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INEQUALITIES FORMULATED BY A SPECIAL CLASS OF BAZILEVIČ FUNCTIONS COMBINING THE BELL SERIES

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ABSTRACT. We study a family of inequalities formed by the Fekete-Szegö design, making use of the normalized analytic functions in the open unit disk. We investigate the following functional:

$$\Psi(z) := \frac{z^{1-\vartheta}\psi'(z)}{\psi^{1-\vartheta}(z)},$$

where $\vartheta \geq 0$ acts on a domain having the starlike with respect to the boundary of the unit disk and symmetric with respect to the real axis. In addition, various presentations of the central result for functions formulated by convolution are investigated. As a special instance of this result, Fekete-Szegö issue associated with Special functions (differential operators) is studied. Moreover, by using bounds of the initial Taylor coefficients, we discussed Second Hankel determinant results.

1. INTRODUCTION

We deal with the class of normalized analytic functions denoting by Λ taking the construction series

$$\psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} \mid |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of Λ owing the univalent functions. Moreover, there is another class of analytic function in Λ taking the series

$$\varphi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots \quad (L_1 > 0),$$

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such that $\varphi(0) = 1$, $\varphi'(0) > 0$, which maps the unit disk Δ onto a starlike domain, which is symmetric with respect to the real axis. A subclass of $\mathcal{S}$ symbolized by $S^*(\varphi)$ and recognized by

$$\frac{z\psi'(z)}{\psi(z)} \prec \varphi(z) \quad (z \in \Delta).$$

Furthermore, a subclass of $\mathcal{S}$ symbolized by $C(\varphi)$ and defined by

$$1 + \frac{z\psi''(z)}{\psi'(z)} \prec \varphi(z) \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were formulated and investigated by Ma and Minda [14]. They have found the Fekete-Szegö inequality for the function in the class $C(\varphi)$. Since

$$\psi \in C(\varphi) \;\Leftrightarrow\; z\psi'(z) \in S^*(\varphi),$$

we get the Fekete-Szegö inequality for functions in the class $S^*(\varphi)$. A brief explanation of the Fekete-Szegö problem for the class of starlike, convex, and close-to-convex functions can be found in the most recent publication by Srivastava et al. [22]. Motivated by the classes defined above, we consider a function associated with the Bell numbers.

The Bell numbers (BNs) $\flat_n$ having the following binomial coefficients $\flat_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \flat_k$. Clearly, [3–6, 25–27]

$$\flat_0 = \flat_1 = 1, \quad \flat_2 = 2, \quad \flat_3 = 5, \quad \flat_4 = 15, \quad \flat_5 = 52 \quad \text{and} \quad \flat_6 = 203.$$

Cho et al. [9] and Kumar et al. [12] considered the function

$$\phi(z) := e^{e^z} - 1 = \sum_{n=0}^{+\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \quad (z \in \Delta), \tag{1.2}$$

which is starlike and its coefficients generate the BNs and established the first order differential subordination relations between functions with a positive real part and starlike functions related to the Bell numbers. We now consider the function $\phi(z) := e^{e^z} - 1$ with its domain of definition as the open unit disk Δ. We shall see that the function ϕ_1, defined by

$$\phi_1(z) = z \exp\left(\int_0^z \frac{\phi(t) - 1}{t} \mathrm{d}t \right) = z + z^2 + z^3 + \frac{17}{18}z^4 - \frac{245}{288}z^5 + \cdots ,$$

would serve as an extreme function in many problems.

The Fekete-Szegö inequality is obtained in this study for functions in a more extended class $\mathfrak{B}^{\vartheta}(\phi)$ of Bazilevič functions, which we describe below. Furthermore, we provide our findings with implementations to specific functions specified by the convolution class.

Definition 1.1. Let $\phi(z)$ be a starlike function given by (1.2). A function $\psi \in \Lambda$ belongs to the class $\mathfrak{B}^{\vartheta}(\phi)$ if

$$\Psi(z) = \frac{z^{1-\vartheta}\psi'(z)}{[\psi(z)]^{1-\vartheta}} \prec \phi(z) \quad (\vartheta \geq 0).$$

By fixing $\vartheta = 0$ and $\vartheta = 1$ we state the following.

Example 1.1. Let $\phi(z)$ be given by (1.2) and starlike function. A function $\psi \in \Lambda$ belongs to the class $\mathfrak{S}(\phi)$ if

$$\frac{z\psi'(z)}{\psi(z)} \prec \phi(z).$$

Example 1.2. Let $\phi(z)$ be a starlike function given by (1.2). A function $\psi \in \Lambda$ belongs to the class $\mathfrak{Q}(\phi)$ if

$$\psi'(z) \prec \phi(z) \quad (\vartheta \geq 0).$$

We request the next result.

Lemma 1.1 ([13])**.** *If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *with* $\mathrm{Re}\,(p(z)) > 0$, *then the following sharp estimate holds*

$$(1.3) \qquad \begin{aligned} |c_n| &\leq 2 \quad (n = 1, 2, 3, \ldots), \\ |c_2 - v c_1^2| &\leq 2 \max\{1, |2v - 1|\}, \end{aligned}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 1.2 ([14])**.** *Suppose that* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part in* Δ. *Then,*

- *for* $v < 0$ *or* $v > 1$, *the equality*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v - 2, & \text{if } v \geq 1, \end{cases}$$

  holds if and only if $p_1(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations;*
- *for* $0 < v < 1$, *then equality holds if and only if*

$$p_1(z) = \frac{1+z^2}{1-z^2}$$

  or one of its rotations;
- *for* $v = 0$, *the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

  or one of its rotations;

- *for $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.*

Furthermore, the top bound above is sharp; it may be made better when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 0.5)$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad (0.5 < v \leq 1).$$

2. Coefficient Bounds and Fekete-Szegö Problem

Our main result is the following.

Theorem 2.1. *Let $\phi(z)$ be given by (1.2). If $\psi(z)$ given by (1.1) belongs to $\mathfrak{B}^{\vartheta}(\phi)$, then*

$$\begin{aligned} |a_2| \leq & \frac{1}{\vartheta + 1}, \\ |a_3| \leq & \frac{1}{\vartheta + 2} \max\left\{1, \left|\frac{\vartheta^2 + 3\vartheta + 4}{2(\vartheta + 1)^2}\right|\right\}. \end{aligned}$$

Further,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2+\vartheta} - \frac{\mu}{(1+\vartheta)^2} + \frac{1-\vartheta}{2(1+\vartheta)^2}, & \text{if } \mu \leq \sigma_1, \\ \frac{1}{2(2+\vartheta)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{2+\vartheta} + \frac{\mu}{(1+\vartheta)^2} - \frac{1-\vartheta}{2(1+\vartheta)^2}, & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 :=& \frac{(1 - \vartheta)(2 + \vartheta)}{2(2 + \vartheta)}, \\ \sigma_2 :=& \frac{2(1 + \vartheta)^2 + (1 - \vartheta)(2 + \vartheta)}{2(2 + \vartheta)}. \end{aligned}$$

The result is sharp.

Proof. Since $\psi \in \mathfrak{B}^{\vartheta}(\phi)$ there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\frac{z^{1-\vartheta}\psi'(z)}{(\psi(z))^{1-\vartheta}} = \phi(w(z)). \tag{2.1}$$

Define the function p_1 by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

or, equivalently

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \cdots\right]. \tag{2.2}$$

Given $p_1(0) = 1$ and $\operatorname{Re}(p_1(z)) > 0$, p_1 is analytic in Δ. Obviously, we have

$$\phi(w(z)) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{c_1}{2}z + \frac{c_2}{2}z^2 + \frac{1}{2}\left(c_3 - \frac{11}{24}c_1^3\right)z^3 + \cdots . \tag{2.3}$$

Since

$$\begin{aligned}\frac{z^{1-\vartheta}\psi'(z)}{(\psi(z))^{1-\vartheta}} =&1 + a_2(\vartheta+1)z + \left(\frac{(\vartheta-1)(\vartheta+2)}{2}a_2^2 + (\vartheta+2)a_3\right)z^2 \\ &+ \left\{a_4(\vartheta+3) + (\vartheta+1)(\vartheta-3)a_2a_3 + \frac{(\vartheta-1)(\vartheta-2)(\vartheta+3)}{6}a_2^3\right\}z^3 + \cdots . \qquad (2.4)\end{aligned}$$

Equating the coefficients of z, z^2 in (2.3) and (2.4), we get

$$a_2 =\frac{c_1}{2(\vartheta+1)}, \tag{2.5}$$

$$\begin{aligned}a_3 =&\frac{1}{8(\vartheta+2)}\left[4c_2 - \frac{(\vartheta-1)(\vartheta+2)}{(\vartheta+1)^2}c_1^2\right] \qquad (2.6)\\ =&\frac{1}{2(\vartheta+2)}\left[c_2 - c_1^2\frac{(\vartheta-1)(\vartheta+2)}{4(\vartheta+1)^2}\right].\end{aligned}$$

In view of Lemma 1.1, we have

$$|a_2| \le \frac{1}{\vartheta+1}$$

and

$$|a_3| \le \frac{1}{\vartheta+2}\max\left\{1, \left|\frac{\vartheta^2+3\vartheta+4}{2(\vartheta+1)^2}\right|\right\}.$$

Further, we have

$$a_3 - \mu a_2^2 = \frac{1}{2(2+\vartheta)}\left\{c_2 - vc_1^2\right\}, \tag{2.7}$$

where

$$v := \frac{(2+\vartheta)(2\mu+\vartheta-1)}{4(1+\vartheta)^2}.$$

Our result now follows by Lemma 1.2. The upper bounds are sharp in terms of the following conclusion. Formulate the functional $K_{\phi n}^{\vartheta}$, $n = 2, 3, \ldots$, by

$$\frac{z^{1-\vartheta}[K_{\phi n}^{\vartheta}]'(z)}{\left[K_{\phi n}^{\vartheta}(z)\right]^{1-\vartheta}} = \phi(z^{n-1}), \quad K_{\phi n}^{\vartheta}(0) = 0 = [K_{\phi n}^{\vartheta}]'(0) - 1,$$

and the function F_λ^ϑ and G_λ^ϑ, $0 \le \lambda \le 1$, by

$$\frac{z^{1-\vartheta}[F_\lambda^\vartheta]'(z)}{\left[F_\lambda^\vartheta(z)\right]^{1-\vartheta}} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F_\lambda(0) = 0 = F_\lambda'(0) - 1$$

and

$$\frac{z^{1-\vartheta}[G_\lambda^\vartheta]'(z)}{\left[G_\lambda^\vartheta(z)\right]^{1-\vartheta}} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly, the functions $K_{\phi n}^\vartheta, F_\lambda^\vartheta, G_\lambda^\vartheta \in \mathfrak{B}^\vartheta(\phi)$. Also, we write $K_\phi^\vartheta := K_{\phi 2}^\vartheta$. We have the following inequalities

- $\mu < \sigma_1$ or $\mu > \sigma_2$ if and only if $\psi \in K_\phi^\vartheta$ or its rotations;
- $\sigma_1 < \mu < \sigma_2$ if and only if $\psi \in K_{\phi 3}^\vartheta$ or its rotations;
- $\mu = \sigma_1$ if and only if $\psi \in F_\lambda^\vartheta$ or its rotations;
- $\mu = \sigma_2$ if and only if $\psi \in G_\lambda^\vartheta$ or its rotations.

□

Remark 2.1. • For $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 1.2, Theorem 2.1 can be modified.

• For σ_3, we have

$$\sigma_3 := \frac{(1+\vartheta)^2 + (1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}.$$

• For $\sigma_1 \leq \mu \leq \sigma_3$, we get

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2}{(2+\vartheta)}\left[\frac{(2\mu+\vartheta-1)(2+\vartheta)}{2(1+\vartheta)}\right]|a_2|^2 \leq \frac{1}{2+\vartheta}.$$

• If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2}{(2+\vartheta)}\left[2 - \frac{(2\mu+\vartheta-1)(2+\vartheta)}{2(1+\vartheta)}\right]|a_2|^2 \leq \frac{1}{2+\vartheta}.$$

Theorem 2.2. *Assume that* $\phi(z)$ *is defined by* (1.2). *If* $\psi(z)$ *as in* (1.1) *belongs to* $\mathfrak{B}^\vartheta(\phi)$, *then for a complex number* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{(2+\vartheta)\mu}{(1+\vartheta)^2} - \frac{\vartheta^2+3\vartheta+4}{2(1+\vartheta)^2}\right|\right\}. \tag{2.8}$$

In particular,

$$|a_3 - a_2^2| = \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{\vartheta}{2(1+\vartheta)}\right|\right\}.$$

Proof. Using (2.5), (2.6) and (2.7) we have

$$a_3 - \mu a_2^2 = \frac{1}{2(2+\vartheta)}\left\{c_2 - v c_1^2\right\},$$

where

$$v := \left[\frac{(2+\vartheta)(2\mu+\vartheta-1)}{4(1+\vartheta)^2}\right].$$

In view of Lemma 1.1, we have the desired assertion. □

3. Application to Functions associated with Special functions

Let $\psi(z)=z+\sum_{n=2}^{+\infty}\psi_n z^n$, $\psi_n>0$. Since $f(z)=z+\sum_{n=2}^{+\infty}a_n z^n\in B_g^{\vartheta}(\phi)$ if and only if $(f*\psi)=z+\sum_{n=2}^{+\infty}\psi_n a_n z^n\in\mathfrak{B}_{\psi}^{\vartheta}(\phi)$, we obtain the coefficient estimate for functions in the class $\mathfrak{B}_{\psi}^{\vartheta}(\phi)$ from the corresponding estimate for functions in the class $\mathfrak{B}^{\vartheta}(\phi)$. Applying Theorem 2.1 to the function $(f*\psi)(z)=z+\psi_2a_2z^2+\psi_3a_3z^3+\cdots$, we get the following results, Theorem 3.1 after an obvious change of the parameter μ. For various choices of $\psi(z)$ we get different operators, which are listed below.

(a) For $\psi(z)=z+\sum_{n=2}^{+\infty}\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1},\dots,(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1},\dots,(\beta_s)_{n-1}(1)_{n-1}}z^n$, we get the Dziok–Srivastava operator $H_{q,s}(\alpha)f(z)$ introduced by Dziok and Srivastava [11].

(b) For $\psi(z)=\phi(a,c,z)=\sum_{n=0}^{+\infty}\frac{(a)_n}{(c)_n}z^n$, we get the Carlson-Shaffer operator $L(a,c)f(z)$ introduced by Carlson-Shaffer [7].

(c) For $\varphi(z)=\frac{z}{(1-z)^{\lambda+1}}$, we get the Ruscheweyh operator $D^{\lambda}f(z)$ introduced by Ruschweyh [20].

(d) For $\psi(z)=z+\sum_{n=2}^{+\infty}n^m z^n$, $m\geq 0$, we get the Sălăgean operator $D^m f(z)$ introduced by Sălăgean [21].

(e) For $\psi(z)=z+\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, $\lambda\geq 0$, $k\in\mathbb{Z}$, we get the multiplier transformation $I(\lambda,k)$ introduced by Cho and Srivastava [10].

(f) For $\varphi(z)=z+\sum_{n=2}^{+\infty}n\left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, $\lambda\geq 0$, $k\in\mathbb{Z}$, the multiplier transformation $J(\lambda,k)$ introduced by Cho and Kim [8].

(g) For $\psi(z)=z+\sum_{n=2}^{+\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^n$, where $z,\alpha,\beta\in\mathbb{C}$, $\beta\neq 0,-1,-2,\dots$ and $\mathrm{Re}\,(\beta)>0$, $\mathrm{Re}\,(\alpha)>0$, the Mittag-Leffler function denoted by $E_{\alpha,\beta}(\zeta)$ (see [2,17]).

(h) For $\lambda\neq 2,3,4,\dots$, let $\psi(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n$, we get fractional derivatives and fractional integrals operator $(\Omega^{\lambda}f)(z)$ (also see [23,24].

(i) A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0,1,2,3,\dots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, $\dots$, respectively, where m is called the parameter. Thus, $P(\mathcal{X}=r)=\frac{m^r e^{-m}}{r!}$, $r=0,1,2,3,\dots$. In [18], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z)=z+\sum_{n=2}^{+\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n\quad(z\in\Delta),$$

where $m>0$. By ratio test the radius of convergence of above series is infinity. Due to Porwal [18] (see [15,16,19]) we have

$$f*\mathcal{K}_m(z)=z+\sum_{n=2}^{+\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n\quad(z\in\Delta).$$

(j) The symmetric differential operator [28]

$$(\eth\psi)_{\alpha}^{m}(z) = z + \sum_{k=2}^{+\infty}[k(\alpha + (1-\alpha)(-1)^k)]^m \psi_k z^k.$$

(k) The conformable differential operator [29]

$$\begin{aligned}(\Omega\psi)^{\wp}(z) =& \frac{\kappa_1(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}\psi(z) + \frac{\kappa_0(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}(z\psi'(z)) \\ =& z + \sum_{k=2}^{+\infty}\left(\frac{\kappa_1(\wp, z) + k\kappa_0(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}\right)\psi_k z^k.\end{aligned}$$

(l) The hybrid fractional integro-differential operator is given by [30]

$$(\Upsilon\psi)^{\wp}(z) = z + \sum_{k=2}^{+\infty}\mathcal{K}_k(\wp)\left(\frac{\Gamma(3-\wp)\Gamma(k+1)}{\Gamma(k+2-\wp)}\right)\psi_k z^k,$$

where $\mathcal{K}_k(\wp) := \frac{\kappa_1(\wp)+k\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}$.

Theorem 3.1. *Assume that $\phi(z)$ is defined by* (1.2)*. If $f(z)$ as in* (1.1) *belongs to $\mathfrak{B}_{\psi}^{\vartheta}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{\psi_3}\left[\frac{1}{2+\vartheta} - \frac{\mu\psi_3}{(1+\vartheta)^2\psi_2^2} + \frac{(1-\vartheta)}{2(1+\vartheta)^2}\right], & \text{if } \mu \leq \sigma_1, \\ \frac{1}{\psi_3}\cdot\frac{1}{2(2+\vartheta)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{\psi_3}\left[-\frac{1}{2+\vartheta} + \frac{\mu\psi_3}{(1+\vartheta)^2\psi_2^2} - \frac{(1-\vartheta)}{2(1+\vartheta)^2}\right], & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned}\sigma_1 :=& \frac{\psi_2^2}{\psi_3}\cdot\frac{(1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}, \\ \sigma_2 :=& \frac{\psi_2^2}{\psi_3}\cdot\frac{2(1+\vartheta)^2 + (1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}.\end{aligned}$$

The result is sharp.

Theorem 3.2. *Let $\phi(z)$ be given by* (1.2)*. If $f(z)$ given by* (1.1) *belongs to $\mathfrak{B}_{\psi}^{\vartheta}(\phi)$, then for complex μ we have*

$$|a_3 - \mu a_2^2| = \frac{1}{(2+\vartheta)\psi_3}\max\left\{1, \left|\frac{(2+\vartheta)\psi_3\mu}{(1+\vartheta)^2\psi_2^2} - \frac{\vartheta^2 + 3\vartheta + 4}{2(1+\vartheta)^2}\right|\right\}. \tag{3.1}$$

4. Second Hankel Determinant of Analytic Functions

Here, the Second Hankel determinant of analytic functions $\psi \in \mathfrak{B}^{\vartheta}(\phi)$.

Lemma 4.1 ([13])**.** *Let $p \in \mathcal{P}$ with $c_1 \geq 0$. Then it satisfies the series*

$$\begin{aligned}2c_2 =& c_1^2 + x(4 - c^2), \\ 4c_3 =& c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2 z),\end{aligned} \tag{4.1}$$

for some x, z, with $|x| \leq 1$ and $|z| \leq 1$.

Theorem 4.1. *Let the function $f \in \mathfrak{B}^{\vartheta}(\phi)$ be given by (1.1).*

(1) *If $\vartheta \leq 0$,*

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| + \frac{4\vartheta-36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)} \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{4(\vartheta+1)^2}.$$

(2) *If $\vartheta \geq 0$,*

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| - \left(\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right) \geq 0$$

or the conditions $\vartheta \leq 0$,

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| + \frac{4\vartheta-36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)} \geq 0,$$

then the second Hankel determinant satisfies

$$\begin{aligned}&\left|a_2a_4-a_3^2\right| \\ \leq& \frac{1}{12}\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| \\ &+ \frac{96\vartheta(\vartheta+4)-32\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}.\end{aligned}$$

(3) *If $\vartheta > 0$,*

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| - \left(\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right) \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{12}\cdot\frac{3\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right|+\frac{88\vartheta-108}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}}{\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right|-\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}}.$$

Proof. Since $f \in \mathfrak{B}^{\vartheta}(\phi)$ and equating the coefficients of z, z^2, z^3 in (2.3) and (2.4) we get

$$a_2 = \frac{c_1}{2(\vartheta+1)}, \quad a_3 = \frac{1}{2(\vartheta+2)}\left[c_2 - \frac{c_1^2}{4}\left\{\frac{(\vartheta-1)(\vartheta+2)}{(\vartheta+1)^2}\right\}\right],$$

and further by using the above, we get

$$\begin{aligned}a_4 =& \frac{1}{48(\vartheta+3)}\left[\left\{-1+\frac{(\vartheta-1)}{(\vartheta+1)^3}(-(\vartheta-2)(\vartheta+3)+3(\vartheta+1)(\vartheta-3))\right\}c_1^3\right.\\&\left.-\left(\frac{12(\vartheta-3)}{(\vartheta+2)}\right)c_1c_2+24c_3\right]\\=&\frac{1}{48(\vartheta+3)}\left[\left\{\frac{(\vartheta-1)(2\vartheta^2-7\vartheta-3)}{(\vartheta+1)^3}-1\right\}c_1^3-\left(\frac{12(\vartheta-3)}{(\vartheta+2)}\right)c_1c_2+24c_3\right].\end{aligned}$$

Therefore,

$$\begin{aligned}a_2a_4-a_3^2 =&\frac{1}{96}\left[c_1^4\left\{\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{6}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right.\right.\\&\left.+\frac{5}{(\vartheta+1)(\vartheta+3)}-\frac{6}{(\vartheta+2)^2}\right\}+\left(\frac{24\vartheta}{(\vartheta+1)^2(\vartheta+2)(\vartheta+3)}\right)2c_2c_1^2\\&\left.+\frac{24c_1c_3}{(\vartheta+1)(\vartheta+3)}-\frac{24c_2^2}{(\vartheta+2)^2}\right].\end{aligned}$$

Let

$$\begin{aligned}d_1 =&\frac{24}{(\vartheta+1)(\vartheta+3)},\\d_2 =&\frac{48\vartheta(\vartheta+2)}{(\vartheta+1)^2(\vartheta+3)(\vartheta+2)^2},\\d_3 =&\frac{-24}{(\vartheta+2)^2},\quad T=\frac{1}{96},\\d_4 =&\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{6}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}+\frac{5}{(\vartheta+1)(\vartheta+3)}-\frac{6}{(\vartheta+2)^2}\\=&\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{1}{(\vartheta+1)(\vartheta+3)}.\end{aligned}\tag{4.2}$$

Then,

$$|a_2a_4-a_3^2|=T|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|,\tag{4.3}$$

since the function $p(e^{i\theta})$, $\theta\in R$, belongs to the class P for any $p\in P$, there is no loss of generality in assuming $c_1\geq 0$. Write $c_1=c$, $c\in[0,2]$. Substituting the values of c_2 and c_3, respectively, from (1.4) and (1.5) in (2.2), we obtain

$$\begin{aligned}|a_2a_4-a_3^2| =&\frac{T}{4}\left|c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)\right.\\&\left.+(4-c^2)x^2(-d_1c^2+d_3(4-c^2))+2d_1c(4-c^2)(1-|x|^2)z\right|.\end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.6) yield

$$\begin{aligned}|a_2a_4-a_3^2|\leq&\frac{T}{4}\left[c^4\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}\right.\right.\\&\left.+\frac{48\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)^2}-\frac{24}{(\vartheta+2)^2}\right|+\frac{48c(4-c^2)(1-\mu^2)}{(\vartheta+1)(\vartheta+3)}\end{aligned}$$

$$
\begin{aligned}
&+4\mu c^2(4-c^2)\left(\frac{12+24\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
&-\mu^2(4-c^2)\left(\frac{24c^2+96(\vartheta^2+4\vartheta+3)}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}\right)\Bigg] \qquad (4.4)\\
&= T\left[\frac{c^4}{4}\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}\right.\right. \\
&\left.+\frac{96\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)^2}-\frac{24}{(\vartheta+2)^2}\right|+\frac{12c(4-c^2)}{(\vartheta+1)(\vartheta+3)} \\
&+\mu c^2(4-c^2)\left(\frac{12+2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
&\left.+\frac{6\mu^2(4-c^2)}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}[(c-2)(-2\vartheta(\vartheta+4)+(c-6))]\right] \\
&\equiv F(c,\mu,\vartheta). \qquad (4.5)
\end{aligned}
$$

Note that for $(c,\mu,\vartheta)\in[0,2]\times[0,1]$, differentiating $F(c,\mu,\vartheta)$ in (2.8) partially with respect to μ yields

$$
\frac{\partial F}{\partial\mu}=T\left[\frac{c^2(4-c^2)}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}(12-2\vartheta)+\frac{(c-2)(-2\vartheta(\vartheta+4)+(c-6))}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}[12\mu(4-c^2)]\right], \tag{4.6}
$$

then for $0<\mu<1$ and for any fixed c with $0<c<2$, it is clear from (2.9) that $\frac{\partial F}{\partial\mu}>0$, that is, $F(c,\mu,\vartheta)$ is an increasing function of μ. Therefore, for some special fixed parameters, we get $\max F(c,\mu,\vartheta)=F(c,1,\vartheta)\equiv G(c)$. In addition, we have

$$
\begin{aligned}
G(c)=\frac{1}{96}&\left[\frac{c^4}{4}\left\{\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}+\frac{88\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right.\right.\right. \\
&\left.\left.-\frac{24}{(\vartheta+2)^2}\right|-\left(\frac{72}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right)\right\}+4c^2\left(\frac{2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
&\left.+\frac{24(12+4\vartheta(\vartheta+4))}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right].
\end{aligned}
$$

Moreover, by the conclusion

$$
\begin{aligned}
P&=\frac{1}{4}\left\{\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}+\frac{88\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right.\right. \\
&\left.\left.-\frac{24}{(\vartheta+2)^2}\right|-\left(\frac{72}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right)\right\}, \\
Q&=4\left(\frac{2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right), \\
R&=\frac{24(12+4\vartheta(\vartheta+4))}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}. \qquad (4.7)
\end{aligned}
$$

We have,

$$|a_2a_4 - a_3^2| \leq \frac{1}{96}\begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \leq 0, P \geq -\frac{Q}{4}, \\ \frac{4PR-Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8}, \end{cases}$$

where P, Q and R are given by (4.7). □

5. Conclusion

In this investigation, we define a new subclass $\mathfrak{B}^{\vartheta}(\phi)$ of normalized analytic functions in the open unit disk Δ, which is associated with Bell Numbers. We then successfully investigate several properties and characteristics, such as estimates for the first few Taylor-Maclaurin coefficients, the Fekete-Szegö problem, and the second-order Hankel determinant $H_2(2)$. Finally, we indicate a number of known operators (or special functions) listed in Section 3, that are already available in the literature on the subject and their application. The appropriate approximation for functions in the class $\mathfrak{B}^{\vartheta}(\phi)$ is used to estimate the coefficients for functions in $\mathfrak{B}^{\vartheta}_{\psi}(\phi)$.

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