

THE HARMONIC INDEX OF EDGE-SEMITOTAL GRAPHS, TOTAL GRAPHS AND RELATED SUMS

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ABSTRACT. For a connected graph G , there are several related graphs such as line graph $L(G)$, subdivision graph $S(G)$, vertex-semitotal graph $R(G)$, edge-semitotal graph $Q(G)$ and total graph $T(G)$ [I. Gutman, B. Furtula, Ž. K. Vukićević and G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem. **74** (2015), 5–16, W. Yan, B. -Y. Yang and Y. -N. Yeh, *The behavior of Wiener indices and polynomials of graphs under five graph decorations*, Appl. Math. Lett. **20** (2007), 290–295]. Let F be one of symbols S , R , Q or T . The F -sum $G_1 +_F G_2$ of two connected graphs G_1 and G_2 is a graph with vertex set $(V(G_1) \cup E(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = u_2 \in V(G_1) \text{ and } v_1 v_2 \in E(G_2)]$ or $[v_1 = v_2 \text{ and } u_1 u_2 \in E(F(G))]$ [M. Eliasi and B. Taeri, *Four new sums of graphs and their Wiener indices*, Discrete Appl. Math. **157** (2009), 794–803]. In this paper, we investigate the harmonic index of edge-semitotal graphs, total graphs and F -sum of graphs, where $F = Q$ or T .

1. INTRODUCTION

Throughout this paper, all graphs are finite, simple, undirected and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, the degree of a vertex u in G is denoted by $\deg_G(u)$. We will use P_n to denote the path of order n .

The edge-semitotal graph $Q(G)$ is obtained from G by inserting a new vertex into each edge of G , then joining with edges those pairs of new vertices on adjacent edges of G (see Figure 1). The total graph $T(G)$ has as its vertices the edges and vertices of G ; adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G (see Figure 1).

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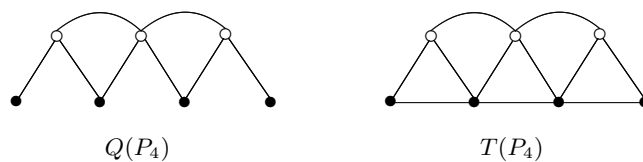


FIGURE 1. $Q(P_4)$ and $T(P_4)$

Let G_1 and G_2 be two graphs and $F \in \{Q, T\}$. Then, the F -sum $G_1 +_F G_2$ has $|V(G_2)|$ copies of the graph $F(G_1)$ and we can label these copies by vertices of G_2 . The vertices in each copy have two situations: the vertices in $V(G_1)$ (black vertices) and the vertices in $E(G_1)$ (white vertices). Now, we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2 (see Figure 2).

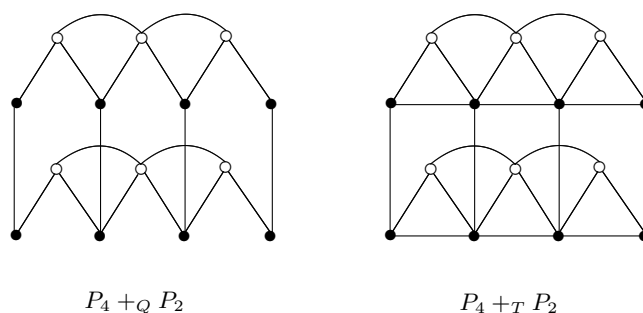


FIGURE 2. $P_4 +_Q P_2$ and $P_4 +_T P_2$

The first Zagreb index, the inverse degree and the harmonic index of a graph G are important vertex-degree-based indices related to G , where denoted by $M_1(G)$, $r(G)$ and $H(G)$, respectively, and defined as follows [3, 4]:

$$M_1(G) = \sum_{u \in V(G)} \deg_G^2(u), \quad r(G) = \sum_{u \in V(G)} \frac{1}{\deg_G(u)}, \quad H(G) = \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}.$$

The first Zagreb index can be expressed as $M_1(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))$ [8].

In this paper, we consider the harmonic index. In recent years, this topological index has been extensively studied. Shwetha et al. [11] derived expressions for the harmonic index of the join, corona product, Cartesian product, composition and symmetric difference of graphs. Recently, Onagh investigated the harmonic index of product graphs, subdivision graphs, t -subdivision graphs and F -sum of graphs, where $F \in \{S, S_t\}$ [9, 10]. More results on the harmonic index can be found in [1, 6, 7, 12–15].

In this paper, we study the harmonic index of edge-semitotal graphs, total graphs and F -sum of graphs, where $F = Q$ or T .

2. THE HARMONIC INDEX OF EDGE-SEMITOTAL AND TOTAL GRAPHS

In this section, we give some upper bounds for the harmonic index of edge-semitotal and total graphs. Hereafter, we deal with nontrivial graphs.

Note that $\deg_{Q(G)}(u) = \deg_G(u)$ for all $u \in V(G)$ and $\deg_{Q(G)}(w) = \deg_G(u) + \deg_G(v) = 2 + \deg_{L(G)}(w)$ for all $w \in V(Q(G)) - V(G)$, where w is the vertex inserted into the edge uv of G .

Theorem 2.1. *Let G be a graph of order n and size m . Then*

$$H(Q(G)) < \frac{1}{2}H(G) + \frac{1}{4}H(L(G)) + \frac{1}{16}M_1(G) + \frac{1}{2}n - \frac{1}{8}m.$$

Proof. By definition of the harmonic index, we have

$$\begin{aligned} & H(Q(G)) \\ &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_{Q(G)}(u) + \deg_{Q(G)}(x)} + \frac{2}{\deg_{Q(G)}(x) + \deg_{Q(G)}(v)} \right) \\ & \quad (\text{where } x \text{ is the vertex inserted into the edge } uv \text{ of } G) \\ & \quad + \sum_{ww' \in E(L(G))} \frac{2}{\deg_{Q(G)}(w) + \deg_{Q(G)}(w')} \\ &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} + \frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} \right) \\ & \quad + \sum_{ww' \in E(L(G))} \frac{2}{(2 + \deg_{L(G)}(w)) + (2 + \deg_{L(G)}(w'))} \\ &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} + \frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} \right) \\ & \quad + \sum_{ww' \in E(L(G))} \frac{2}{4 + (\deg_{L(G)}(w) + \deg_{L(G)}(w'))} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

By Jensen's inequality, for every $uv \in E(G)$, we have

$$\begin{aligned} \frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} &< \frac{1}{2} \frac{1}{\deg_G(u)} + \frac{1}{4} \frac{2}{\deg_G(u) + \deg_G(v)}, \\ \frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} &< \frac{1}{4} \frac{2}{\deg_G(u) + \deg_G(v)} + \frac{1}{2} \frac{1}{\deg_G(v)}. \end{aligned}$$

So,

$$\sum 1 < \frac{1}{2} \sum_{uv \in E(G)} \left(\frac{1}{\deg_G(u)} + \frac{1}{\deg_G(v)} \right) + \frac{1}{2} \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{u \in V(G)} \left(\frac{1}{\deg_G(u)} \times \deg_G(u) \right) + \frac{1}{2} H(G) \\
&= \frac{1}{2} n + \frac{1}{2} H(G).
\end{aligned}$$

Similarly, for every $ww' \in E(L(G))$,

$$\frac{2}{4 + (\deg_{L(G)}(w) + \deg_{L(G)}(w'))} \leq \frac{1}{8} + \frac{1}{4} \frac{2}{\deg_{L(G)}(w) + \deg_{L(G)}(w')},$$

with equality if and only if $\deg_{L(G)}(w) + \deg_{L(G)}(w') = 4$. This implies that

$$\begin{aligned}
\sum 2 &\leq \frac{1}{8} |E(L(G))| + \frac{1}{4} H(L(G)) \\
&= \frac{1}{8} \left(\frac{1}{2} M_1(G) - m \right) + \frac{1}{4} H(L(G)) \\
&= \frac{1}{16} M_1(G) - \frac{1}{8} m + \frac{1}{4} H(L(G)).
\end{aligned}$$

Therefore,

$$H(Q(G)) < \frac{1}{2} H(G) + \frac{1}{4} H(L(G)) + \frac{1}{16} M_1(G) + \frac{1}{2} n - \frac{1}{8} m.$$

This completes the proof. \square

Example 2.1. For any $n \geq 4$,

$$H(Q(P_n)) = \left(\frac{9}{5} + \frac{2}{3}(n-3) \right) + \left(\frac{4}{7} + \frac{1}{4}(n-4) \right) = \frac{83}{35} + \frac{2}{3}(n-3) + \frac{1}{4}(n-4).$$

Note that $\deg_{T(G)}(u) = 2 \deg_G(u)$ for all $u \in V(G)$ and $\deg_{T(G)}(w) = \deg_G(u) + \deg_G(v) = 2 + \deg_{L(G)}(w)$ for all $w \in V(T(G)) - V(G)$, where w is the vertex inserted into the edge uv of G .

Theorem 2.2. *Let G be a graph of order n and size m . Then*

$$H(T(G)) \leq H(G) + \frac{1}{4} H(L(G)) + \frac{1}{16} M_1(G) + \frac{1}{4} n - \frac{1}{8} m,$$

with equality if and only if $G \cong C_n$.

Proof. Note that

$$\begin{aligned}
&H(T(G)) \\
&= \sum_{uv \in E(G)} \frac{2}{\deg_{T(G)}(u) + \deg_{T(G)}(v)} \\
&\quad + \sum_{uv \in E(G)} \left(\frac{2}{\deg_{T(G)}(u) + \deg_{T(G)}(x)} + \frac{2}{\deg_{T(G)}(x) + \deg_{T(G)}(v)} \right) \\
&\quad \text{(where } x \text{ is the vertex inserted into the edge } uv \text{ of } G)
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{ww' \in E(L(G))} \frac{2}{\deg_{T(G)}(w) + \deg_{T(G)}(w')} \\
 = & \sum_{uv \in E(G)} \frac{2}{2 \deg_G(u) + 2 \deg_G(v)} \\
 & + \sum_{uv \in E(G)} \left(\frac{2}{2 \deg_G(u) + (\deg_G(u) + \deg_G(v))} \right. \\
 & \left. + \frac{2}{(\deg_G(u) + \deg_G(v)) + 2 \deg_G(v)} \right) \\
 & + \sum_{ww' \in E(L(G))} \frac{2}{(2 + \deg_{L(G)}(w)) + (2 + \deg_{L(G)}(w'))} \\
 = & \frac{1}{2} H(G) \\
 & + \sum_{uv \in E(G)} \left(\frac{2}{2 \deg_G(u) + (\deg_G(u) + \deg_G(v))} \right. \\
 & \left. + \frac{2}{(\deg_G(u) + \deg_G(v)) + 2 \deg_G(v)} \right) \\
 & + \sum_{ww' \in E(L(G))} \frac{2}{4 + (\deg_{L(G)}(w) + \deg_{L(G)}(w'))} \\
 := & \frac{1}{2} H(G) + \sum 1 + \sum 2.
 \end{aligned}$$

By Jensen's inequality, for every $uv \in E(G)$, we have

$$(2.1) \quad \frac{2}{2 \deg_G(u) + (\deg_G(u) + \deg_G(v))} \leq \frac{1}{4} \frac{1}{\deg_G(u)} + \frac{1}{4} \frac{2}{\deg_G(u) + \deg_G(v)},$$

with equality if and only if $\deg_G(u) = \deg_G(v)$. Also,

$$(2.2) \quad \frac{2}{(\deg_G(u) + \deg_G(v)) + 2 \deg_G(v)} \leq \frac{1}{4} \frac{2}{\deg_G(u) + \deg_G(v)} + \frac{1}{4} \frac{1}{\deg_G(v)},$$

with equality if and only if $\deg_G(u) = \deg_G(v)$. Thus,

$$\begin{aligned}
 \sum 1 & \leq \frac{1}{4} \sum_{uv \in E(G)} \left(\frac{1}{\deg_G(u)} + \frac{1}{\deg_G(v)} \right) + \frac{1}{2} \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)} \\
 & = \frac{1}{4} n + \frac{1}{2} H(G).
 \end{aligned}$$

On the other hand,

$$(2.3) \quad \sum 2 \leq \frac{1}{4} H(L(G)) + \frac{1}{16} M_1(G) - \frac{1}{8} m,$$

with equality if and only if $\deg_{L(G)}(w) + \deg_{L(G)}(w') = 4$ for all $ww' \in E(L(G))$. Therefore,

$$H(T(G)) \leq H(G) + \frac{1}{4}H(L(G)) + \frac{1}{16}M_1(G) + \frac{1}{4}n - \frac{1}{8}m.$$

One can see that equality holds in above inequality if and only if the inequalities (2.1), (2.2) and (2.3) be equalities, i.e., G is a k -regular such that $\deg_{L(G)}(w) + \deg_{L(G)}(w') = 4$ (*) for all $ww' \in L(G)$. Since $L(G)$ is $(2k - 2)$ -regular, by (*), we can get $k = 2$.

This completes the proof. □

Example 2.2. For any $n \geq 4$,

$$\begin{aligned} H(T(P_n)) &= \left(\frac{2}{3} + \frac{1}{4}(n - 3)\right) + \left(\frac{48}{35} + \frac{1}{2}(n - 3)\right) + \left(\frac{4}{7} + \frac{1}{4}(n - 4)\right) \\ &= \frac{274}{105} + \frac{3}{4}(n - 3) + \frac{1}{4}(n - 4). \end{aligned}$$

3. THE HARMONIC INDEX FOR Q -SUM AND T -SUM OF GRAPHS

In the following theorem, we give an upper bound for the harmonic index of $G_1 +_Q G_2$ in terms of $H(Q(G_1))$, $H(L(G_1))$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 3.1. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned} H(G_1 +_Q G_2) &< \frac{1}{4}n_2H(Q(G_1)) + \frac{3}{16}n_2H(L(G_1)) + \frac{1}{4}n_1H(G_2) \\ &\quad + \frac{3}{64}n_2M_1(G) + \frac{1}{4}m_2r(G_1) + m_1r(G_2) - \frac{3}{32}n_2m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u, v) = \deg_{G_1 +_Q G_2}(u, v)$ be the degree of a vertex (u, v) in $G_1 +_Q G_2$. Then,

$$\begin{aligned} H(G_1 +_Q G_2) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(Q(G_1))} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

Note that

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\left(\deg_{Q(G_1)}(u) + \deg_{G_2}(v_1)\right) + \left(\deg_{Q(G_1)}(u) + \deg_{G_2}(v_2)\right)} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2 \deg_{Q(G_1)}(u) + \left(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\right)} \end{aligned}$$

$$= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))}.$$

By Jensen's inequality, for every $u \in V(G_1)$ and every $v_1 v_2 \in E(G_2)$, we have
 (3.1)

$$\frac{2}{2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))} \leq \frac{1}{4} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)},$$

with equality if and only if $2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$. Thus,

$$\begin{aligned} \sum 1 &\leq \frac{1}{4} \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)} \\ &= \frac{1}{4} \sum_{u \in V(G_1)} \left(m_2 \times \frac{1}{\deg_{G_1}(u)} \right) + \frac{1}{4} \sum_{u \in V(G_1)} H(G_2) \\ &= \frac{1}{4} m_2 r(G_1) + \frac{1}{4} n_1 H(G_2). \end{aligned}$$

Also,

$$\begin{aligned} \sum 2 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{(\deg_{Q(G_1)}(u_1) + \deg_{G_2}(v)) + \deg_{Q(G_1)}(u_2)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)}. \end{aligned}$$

For every $v \in V(G_2)$ and every $u_1 u_2 \in E(Q(G_1))$ with $u_1 \in V(G_1)$ and $u_2 \in V(Q(G_1)) - V(G_1)$, we have

$$\begin{aligned} (3.2) \quad \frac{2}{(\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)) + \deg_{G_2}(v)} &\leq \frac{1}{4} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\ &\quad + \frac{1}{2} \frac{1}{\deg_{G_2}(v)}, \end{aligned}$$

with equality if and only if $\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2) = \deg_{G_2}(v)$. Thus,

$$\sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{(\deg_{Q(G_1)}(u_1) + \deg_{G_2}(v)) + \deg_{Q(G_1)}(u_2)}$$

$$\begin{aligned}
&\leq \frac{1}{4} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&\quad + \frac{1}{2} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{1}{\deg_{G_2}(v)} \\
&= \frac{1}{4} \sum_{u_1 u'_1 \in E(G_1)} \left(\frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} + \frac{2}{\deg_{Q(G_1)}(u_2) + \deg_{Q(G_1)}(u'_1)} \right) \\
&\quad \text{(where } u_2 \text{ is the vertex inserted into the edge } u_1 u'_1 \text{ of } G_1) \\
&\quad + \frac{m_1}{\deg_{G_2}(v)} \\
&= \frac{1}{4} \left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right) + \frac{m_1}{\deg_{G_2}(v)}.
\end{aligned}$$

So,

$$\begin{aligned}
\sum 2 &\leq \frac{1}{4} \sum_{v \in V(G_2)} \left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right) \\
&\quad + \sum_{v \in V(G_2)} \frac{m_1}{\deg_{G_2}(v)} \\
&\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) \\
&\quad - \frac{1}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \\
&\quad + n_2 \sum_{u_1 u_2 \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) + \frac{3}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) \\
&\quad + \frac{3}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{4 + (\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w'))}.
\end{aligned}$$

On the other hand,

$$(3.3) \quad \sum_{ww' \in E(L(G_1))} \frac{2}{4 + (\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w'))} \leq \frac{1}{4}H(L(G_1)) + \frac{1}{16}M_1(G_1) - \frac{1}{8}m_1,$$

with equality if and only if $\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w') = 4$ for all $ww' \in E(L(G_1))$. Therefore,

$$\begin{aligned} H(G_1 +_Q G_2) &\leq \frac{1}{4}n_2H(Q(G_1)) + \frac{3}{16}n_2H(L(G_1)) + \frac{1}{4}n_1H(G_2) \\ &\quad + \frac{3}{64}n_2M_1(G) + \frac{1}{4}m_2r(G_1) + m_1r(G_2) - \frac{3}{32}n_2m_1. \end{aligned}$$

Now, suppose that there exist two graphs G_1 and G_2 such that equality holds in above inequality. Then, the inequalities (3.1), (3.2) and (3.3) must be equalities. So, G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $2k_1 = k_2 + k_2$, $k_1 + (k_1 + k_1) = k_2$ and $4 = (2k_1 - 2) + (2k_1 - 2)$, a contradiction.

This completes the proof. □

Example 3.1. For any $n \geq 4$,

$$\begin{aligned} H(P_n +_Q P_2) &= \left(1 + \frac{1}{3}(n - 2)\right) + 2\left(\frac{22}{15} + \frac{4}{7}(n - 3) + \frac{4}{7} + \frac{1}{4}(n - 4)\right) \\ &= \frac{533}{105} + \frac{1}{3}(n - 2) + \frac{8}{7}(n - 3) + \frac{1}{2}(n - 4). \end{aligned}$$

Now, we obtain an upper bound for the harmonic index of $G_1 +_T G_2$ in terms of $H(T(G_1))$, $H(L(G_1))$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 3.2. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned} H(G_1 +_T G_2) &< \frac{1}{4}n_2H(T(G_1)) + \frac{3}{16}n_2H(L(G_1)) + \frac{1}{4}n_1H(G_2) \\ &\quad + \frac{3}{64}n_2M_1(G) + \frac{1}{8}m_2r(G_1) + \frac{5}{4}m_1r(G_2) - \frac{3}{32}n_2m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u, v) = \deg_{G_1 +_T G_2}(u, v)$ be the degree of a vertex (u, v) in $G_1 +_T G_2$. By definition of harmonic index, we have

$$\begin{aligned} H(G_1 +_T G_2) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(T(G_1))} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

Note that

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\left(\deg_{T(G_1)}(u) + \deg_{G_2}(v_1)\right) + \left(\deg_{T(G_1)}(u) + \deg_{G_2}(v_2)\right)} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2 \deg_{T(G_1)}(u) + \left(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\right)} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{4 \deg_{G_1}(u) + \left(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\right)}. \end{aligned}$$

By similar argument as in the proof of Theorem 3.1, one can show that

$$(3.4) \quad \sum 1 \leq \frac{1}{8} m_2 r(G_1) + \frac{1}{4} n_1 H(G_2),$$

with equality if and only if $4 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$ for all $u \in V(G_1)$ and all $v_1 v_2 \in E(G_2)$.

On the other hand,

$$\begin{aligned} \sum 2 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(G_1)}} \frac{2}{\left(\deg_{T(G_1)}(u_1) + \deg_{G_2}(v)\right) + \left(\deg_{T(G_1)}(u_2) + \deg_{G_2}(v)\right)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\left(\deg_{T(G_1)}(u_1) + \deg_{G_2}(v)\right) + \deg_{T(G_1)}(u_2)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)} \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(G_1)}} \frac{2}{\left(\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)\right) + 2 \deg_{G_2}(v)} \\ &\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\left(\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)\right) + \deg_{G_2}(v)} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(L(G_1))} \frac{2}{\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)} \\
 = & \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{2}{(2 \deg_{G_1}(u_1) + 2 \deg_{G_1}(u_2)) + 2 \deg_{G_2}(v)} \\
 & + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{(\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)) + \deg_{G_2}(v)} \\
 & + \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(L(G_1))} \frac{2}{\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)}.
 \end{aligned}$$

One can prove that

$$\begin{aligned}
 \sum 2 \leq & \frac{1}{8} n_2 H(G_1) + \frac{1}{4} m_1 r(G_2) \\
 & + \frac{1}{4} n_2 H(T(G_1)) - \frac{1}{8} n_2 H(G_1) + m_1 r(G_2) \\
 & + \frac{3}{4} n_2 \left(\frac{1}{4} H(L(G_1)) + \frac{1}{16} M_1(G_1) - \frac{1}{8} m_1 \right).
 \end{aligned}$$

Equality holds in above inequality if and only if

$$(3.5) \quad \deg_{G_1}(u_1) + \deg_{G_1}(u_2) = \deg_{G_2}(v),$$

for all $v \in V(G_2)$ and all $u_1 u_2 \in E(G_1)$,

$$(3.6) \quad \deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2) = \deg_{G_2}(v),$$

for all $v \in V(G_2)$ and all $u_1 u_2 \in E(T(G_1))$ with $u_1 \in V(G_1)$ and $u_2 \in V(T(G_1)) - V(G_1)$, and

$$(3.7) \quad \deg_{L(G_1)}(w) + \deg_{L(G_1)}(w') = 4,$$

for all $ww' \in E(L(G_1))$. Therefore,

$$\begin{aligned}
 H(G_1 +_T G_2) \leq & \frac{1}{4} n_2 H(T(G_1)) + \frac{3}{16} n_2 H(L(G_1)) + \frac{1}{4} n_1 H(G_2) \\
 (3.8) \quad & + \frac{3}{64} n_2 M_1(G) + \frac{1}{8} m_2 r(G_1) + \frac{5}{4} m_1 r(G_2) - \frac{3}{32} n_2 m_1.
 \end{aligned}$$

Moreover, equality holds in (3.8) if and only the inequalities (3.4), (3.5), (3.6) and (3.7) be equalities, i.e., G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $4k_1 = k_2 + k_2$, $k_1 + k_1 = k_2$, $2k_1 + (k_1 + k_1) = k_2$ and $(2k_1 - 2) + (2k_1 - 2) = 4$, a contradiction. This completes the proof. \square

Example 3.2. For any $n \geq 4$,

$$\begin{aligned}
 H(P_n +_T P_2) & = \left(\frac{2}{3} + \frac{1}{5}(n - 2) \right) + 2 \left(\frac{1}{2} + \frac{1}{5}(n - 3) + \frac{7}{6} + \frac{4}{9}(n - 3) + \frac{4}{7} + \frac{1}{4}(n - 4) \right) \\
 & = \frac{36}{7} + \frac{1}{5}(n - 2) + \frac{58}{45}(n - 3) + \frac{1}{2}(n - 4).
 \end{aligned}$$

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