}^n$ and $\operatorname{conv}(f(\alpha))$ is a corresponding geometric simplex in $|K|$. In addition to that, for all faces α and β of K, we must have that $\operatorname{conv}(f(\alpha)) \cap \operatorname{conv}(f(\beta)) = \operatorname{conv}(f(\alpha) \cap f(\beta))$. We remark that a geometric realization is unique up to homeomorphism.

Definition 2.1. Suppose that α and β are two simplicies of an abstract simplicial complex K such that α is a free face and $\beta \supseteq \alpha$ is a maximal face of K. A *simplicial collapse*, denoted as $K \downarrow^{\alpha}$, is the removal of all γ such that $\alpha \subseteq \gamma \subseteq \beta$. A collapse is elementary if $\dim \alpha = \dim \beta - 1$.

It is important to point out that the spaces $|K|$ and $|K \downarrow^{\alpha}|$ are homotopy equivalent, thus by a series of simplicial collapses we can get a simpler complex of the same homotopy type.

Definition 2.2. Let K be any abstract simplicial complex and let α be any face of K. The *star* of α, denoted by $\operatorname{St}_K(\alpha)$, is the subcomplex of K consisting of all simplices β containing α and of all faces of β, i.e.,

$$\operatorname{St}_K(\alpha) = \{s \in K \mid \text{there exists } \beta \in K \text{ such that } \alpha \subseteq \beta \text{ and } s \subseteq \beta\}.$$

Definition 2.3. The *link* of α in K, denoted by $\operatorname{Lk}_K(\alpha)$, is the subcomplex of K consisting of all faces in $\operatorname{St}_K(\alpha)$ that do not intersect α, i.e.,

$$\operatorname{Lk}_K(\alpha) = \{\, \beta \in K \mid \beta \in \operatorname{St}_K(\alpha) \text{ and } \alpha \cap \beta = \emptyset\}.$$

In order to simplify notation, if $\alpha = \{v\}$, where v is a vertex, we write $\operatorname{St}_K(v)$ instead of $\operatorname{St}_K(\{v\})$ and $\operatorname{Lk}_K(v)$ instead of $\operatorname{Lk}_K(\{v\})$. Also, we denote the union $\bigcup_{i=1}^k \operatorname{St}(v_i)$ by $\operatorname{St}_K(v_1, \dots, v_k)$, and $\bigcup_{i=1}^k \operatorname{Lk}(v_i)$ by $\operatorname{Lk}_K(v_1, \dots, v_k)$.

3. Complex $K_\emptyset$ Associated to Polynomial Φ_{pqr}

In this section, we investigate the structure of the complex $K_\emptyset$ associated to the polynomial Φ_{pqr}, where $p < q < r$ are odd primes. Let $n := pqr$. Simplicial complex $K_{p,q,r}$ has $p + q + r$ vertices:

$$\begin{array}{llll} 0 \bmod p, & 1 \bmod p, & \dots, & (p-1) \bmod p, \\ 0 \bmod q, & 1 \bmod q, & \dots, & (q-1) \bmod q, \\ 0 \bmod r, & 1 \bmod r, & \dots, & (r-1) \bmod r, \end{array}$$

which we shortly denote by $0, \dots, p-1, p, \dots, p+q-1, p+q, \dots, p+q+r-1$, respectively.

This subcomplex $K_\emptyset$ is a two-dimensional simplicial complex consisting of facets $F_{j \bmod n}$, for each $j \in \{\phi(n)+1, \dots, n-1\}$. Therefore, the complex $K_\emptyset$ has $pq + pr +$

$qr-p-q-r$ facets. Note that if $r > pq-p-q$, we will show that the number of facets of $K_\emptyset$ can be reduced by a series of elementary collapses.

Namely, when $r > pq-p-q$ then $qr > pr = pr+r-r > pr+pq-p-q-r$. Now for all $j \in \{\phi(n)+pq+pr-p-q-r+1,\dots,\phi(n)+qr\}$, 1-simplex $j \bmod qr = (j \bmod q, j \bmod r)$ is a face of facet $F_{j \bmod n}$, but cannot be a face of any other facet in $K_\emptyset$ because $j-qr \leq \phi(n)+qr-qr = \phi(n)$ and $j+qr \geq \phi(n)+pq+pr-p-q-r+1+qr = n$. Therefore, 1-simplex $j \bmod qr$ is a free face of facet $F_{j \bmod n}$, for all $j \in \{\phi(n)+pq+pr-p-q-r+1,\dots,\phi(n)+qr\}$. According to this, there exists a sequence of elementary collapses $K_\emptyset \downarrow^{\{j \bmod qr\}}$, where $j \in \{\phi(n)+pq+pr-p-q-r+1,\dots,\phi(n)+qr\}$. Denote by

$$\widetilde{K_\emptyset} := K_\emptyset \downarrow^{\{j \bmod qr\}}, \quad j \in \{\phi(n)+pq+pr-p-q-r+1,\dots,\phi(n)+qr\} \tag{3.1}$$

the complex obtained from $K_\emptyset$ by these elementary collapses. Complex $\widetilde{K_\emptyset}$ consists of disjoint sets

$$\{F_{(\phi(n)+dr+i) \bmod n} : d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}, i \in \{1,\dots,pq-p-q\}\}$$

and

$$\{F_{(\phi(n)+dr+i) \bmod n} : d \in \{0,\dots,p-2\} \cup \{q,\dots,p+q-2\}, i \in \{pq-p-q+1,\dots,r\}\}.$$

It follows that

$$\begin{aligned}
\phi(n)+dr+i &= (p-1)(q-1)(r-1)+dr+i \\
&= pqr-qr-pr+r-pq+q+p-1+dr+i \\
&= (pq-q-p+1+d)r+i-(pq-q-r+1),
\end{aligned}$$

and, consequently,

$$(\phi(n)+dr+i) \bmod r = \begin{cases} r+i-(pq-q-p+1), & i \in \{1,\dots,pq-p-q\}, \\ i-(pq-q-p+1), & i \in \{pq-p-q+1,\dots,r\}. \end{cases}$$

According to the previous, if $i \in \{1,\dots,pq-p-q\}$, then every facet from the set of facets $\left\{F_{(\phi(n)+dr+i) \bmod n} : d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}\right\}$, contains the vertex $(r+i-(pq-q-p+1)) \bmod r$, which is labeled by $r+2p+2q-pq+i-1$. Similarly, for $i \in \{pq-p-q+1,\dots,r\}$, the vertex $(i-(pq-q-p+1)) \bmod r$, which is labeled by $2p+2q-pq+i-1$, belongs to each facet from the set of facets $\left\{F_{(\phi(n)+dr+i) \bmod n} : d \in \{0,\dots,p-2\} \cup \{q,\dots,p+q-2\}\right\}$. As every facet of $\widetilde{K_\emptyset}$ contains exactly one vertex from the set of vertices $\{p+q,\dots,p+q+r-1\}$, we conclude that

$$\widetilde{K_\emptyset} = \bigcup_{t=p+q}^{p+q+r-1} \mathrm{St}_{\widetilde{K_\emptyset}}(t).$$

For simplicity, we introuduce following notation:

$$\begin{aligned}
F_{p,q,r} :=& \{p+q,\dots,r+2p+2q-pq-1\}, \\
S_{p,q,r} :=& \{r+2p+2q-pq,\dots,p+q+r-1\}.
\end{aligned}$$

As facet $F_{(\phi(n)+dr+i) \bmod n}$ contains the vertex $t = 2p+2q-pq+i-1$, when $i \in \{pq-p-q+1,\dots,r\}$, $d \in \{0,\dots,p-2\} \cup \{q,\dots,p+q-2\}$, it follows that

$$\mathrm{St}_{\widetilde{K}_\emptyset}(t) = \left\{F_{(\phi(n)+dr+t-2p-2q+pq+1) \bmod n} : d \in \{0,\dots,p-2\} \cup \{q,\dots,p+q-2\}\right\}, \tag{3.2}$$

for $t \in F_{p,q,r}$.

On the other hand, when $i \in \{1,\dots,pq-p-q\}$, $d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}$, every facet $F_{(\phi(n)+dr+i) \bmod n}$ contains the vertex $t = r+2p+2q-pq+i-1$, and consequently it follows that

$$\mathrm{St}_{\widetilde{K}_\emptyset}(t) = \left\{F_{(\phi(n)+dr+t-r-2p-2q+pq+1) \bmod n} : d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}\right\}, \tag{3.3}$$

for $t \in S_{p,q,r}$.

Next, we consider subcomplexes $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$ for different primes r and the relations between them. Let r and r' be two distinct prime numbers such that $r, r' > pq-p-q$. We denote by $\widetilde{K}_\emptyset$ and $\widetilde{K}'_\emptyset$ the associated complexes for $n = pqr$ and $n' = pqr'$, respectively. Let us consider subcomplexes $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$ and $\mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'})$. Complex $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$ is generated by the set of facets

$$\left\{F_{\phi(n)+dr+i} : d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}, i \in \{1,\dots,pq-p-q\}\right\},$$

while $\mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'})$ is generated by the set of facets

$$\left\{F_{\phi(n')+dr'+i} : d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}, i \in \{1,\dots,pq-p-q\}\right\}.$$

Notice that $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$ and $\mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'})$ have the same cardinality. Additionally, under some conditions, these two complexes are isomorphics.

Namely, if $r \equiv r' \pmod{pq}$, then

$$\phi(n) + dr + i \equiv \phi(n') + dr' + i \pmod{pq},$$

for all $d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}$ and all $i \in \{1,\dots,pq-p-q\}$. Hence, we can define a simplicial map $\pi : \mathrm{Vert}(\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})) \to \mathrm{Vert}(\mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'}))$ which is given as follows $\pi(\phi(n)+dr+i) = \phi(n')+dr'+i$, for all $d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}$ and all $i \in \{1,\dots,pq-p-q\}$.

In case when $r \equiv -r' \pmod{pq}$, it follows that

$$\phi(n) + dr + i \equiv \phi(n') + dr' + pq - p - q - i \pmod{pq},$$

for all $d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}$ and $i \in \{1,\dots,pq-p-q\}$. The simplicial map $\pi : \mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r}) \to \mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'})$, such that

$$\pi(\phi(n) + dr + i) = \phi(n') + dr' + pq - p - q - i,$$

for all $d \in \{0,\dots,p-1\} \cup \{q,\dots,p+q-1\}$, $i \in \{1,\dots,pq-p-q\}$, shows isomorphism between $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$ and $\mathrm{St}_{\widetilde{K}'_\emptyset}(S_{p,q,r'})$.

Therefore, for fixed prime numbers p, q, there are only finitely many non-isomorphic subcomplexes $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{p,q,r})$.

3.1. **Subcomplex** $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$**.** To understand the structure of complex $\widetilde{K_\emptyset}$ we investigate in detail the structure of its subcomplexes $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$, $t \in F_{p,q,r} \sqcup S_{p,q,r}$. Let us introduce the following notation

$$a_j^t := \begin{cases} (\phi(n) + (j-1)r + t - 2p - 2q + pq + 1) \bmod p, & \text{for } t \in F_{p,q,r}, \\ (\phi(n) + (j-1)r + t - r - 2p - 2q + pq + 1) \bmod p, & \text{for } t \in S_{p,q,r}, \end{cases}$$

and

$$b_j^t := \begin{cases} ((\phi(n) + (j-1)r + t - 2p - 2q + pq + 1) \bmod q) + p, & \text{for } t \in F_{p,q,r}, \\ ((\phi(n) + (j-1)r + t - r - 2p - 2q + pq + 1) \bmod q) + p, & \text{for } t \in S_{p,q,r}. \end{cases}$$

In other words, a_j^t are labels for vertices $\bmod p$ in facets where vertex $\bmod r$ is denoted by t. Similarly, b_j^t are labels for vertices $\bmod q$. Based on the previous, we have 2-simplices

$$\left[a_j^t, b_j^t, t\right] = \begin{cases} F_{(\phi(n)+(j-1)r+t-2p-2q+pq+1) \bmod n}, & \text{for } t \in F_{p,q,r}, \\ F_{(\phi(n)+(j-1)r+t-r-2p-2q+pq+1) \bmod n}, & \text{for } t \in S_{p,q,r}. \end{cases}$$

For example, for $n = 165 = 3 \cdot 5 \cdot 11$, let us consider facets $F_{90 \bmod 165}$ and $F_{101 \bmod 165}$. For facet $F_{90 \bmod 165}$ we have $\phi(n)+dr+i = \phi(165)+d\cdot 11+i = 90$, for $d = 0$, $i = 10$ and this facet contains vertex denoted as $t = 2p+2q-pq+i-1 = 2\cdot 3+2\cdot 5-3\cdot 5+10-1 = 10$. As $90 = \phi(165)+(1-1)\cdot 11+10-2\cdot 3-2\cdot 5+3\cdot 5+1$, the remaining two vertices are labeled by $a_1^{10} = 0$ and $b_1^{10} = 3$. Similarly, $101 = \phi(165)+(2-1)\cdot 11+10-2\cdot 3-2\cdot 5+3\cdot 5+1$, facet $F_{101 \bmod 165}$ has vertices labeled by $a_2^{10} = 2$, $b_2^{10} = 4$ and $t = 10$.

According to (3.2) and (3.3), it is clear that

$$\mathrm{St}_{\widetilde{K_\emptyset}}(t) = \{[a_j^t, b_j^t, t] : j \in \{1, \dots, p-1\} \cup \{q+1, \dots, p+q-1\}\}, \tag{3.4}$$

for $t \in F_{p,q,r}$, and

$$\mathrm{St}_{\widetilde{K_\emptyset}}(t) = \{[a_j^t, b_j^t, t] : j \in \{1, \dots, p\} \cup \{q+1, \dots, p+q\}\}, \tag{3.5}$$

for $t \in S_{p,q,r}$.

Let $l : \mathbb{N} \to \{1, \dots, p\}$ and $k : \mathbb{N} \to \{1, \dots, q\}$ be functions defined as

$$l(i) = \begin{cases} i \bmod p, & \text{if } p \nmid i, \\ p & \text{if } p \mid i, \end{cases} \quad \text{and} \quad k(i) = \begin{cases} i \bmod q, & \text{if } q \nmid i, \\ q & \text{if } q \mid i. \end{cases}$$

Consider the facets $[a_i^t, b_i^t, t]$ and $[a_{i+j}^t, b_{i+j}^t, t]$ in subcomplex $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$, $t \in F_{p,q,r}$. If $[a_i^t, b_i^t, t] = F_{d \bmod n}$, for $d \in \{\phi(n)+1, \dots, n-1\}$, observe that $[a_{i+j}^t, b_{i+j}^t, t] = F_{(d+jr) \bmod n}$. Namely, we have that $d \equiv d + jr \pmod p$, when $p \mid j$. Analogously, if $q \mid j$, we have that $d \equiv d + jr \pmod p$. Therefore, we obtain that

$$a_i^t = a_{l(i)}^t \quad \text{and} \quad b_i^t = b_{k(i)}^t. \tag{3.6}$$

The numbers p and r are coprime, thus the vertices labeled by $a_1^t, a_2^t, \dots, a_p^t \in K_p$ are pairwise distinct. Also, vertices labeled by $b_1^t, b_2^t, \dots, b_q^t \in K_q$ are pairwise distinct, because q and r are coprime as well.

Based on previous considerations, we can conclude that

$$\begin{aligned}\mathrm{St}_{\widetilde{K_\emptyset}}(t) = \{&[a_1^t, b_1^t, t], [a_2^t, b_2^t, t], \ldots, [a_{p-1}^t, b_{p-1}^t, t],\\ &[a_{l(q+1)}^t, b_1^t, t], [a_{l(q+2)}^t, b_2^t, t], \ldots, [a_{l(p+q-1)}^t, b_{p-1}^t, t]\},\end{aligned}$$

for $t \in F_{p,q,r}$.

The following simple observations give us detailed insight into the structure of the subcomplex $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$, for $t \in F_{p,q,r}$.

Indices $\{l(q+1), l(q+2), \ldots, l(p+q-1)\} \subset \{1, 2, \ldots, p\}$ are pairwise distinct and $l(q) \notin \{l(q+1), l(q+1), \ldots, l(p+q-1)\}$. Additionally, as $p \nmid q$, we have $l(q) \neq p$, and consequently, $l(q) \in \{1, \ldots, p-1\}$. Thus, $p \in \{l(q+1), l(q+1), \ldots, l(p+q-1)\}$.

Note that there are exactly two 2-simplices whose face is $[b_j^t, t]$, for all $j \in \{1, \ldots, p-1\}$. These simplices are $[a_j^t, b_j^t, t]$ and $[a_{l(q+j)}^t, b_j^t, t]$. As $l(q) \in \{1, \ldots, p-1\}$ and $l(q) \notin \{l(q+1), l(q+1), \ldots, l(p+q-1)\}$, 1-simplex $[a_{l(q)}^t, t]$ is a face of exactly one 2-simplex $[a_{l(q)}^t, b_{l(q)}^t, t]$. Similarly, as $p \in \{l(p+1), \ldots, l(p+q-1)\}$ and $p \notin \{1, \ldots, p-1\}$, the only 2-simplex that has 1-simplex $[a_p^t, t]$ as a face is $[a_p^t, b_{l(p-q)}^t, t]$. Each of the remaining 1-simplices $[a_j^t, t]$, $j \in \{1, \ldots, p-1\} \setminus \{l(q)\}$, is a face of exactly two 2-simplices, $[a_j^t, b_j^t, t]$ and $[a_j^t, b_{l(j-q)}^t, t]$.

Figure 4 shows the geometric realization of the subcomplex $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$, $t \in F_{p,q,r}$. As $p \nmid q$, it is easy to see that $\{l(q), l(2q), \ldots, l((p-1)q)\} = \{1, 2, \ldots, p-1\}$.

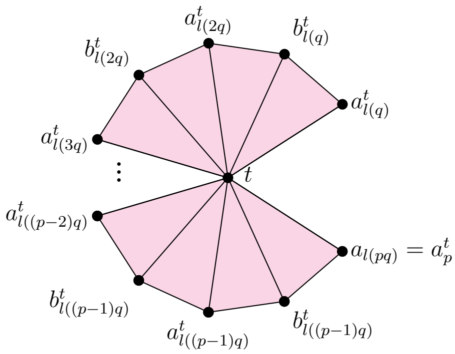

FIGURE 4. Subcomplex $\mathrm{St}_{\widetilde{K_\emptyset}}(t)$, $t \in F_{p,q,r}$

In case when $t \in S_{p,q,r}$, it follows by (3.5) and (3.6) that

$$\begin{aligned}\mathrm{St}_{\widetilde{K_\emptyset}}(t) = \{&[a_1^t, b_1^t, t], [a_2^t, b_2^t, t], \ldots, [a_{p-1}^t, b_{p-1}^t, t], [a_p^t, b_p^t, t],\\ &[a_{l(q+1)}^t, b_1^t, t], [a_{l(q+2)}^t, b_2^t, t], \ldots, [a_{l(p+q-1)}^t, b_{p-1}^t, t], [a_{l(p+q)}^t, b_p^t, t]\}.\end{aligned}$$

It is straightforward to see that $\{l(q+1), \ldots, l(q+p)\} = \{1, \ldots, p\}$. Therefore, each of 1-simplices $[a_j^t, t]$ and $[b_j^t, t]$, $t \in \{1, \ldots, p\}$, is a face of exactly two 2-simplices. Simplex $[a_j^t, t]$ is a face of 2-simplices $[a_j^t, b_j^t, t]$ and $[a_j^t, b_{l(j-q)}^t, t]$, and simplex $[b_j^t, t]$ is a face of 2-simplices $[a_j^t, b_j^t, t]$ and $[a_{l(q+j)}^t, b_j^t, t]$.

The geometric realization of the subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, $t \in S_{p,q,r}$, is given in Figure 5. Since $p \nmid q$, it follows that $\{l(q), l(2q), \dots, l(pq)\} = \{1, 2, \dots, p\}$. Generally, the subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, for $t \in S_{p,q,r}$, is an $(2p)$-gon.

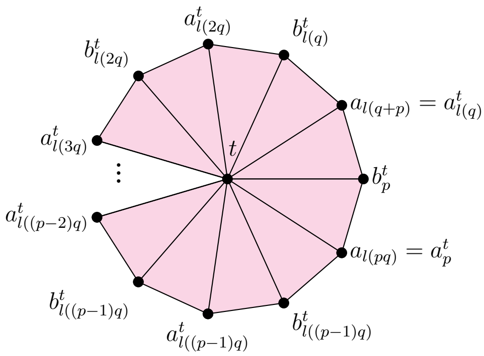

FIGURE 5. Subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, $t \in S_{p,q,r}$

Example 3.1. To illustrate what has been shown concerning star subcomplexes, let us consider the complex $\widetilde{K}_\emptyset$ associated to the cyclotomic polynomial $\Phi_{3\cdot5\cdot11}$. This complex is generated by the facets:

$[0,4,12]$, $[1,5,13]$, $[2,6,14]$, $[0,7,15]$, $[1,3,16]$, $[2,4,17]$, $[0,5,18]$, $[1,6,8]$, $[2,7,9]$, $[0,3,10]$,
$[1,4,11]$, $[2,5,12]$, $[0,6,13]$, $[1,7,14]$, $[2,3,15]$, $[0,4,16]$, $[1,5,17]$, $[2,6,18]$, $[0,7,8]$, $[1,3,9]$,
$[2,4,10]$, $[0,5,11]$, $[1,6,12]$, $[2,7,13]$, $[0,3,14]$, $[1,4,15]$, $[2,5,16]$, $[0,6,17]$, $[1,7,18]$,
$[1,4,12]$, $[2,5,13]$, $[0,6,14]$, $[1,7,15]$, $[2,3,16]$, $[0,4,17]$, $[1,5,18]$, $[2,6,8]$, $[0,7,9]$, $[1,3,10]$,
$[2,4,11]$, $[0,5,12]$, $[1,6,13]$, $[2,7,14]$, $[0,3,15]$, $[1,4,16]$, $[2,5,17]$, $[0,6,18]$, $[1,7,8]$, $[2,3,9]$,
$[0,4,10]$, $[1,5,11]$, $[2,6,12]$, $[0,7,13]$, $[1,3,14]$, $[2,4,15]$, $[0,5,16]$, $[1,6,17]$, $[2,7,18]$.

Therefore, $\widetilde{K}_\emptyset$ is the union of the star complexes shown in Figure 6.

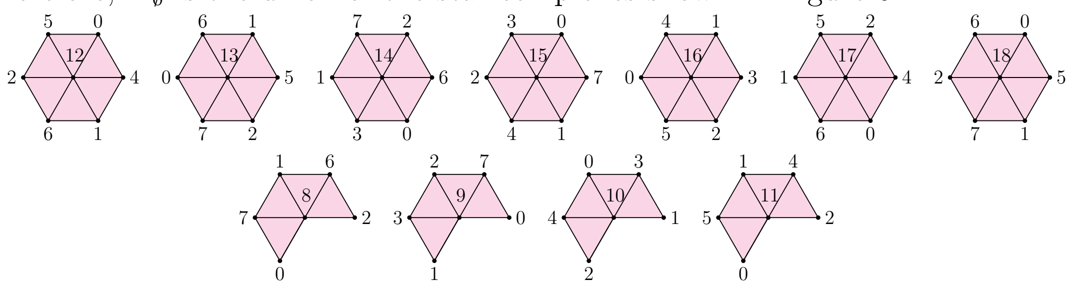

FIGURE 6. The complex $\widetilde{K}_\emptyset$ associated to the polynomial $\Phi_{3\cdot5\cdot11}$

4. REDUCTION OF THE COMPLEX $\widetilde{K}_\emptyset$

In order to determine the homotopy type of the complex $\widetilde{K}_\emptyset$ we will reduce the number of 2-simplices in $\widetilde{K}_\emptyset$ as much as possible. This section is devoted to some technical results which concern the reduction of the complex $\widetilde{K}_\emptyset$.

First, we consider the subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, for $t \in F_{p,q,r}$. Notice that this two-dimensional star subcomplex is homotopy equivalent to the one dimensional complex $\mathrm{Lk}_{\widetilde{K}_\emptyset}(t)$. Namely, simplex $[a_p^t, t]$ is a free face in $\widetilde{K_\emptyset}$, so elementary collapse $\mathrm{St}_{\widetilde{K}_\emptyset}(t) \downarrow^{[a_p^t,t]}$ can be performed. Additionally, $[b_{l((p-1)q)}^t, t]$ is a free face in $\widetilde{K_\emptyset} \setminus \{[a_p^t, t]\}$, hence, we can continue to simplify subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$ (for more details see Figure 7).

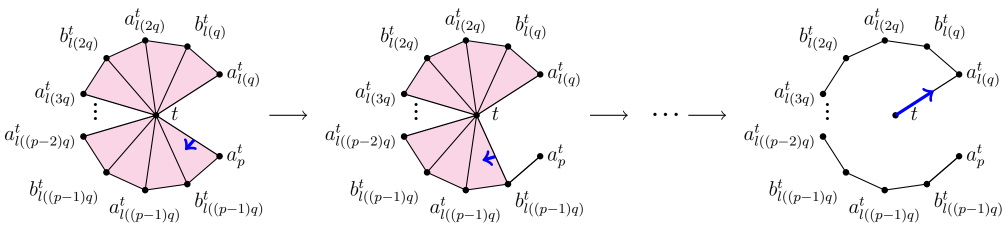

FIGURE 7. Simplicial collapse of $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$ on $\mathrm{Lk}_{\widetilde{K}_\emptyset}(t)$, $t \in F_{p,q,r}$

Now we can proceed to the case of subcomplexes $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, $t \in S_{p,q,r}$. If there exists 1-simplex $[a_i^t, b_j^t] \in \mathrm{Lk}_{\widetilde{K}_\emptyset}(t)$ which is a free face in $\widetilde{K_\emptyset}$, then $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$ is homotopy equivalent to its subcomplex $\mathrm{Lk}_{\widetilde{K}_\emptyset}(t) \setminus \{[a_i^t, b_j^t]\}$. The reasoning for this is similar as in the case of subcomplexes $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$, $t \in F_{p,q,r}$. We can perform a sequence of elementary collapses starting with the collapse $\mathrm{St}_{\widetilde{K}_\emptyset}(t) \downarrow^{[a_i^t,b_j^t]}$ and ending with $\mathrm{St}_{\widetilde{K}_\emptyset}(t) \downarrow^{[t]}$ (see Figure 8). If L is a subcomplex of the complex $\widetilde{K_\emptyset}$, we denote by $L \downarrow^{[a_i^t,b_j^t],t}$ the said sequence of elementary collapses which collapse the subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(t) \subseteq L$ onto its boundary $\mathrm{Lk}_{\widetilde{K}_\emptyset}(t) \setminus \{[a_i^t, b_j^t]\}$ inside the complex L.

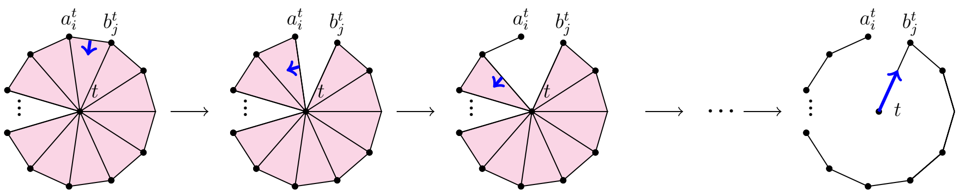

FIGURE 8. Simplicial collapse of $\mathrm{St}_{\widetilde{K}_\emptyset}(t)$ on $\mathrm{Lk}_{\widetilde{K}_\emptyset}(t) \setminus \{[a_i^t, b_j^t]\}$, $t \in S_{p,q,r}$

5. Homotopy Type

Based on analysis given in previous sections, in what follows we examine the homotopy type of the complexes $K_\emptyset$ and $K_{\{j\}}$ associated to the cyclotomic polynomial Φ_{pqr}.

Theorem 5.1. *Let $n = pqr$, where p, q, r are prime numbers, $p < q < r$. If $r \equiv \pm 1 \pmod{pq}$, then the following hold.*

(1) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.*

(2) *If* $\Phi_{pqr}(x) = \sum_{j=0}^{\phi(x)} c_j x^j$*, then for* $j \in \{0, 1, \ldots, \phi(n)\}$*, the complex* $K_{\{j\}}$ *is homotopy equivalent to* $\mathbb{S}^1 \cup_{f_j} \mathbb{B}^2$*, where* $\deg(f_j) = c_j$.

Proof. First, notice that if $r \equiv \pm 1 \pmod{pq}$, then $r \geq pq - 1 > pq - p - q$, hence we can observe the complex $\widetilde{K_\emptyset}$ instead of the complex $K_\emptyset$.

In order to prove that $\widetilde{K_\emptyset} \simeq \mathbb{S}^1$, we will show that $\widetilde{K_\emptyset}$ is homotopy equivalent to a one-dimensional simplicial complex. Together with the result from Theorem 1.1, where we have that $H_1(\widetilde{K_\emptyset}) \cong \mathbb{Z}$, it will then follow that the complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.

Based on the discussion presented in Section 3, it is sufficient to assume that $r \equiv 1 \pmod{pq}$.

Taking $m = |S_r| = pq - p - q$, we denote elements of the set S_r by $t_1, t_2, \ldots, t_m$, respectively.

Let $i' > i$, where $i, i' \in \{1, \ldots, m\}$. Since $1 \leq i' - i < m = pq - p - q$, for all $j \in \{1, \ldots, p\} \cup \{q + 1, \ldots, p + q\}$, it follows that

$$\phi(n) + i' + (j - 1)r \not\equiv \phi(n) + i \pmod{pq}.$$

Otherwise, we would have $i' - i = lpq - (j - 1)$ for some $l > 0$. As $lpq - (j - 1) > pq - p - q + 1$ and $i' - i < pq - p - q$, this is not possible. Therefore,

$$[a_1^{t_i}, b_1^{t_i}] \neq [a_j^{t_{i'}}, b_j^{t_{i'}}],$$

for all $j \in \{1, \ldots, p\} \cup \{q + 1, \ldots, p + q\}$. In other words, it follows that

$$[a_1^{t_i}, b_1^{t_i}] \notin \mathrm{St}_{\widetilde{K_\emptyset}}(t_{i'}).$$

According to the previous, 1-simplex $[a_1^{t_1}, b_1^{t_1}]$ is a free face in $\widetilde{K_\emptyset}$, $[a_2^{t_2}, b_2^{t_2}]$ is a free face in $\widetilde{K_\emptyset} \downarrow^{[a_1^{t_1}, b_1^{t_1}]}$, and so on. Thus, there is a sequence of simplicial collapses

$$\widetilde{K_\emptyset} \downarrow^{[a_1^{t_1}, b_1^{t_1}], t_1} \downarrow^{[a_2^{t_2}, b_2^{t_2}], t_2} \downarrow^{[a_3^{t_3}, b_3^{t_3}], t_3} \cdots \downarrow^{[a_m^{t_m}, b_m^{t_m}], t_m} .$$

In other words, $\widetilde{K_\emptyset}$ is homotopy equivalent to one-dimensional simplicial complex

$$K_{p,q} \setminus \{[a_1^{t_1}, b_1^{t_1}], [a_2^{t_2}, b_2^{t_2}], \ldots, [a_m^{t_m}, b_m^{t_m}]\},$$

which proves the first part of Theorem 5.1.

The complex $K_{\{j\}}$ where $j \in \{0, 1, \ldots, \phi(n)\}$, is built from its subcomplex $K_\emptyset$ by attaching the facet $F_{j \bmod n}$ along its boundary. Let g be a generator of the fundamental group $\pi_1(K_\emptyset)$. Then, it follows that $\pi_1(K_{\{j\}}) = \langle g \mid g^d = 1 \rangle$, for some $d \in \mathbb{N}$. However, Theorem 1.1 implies that $H_1(K_{\{j\}}) = \mathbb{Z}/c_j\mathbb{Z}$. As $H_1(K_{\{j\}})$ is abelization of the group $\pi_1(K_{\{j\}})$, we can conclude that $d = c_j$. Thus, the second assertion of the theorem holds. □

Example 5.1. The following example serves to illustrate the previous theorem. The subcomplex $\mathrm{St}_{\widetilde{K_\emptyset}}(S_{5,7,71})$ associated to the cyclotomic polynomial $\Phi_{5 \cdot 7 \cdot 71}$ is the union of star complexes shown in Figure 9.

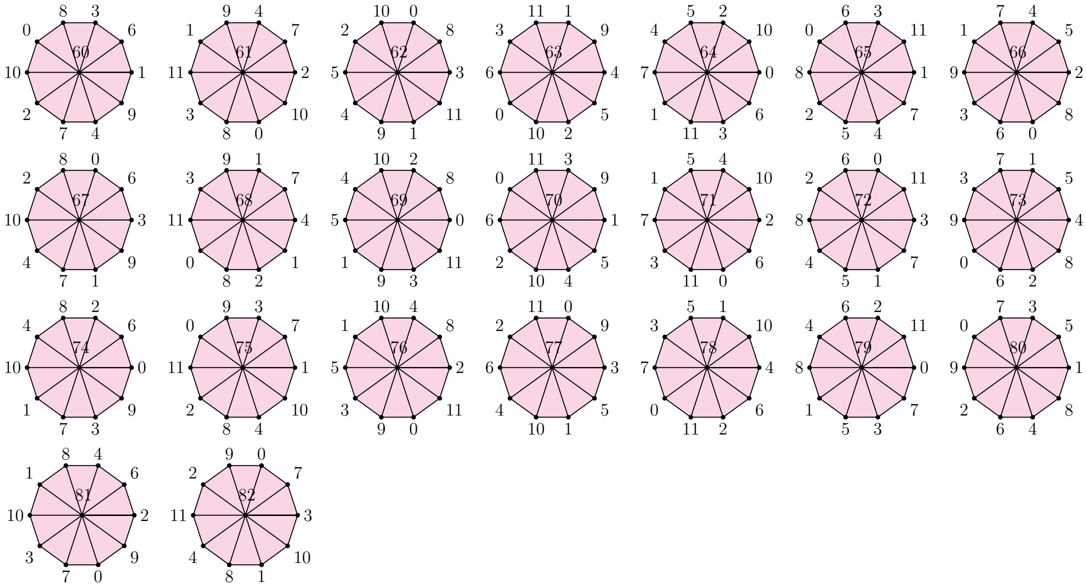

FIGURE 9. Subcomplex $\mathrm{St}_{\widetilde{K}_\emptyset}(S_{5,7,71})$ associated to the polynomial $\Phi_{5\cdot 7\cdot 71}$

There exists a sequence of elementary collapses

$$\begin{aligned}\mathrm{St}_{\widetilde{K}_\emptyset}(S_{5,7,71}) &\downarrow^{[1,6],\,60}\downarrow^{[2,7],\,61}\downarrow^{[3,8],\,62}\downarrow^{[4,9],\,63}\downarrow^{[0,10],\,64}\downarrow^{[1,11],\,65}\downarrow^{[2,5],\,66}\downarrow^{[3,6],\,67}\downarrow^{[4,7],\,68}\\ &\downarrow^{[0,8],\,69}\downarrow^{[1,9],\,70}\downarrow^{[2,10],\,71}\downarrow^{[3,11],\,72}\downarrow^{[4,5],\,73}\downarrow^{[0,6],\,74}\downarrow^{[1,7],\,75}\downarrow^{[2,8],\,76}\downarrow^{[3,9],\,77}\\ &\downarrow^{[4,10],\,78}\downarrow^{[0,11],\,79}\downarrow^{[1,5],\,80}\downarrow^{[2,6],\,81}\downarrow^{[3,7],\,82},\end{aligned}$$

which shows that $\widetilde{K}_\emptyset$ is homotopy equivalent to the one-dimensional simplicial complex given in Figure 10. It follows that $\widetilde{K}_\emptyset \simeq \mathbb{S}^1$.

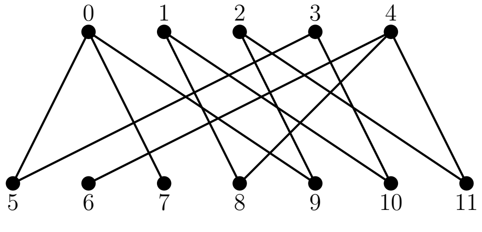

FIGURE 10. The reduction of simplicial complex $\widetilde{K}_\emptyset$ associated to the polynomial $\Phi_{5\cdot 7\cdot 71}$

Example 5.2. This example illustrates a case when $r \equiv k \bmod pq$, for $k \neq \pm 1$. Consider the complex $\widetilde{K}_\emptyset$ associated to the cyclotomic polynomial $\Phi_{3\cdot 7\cdot 23}$. It is a union of star complexes shown in Figure 11.

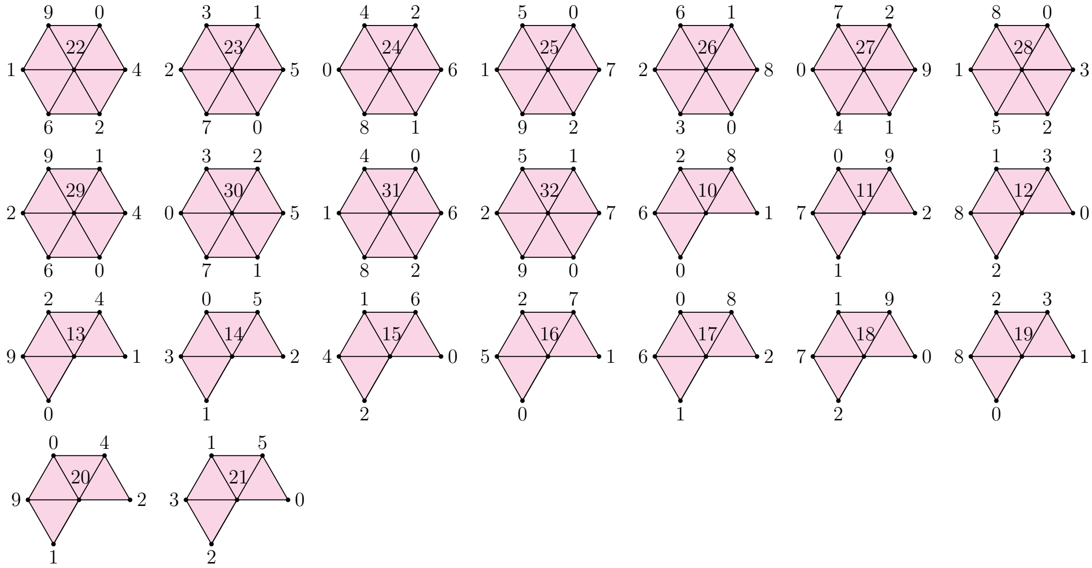

FIGURE 11. The complex $\widetilde{K_\emptyset}$ asssociated to the cyclotomic polynomial $\Phi_{3\cdot 7\cdot 23}$

We have the following sequence of elementary collapses:

$$\mathrm{St}_{\widetilde{K_\emptyset}}(S_{3,7,23}) \downarrow^{[2,8],\,31} \downarrow^{[1,3],\,23} \downarrow^{[0,6],\,29} \downarrow^{[0,5],\,25} \downarrow^{[1,4],\,27} \downarrow^{[1,9],\,22} \downarrow^{[2,9],\,32} \downarrow^{[0,4],\,24} \downarrow^{[1,7],\,30} \downarrow^{[2,6],\,26} \downarrow^{[2,5],\,28},$$

so we obtain that $\mathrm{St}_{\widetilde{K_\emptyset}}(S_{3,7,23})$ can be collapsed onto one-dimensional simplicial complex.

Finally, we can conclude that $\widetilde{K_\emptyset}$ is homotopy equivalent to one-dimensional simplicial complex which is given in Figure 12. Therefore, $\widetilde{K_\emptyset} \simeq \mathbb{S}^1$. In view of section 3, it follows that the complex $\widetilde{K_\emptyset}$ associated to the cyclotomic polynomial $\Phi_{3\cdot 7\cdot r}$ is homotopy equivalent to $\mathbb{S}^1$ whenever $r \equiv \pm 2 \pmod{21}$.

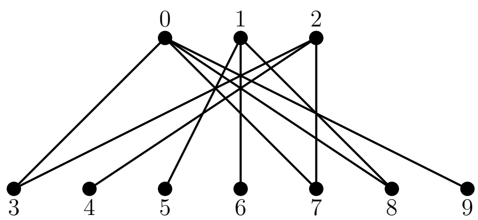

FIGURE 12. Reduction of simplicial complex $\widetilde{K_\emptyset}$ associated to the polynomial $\Phi_{3\cdot 7\cdot 23}$

Motivated by the previous example and results from [5] where it is shown that for certain classes of primes $\widetilde{K_\emptyset}$ is not homotopy equivalent to $\mathbb{S}^1$, we finish this section by posing the following question:

Question 1. For what $j \in \{1, \dots, \frac{pq-1}{2}\}$ we have that for all $r \equiv \pm j \bmod pq$ the complex $\widetilde{K_\emptyset}$ is homotopy equivalent to $\mathbb{S}^1$?

Acknowledgements. The authors are very grateful to the anonymous referee whose comments and suggestions significantly improved the clarity of the paper.

The first and the third author are partially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia Project number 451-03-33/2026-03/200104.

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