Kragujevac Journal of Mathematics
Volume 45(4) (2021), Pages 543–570.

EXISTENCE OF SOLUTIONS FOR A CLASS OF CAPUTO FRACTIONAL q-DIFFERENCE INCLUSION ON MULTIFUNCTIONS BY COMPUTATIONAL RESULTS

MOHAMMAD ESMAEL SAMEI[1], GHORBAN KHALILZADEH RANJBAR[1], AND VAHID HEDAYATI[2]

Abstract. In this paper, we study a class of fractional q-differential inclusion of order $0 < q < 1$ under L^1-Caratheodory with convex-compact valued properties on multifunctions. By the use of existence of fixed point for closed valued contractive multifunction on a complete metric space which has been proved by Covitz and Nadler, we provide the existence of solutions for the inclusion problem via some conditions. Also, we give a couple of examples to elaborate our results and to present the obtained results by some numerical computations.

1. Introduction

Fractional calculus is an important branch in mathematical analysis. However, after Leibniz and Newton invented differential calculus, it has numerous applications in different sciences such as mechanics, electricity, biology, control theory, signal and image processing (for example, see [4, 6, 40]). In recent years the fractional differential equations and the fractional differential inclusions were developed intensively (for more information, see [8, 10, 19, 22, 38]). Also, it has been appeared many work on fractional differential inclusions [11, 14–16, 23, 25, 27, 28]

In 1910, the subject of q-difference equations introduce by Jackson [33]. Later, at the beginning of the last century, studies on q-difference equation, appeared in so many works especially in Carmichael [26], Mason [39], Adams [3], Trjitzinsky [45]. It has been proven that these cases of equations have numerous applications in

Key words and phrases. Existence of solution, fractional q-difference inclusion, integral boundary value problem.

2010 *Mathematics Subject Classification*: Primary: 26A33. Secondary: 34A08, 34K37.

Received: July 05, 2018.

Accepted: March 18, 2019.

diverse domains and thus have evolved into multidisciplinary subjects (for example, see [1, 2, 7, 18, 30, 32, 47] and references therein).

In this paper, motivated by [9,44] and among these achievements, we wish to discuss the existence of solutions for a problem of fractional q-derivative inclusions via the integral boundary value conditions given by

$$\begin{cases} {}^cD_q^\alpha x(t) \in F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right), \\ x(0) + x'(0) + {}^cD_q^\beta x(0) = \int_0^\eta x(s)\, ds, \\ x(1) + x'(1) + {}^cD_q^\beta x(1) = \int_0^\nu x(s)\, ds, \end{cases} \tag{1.1}$$

for real number t in $[0,1]$, where F maps $[0,1] \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact valued multifunction, ${}^cD_q^\alpha$ is the fractional Caputo type q-derivative operator of order $\alpha \in (1,2]$ with q belongs to $(0,1)$, and

$$\Gamma_q(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0,$$

for $\eta, \nu, \beta \in (0,1)$, such that $\alpha - \beta > 1$.

In 2012, Ahmad, Ntouyas and Purnaras investigated the q-difference equation:

$$\begin{cases} \left({}^cD_q^\alpha y\right)(x) = f(x, y(x)), \\ \alpha_1 y(0) - \beta_1 D_q y(0) = \gamma_1 y(e_1), \quad \alpha_2 y(1) + \beta_2 D_q y(1) = \gamma_2 y(e_2), \end{cases}$$

where $0 \leq x \leq 1$, $1 < \alpha \leq 2$ and $\alpha_i, \beta_i, \gamma_i, e_i \in \mathbb{R}$ for all i (see [17]). In 2013, Zhao, Chen and Zhang reviewed the nonlinear fractional q-difference equation:

$$\begin{cases} (D_q^\alpha y)(x) = f(x, y(x)), \\ y(0) = 0, \quad y(1) = \mu I_q^\beta y(e), \end{cases}$$

where $0 < x < 1$, $1 < \alpha \leq 2$, $0 < \beta \leq 2$ and $\mu > 0$ [46]. In 2015, Etemad, Ettefagh, and Rezapour investigated the q-differential equation:

$$\begin{cases} \left({}^cD_q^\alpha y\right)(x) = f(x, y(x), D_q y(x)), \\ \lambda_1 y(0) + \mu_1 D_q y(0) = e_1 I_q^\beta y(x_1), \quad \lambda_2 y(1) + \mu_2 D_q y(1) = e_2 I_q^\beta y(x_2), \end{cases}$$

where $0 \leq x \leq 1$, $1 < \alpha \leq 2$, $q \in (0,1)$, $\beta \in (0,2]$, $x_1, x_2 \in (0,1)$, with $x_1 < x_2$, $\lambda_i, \mu_i, e_i, \in \mathbb{R}$ for $i = 1,2$, and real value map f from $[0,1] \times \mathbb{R}^2$ is continuous [13]. Also, in the same year, Agarwal, Baleanu, Hedayati, and Rezapour founded results for the inclusion Caputo fractional differential:

$$\begin{cases} {}^cD^\alpha f(t) \in T\left(t, f(t), {}^cD^\beta f(t)\right), \\ f(0) = 0, \quad f(1) + f'(1) = \int_0^e f(s) ds, \end{cases}$$

such that $0 < e < 1$, $1 < \alpha \leq 2$, $0 < \beta < 1$, with $\alpha - \beta > 1$, and multifunction T define on $[0,1] \times \mathbb{R}^2$ has a compact valued in $2^{\mathbb{R}}$ [9]. Also, they investigate the existence of solutions for the Caputo fractional differential inclusion ${}^cD^\alpha x(t) \in F(t, x(t))$ such that $x(0) = a \int_0^\nu x(s) ds$ and $x(1) = b \int_0^\eta x(s) ds$, where $0 < \nu, \eta < 1$, $1 < \alpha \leq 2$ and

$a, b \in \mathbb{R}$ [9]. In 2016, Abdeljawad, Alzabut, and Baleanu stated and proved a new discrete q-fractional version of Gronwall inequality:

$$\begin{cases} {}_qC_a^\alpha f(t) = T\left(t, f(t)\right), \\ f(a) = \gamma, \end{cases}$$

such that $\alpha \in (0,1]$, $a \in \mathbb{T}_q = \{q^n \mid n \in \mathbb{Z}\}$, t belongs to $\mathbb{T}_a = [0,\infty)_q = \{q^{-i}a \mid i = 0,1,2,\dots\}$, ${}_qC_a^\alpha$ means the Caputo fractional difference of order α, and $T(t,x)$ fulfills a Lipschitz condition for all t and x [2]. Later, in 2017, Zhou, Alzabut, and Yang provide existence criteria for the solutions of p-Laplacian fractional Langevin differential equations with ansi-periodic boundary conditions:

$$\begin{cases} D_{0+}^\beta \phi_p[(D_{0+}^\alpha + \lambda)x(t)] = f(t, x(t), D_{0+}^\alpha x(t)), \\ x(0) = -x(1), \quad D_{0+}^\alpha x(0) = -D_{0+}^\alpha x(1), \end{cases}$$

and

$$\begin{cases} {}_qD_{0+}^\beta \phi_p[(D_{0+}^\alpha + \lambda)x(t)] = g(t, x(t), {}_qD_{0+}^\alpha x(t)), \\ x(0) = -x(1), \quad {}_qD_{0+}^\alpha x(0) = -{}_qD_{0+}^\alpha x(1), \end{cases}$$

for all $0 \le t \le 1$, where $0 < \alpha, \beta \le 1$, λ is more than or equal to zero, $1 < \alpha + \beta < 2$, $q \in (0,1)$ and $\phi_p(s) = |s|^{p-2}s$, with $p \in (1,2]$ [47]. In this manuscript, by using idea of the works, we study the existence of solutions for the fractional q-derivative inclusions via the integral and q-derivative boundary value conditions.

2. Preliminaries

Here, we recall some discovered facts on fractional q-calculus and their derivatives and integral. For more details on this, we refer the reader to the references [20, 34].

Let $q \in (0,1)$, $a \in \mathbb{R}$, and $\alpha \neq 0$ be a real number. Define $[a]_q = \frac{1-q^a}{1-q}$ (see [33]). The q-analogue of the power function $(a-b)^n$, with $n \in \mathbb{N}_0$, is $(a-b)_q^{(n)} = \prod_{k=0}^{n-1}(a - bq^k)$ and $(a-b)_q^{(0)} = 1$, where a and b in $\mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ (see [43]). Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}.$$

If $b = 0$, then it is clear that $a^{(\alpha)} = a^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$, where x belongs to $\mathbb{R}\backslash\{0,-1,-2,\dots\}$ (see [33]). Note that, $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$. A simplified analysis can be performed to estimate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for calculated q-Gamma function of order n which show in Algorithm 2. For function f, the q-derivative is defined by $(D_qf)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_qf)(0) = \lim_{x\to 0}(D_qf)(x)$ (see [3]). Also, the higher order q-derivative of a function f is defined by $(D_q^nf)(x) = D_q(D_q^{n-1}f)(x)$ for all $n \ge 1$, where $(D_q^0f)(x) = f(x)$ (see [3]). The

q-integral of a function f define on $[0, b]$ by

$$I_q f(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k),$$

for $x \in [0, b]$, provided that the sum converges absolutly [3]. If $a \in [0, b]$, then

$$\int_a^b f(u) d_q u = I_q f(b) - I_q f(a) = (1-q) \sum_{k=0}^{\infty} q^k \left[b f(bq^k) - a f(aq^k) \right],$$

whenever the series exists. The operator I_q^n is given by $I_q^0 f(x) = f(x)$ and $I_q^n f(x) = I_q(I_q^{n-1} f)(x)$ for all $n \geq 1$ (see [3]). It has been proved that $(D_q I_q f)(x) = f(x)$ and $(I_q D_q f)(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ (see [3]). The fractional Riemann-Liouville type q-integral of the function f on $[0, 1]$ is given by

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha - 1)} f(s) d_q s,$$

whenever $\alpha > 0$ and $I_q^0 f(x) = f(x)$ whenever $\alpha = 0$, where $x \leq 1$ is a real number [13]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned} \left({}^c D_q^\alpha f \right)(x) &= \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f \right)(x) \\ &= \frac{1}{\Gamma_q([\alpha] - \alpha)} \int_0^x (x - qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f \right)(s) d_q s, \end{aligned}$$

for $x \in [0, 1]$ and $\alpha > 0$ (see [13]). It has been proved that $\left(I_q^\beta I_q^\alpha f \right)(x) = \left(I_q^{\alpha+\beta} f \right)(x)$, and $\left(D_q^\alpha I_q^\alpha f \right)(x) = f(x)$, where $\alpha, \beta \geq 0$ (see [29]). By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 3.

It is well recognized that the Pompeiu-Hausdorff metric H_d maps $2^X \times 2^X$ into $\mathbb{R}^{\geq 0}$ on metric space (X, d) is defined by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ (also, see [12, 31]). Denote the set of bounded and closed subsets of X, the set of closed subsets of X and the set of compact and convex subsets of X by $CB(X)$, $C(X)$ and $P_{cp,cv}(X)$, respectively. Thus, $(CB(X), H_d)$ and $(C(X), H_d)$ are a metric space and a generalized metric space, respectively (for more details, see [35]). An element x belongs to X is called an fixed point of multifunction T maps X into 2^X whenever x in $T(x)$ (for more information, see [31]). If $\gamma \in (0, 1)$ exists somehow that $H_d(N(x), N(y))$ is less than or equal to $\gamma d(x, y)$ for all x and y in X, then a multifunction T maps X to $C(X)$ is called a contraction.

In 1970, Covitz and Nadler prove that there is a fixed point for each closed valued contractive multifunction on a complete metric space has a fixed point [27]. Let $J = [0, 1]$. A multifunction $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t))$ is measurable for all y belongs to $\mathbb{R}$ [28]. We say that F maps $J \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $t \mapsto F(t, x, y, z)$ is measurable for all x, y, and z in $\mathbb{R}$ and $(x, y, z) \mapsto F(t, x, y, z)$ is upper semi-continuous for all t belongs to J [21, 28, 35]. Also, a Caratheodory multifunction F defines on $J \times \mathbb{R}^3$ to $2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each ρ more than zero, there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x, y, z)\| = \sup_{v \in F(t,x,y,z)} |v| \leq \phi_\rho(t),$$

for all $|x|, |y|, |z| \leq \rho$ and for $t \in J$ (for more details, see [21, 28]). Denote by $AC[0, 1]$ the space of all the absolutely continuous functions defined on J. By using main idea of [15, 16, 41], we define the set of selections of F by

$$S_{F,x} := \left\{ v \in AC(J, \mathbb{R}) \;\middle|\; v(t) \in F\left(t, x(t), {}^c D_q^\beta x(t), x'(t)\right) \text{ for all } t \in J \right\},$$

for all x belongs to $C(J, \mathbb{R})$. Let E be a nonempty closed subset of a Banach space X and G maps E into 2^X a multifunction with nonempty closed values. We say that the multifunction G is lower semi-continuous whenever the set $\{y \in E \mid G(y) \cap B \neq \emptyset\}$ is open for all open set $B \subset X$ [31]. Furthermore, It has been proved that each completely continuous multifunction is lower semi-continuous [31]. Let $AC^2[0, 1] = \{w \in C^1[0, 1] \mid w' \in L[0, 1]\}$. The following lemmas will be used in the sequel.

Lemma 2.1 ([37]). *For Banach space X, consider multifunction F maps $J \times X$ into $P_{cp,cv}(X)$ and function Θ maps $L^1(J, X)$ into $C(J, X)$ such that are L^1-Caratheodory and linear continuous, respectively. The operator*

$$\begin{cases} \Theta o S_F : C(J, X) \to P_{cp,cv}(C(J \times X)), \\ (\Theta o S_F)(x) = \Theta(S_{F,x}), \end{cases}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.2 ([31]). *Suppose that C a closed convex subset of Banach space E, $U \subset C$ is an open such that $0 \in U$. Also, let $F : \overline{U} \to P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

3. Main Results

Now, we would be ready to give theorems for the solution of the q-derivative inclusion problem (1.1). Define $x_v(t) = I_q^\alpha v(t) - c_{0v} - c_{1v} t$, where

$$\begin{aligned} c_{1v} = &-\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} v(m) d_q m ds \\ &- \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} v(s) d_q s \\ &- \frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s - qm)^{(\alpha-1)} v(m) d_q m ds \end{aligned}$$

$$- \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} v(s) d_q s$$
$$- \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s$$

and

$$\begin{aligned} c_{0v} = &- \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\ &+ \frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} v(s) d_q s \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{\alpha-2} v(s) d_q s. \end{aligned}$$

Clearly, $x_v \in AC^2[0,1]$ is well-define and x'_v, ${}^cDx_v$ and $\int_0^\eta x_v(s)\,ds$ exist whenever v belongs to $AC[0,1]$ (for more details, see [36]).

Lemma 3.1. *Let v belongs to $AC[0,1]$, q, β, η and ν in $(0,1)$, $1 < \alpha \le 2$, with $\alpha - \beta > 1$, and*

$$\Gamma_q(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0. \tag{3.1}$$

Then, $x_v(t)$ is the unique solution for the problem ${}^cD_q^\alpha x(t) = v(t)$ with the integral boundary value conditions

$$\begin{cases} x(0) + x'(0) + {}^cD_q^\beta x(0) = \int_0^\eta x(s) ds, \\ x(1) + x'(1) + {}^cD_q^\beta x(1) = \int_0^\nu x(s) ds. \end{cases} \tag{3.2}$$

Proof. It is observed that the general solution of the equation $v(t) = {}^cD_q^\alpha x(t)$ is

$$x(t) = I_q^\alpha v(t) - a_0 - a_1 t = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s - a_0 - a_1 t,$$

where a_0 and a_1 are arbitrary constants and t in J (see [42]). Thus,

$$\begin{aligned} {}^cD_q^\beta x(t) &= I_q^{\alpha-\beta} v(t) - \frac{t^{1-\beta} a_1}{\Gamma_q(2-\beta)} \\ &= \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} v(s) d_q s - \frac{t^{1-\beta} a_1}{\Gamma_q(2-\beta)} \end{aligned}$$

and

$$x'(t) = I_q^{\alpha-1}v(t) - a_1 = \frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}v(s)d_qs - a_1.$$

Hence, by using an easy calculation, we get $x(0) + {}^cD_q^\beta x(0) + x'(0) = -a_0 - a_1$ and

$$\begin{aligned} x(1) + {}^cD_q^\beta x(1) + x'(1) = {} & \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}v(s)d_qs \\ & + \left(\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1 (1-qs)^{(\alpha-\beta-1)}v(s)d_qs\right) \\ & \times \left(\frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs\right) \\ & - \frac{\Gamma_q(2)a_1}{\Gamma_q(2-\beta)} - 2a_1 - a_0. \end{aligned}$$

By using the boundary conditions (3.2), we obtain

$$a_0(\eta-1) - a_1\left(\frac{\eta^2}{2} - 1\right) = \frac{1}{\Gamma_q(\alpha)}\int_0^\eta \int_0^s (s-qm)^{(\alpha-1)}v(m)d_qmds$$

and

$$\begin{aligned} a_0(\nu-1) + a_1\left(\frac{\nu^2}{2} - 2 - \frac{\Gamma_q(2)}{\Gamma_q(2-\beta)}\right) = {} & -\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{\alpha-1}v(s)d_qs \\ & - \frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1 (1-qs)^{(\alpha-\beta-1)}v(s)d_qs \\ & - \frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs \\ & + \frac{1}{\Gamma_q(\alpha)}\int_0^\nu \int_0^s (s-qm)^{(\alpha-1)}v(m)d_qmds. \end{aligned}$$

Thus,

$$\begin{aligned} a_0 = c_{0v} = {} & -\frac{1}{\Gamma_q(\alpha)(1-\eta)}\int_0^\eta \int_0^s (s-qm)^{(\alpha-1)}v(m)d_qmds \\ & + \frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\int_0^\eta \int_0^s (s-qm)^{(\alpha-1)}v(m)d_qmds \\ & + \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}v(s)d_qs \\ & + \frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\int_0^\nu \int_0^s (s-qm)^{(\alpha-1)}v(m)d_qmds \\ & + \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\int_0^1 (1-qs)^{(\alpha-\beta-1)}v(s)d_qs \end{aligned}$$

$$+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs$$

and

$$\begin{aligned}
a_1=c_{1v}=&-\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\
&-\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs,
\end{aligned}$$

where

$$\gamma=(\nu-1)\left(\frac{\eta^2}{2}-1\right)+(\eta-1)\left(\frac{\eta^2}{2}-2-\frac{\Gamma_q(2)}{\Gamma_q(2)-\beta}\right). \tag{3.3}$$

Hence,

$$\begin{aligned}
x(t)=x_vt=&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}v(s)d_qs\\
&+\frac{1}{\Gamma_q(\alpha)(1-\eta)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\
&+\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs\\
&+\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs
\end{aligned}$$

$$+\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds$$
$$+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs$$
$$+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs=I_q^\alpha v(t)-c_{0v}-c_{1v}t.$$

Conversely, it is clear that

$$\begin{cases} x_v'(t)=I_q^{\alpha-1}v(t)+c_{1v},\\ x_v''(t)=\left(I_q^{\alpha-1}v(t)\right)'={}^RD_q^{2-\alpha}v(t),\end{cases}$$

for almost all $t\in J$. Because, $2-\alpha$ belongs to $(0,1]$, we get

$${}^cD_q^\alpha x_v(t)=I_q^{2-\alpha}x_v''(t)=I_q^{2-\alpha}\left({}^RD_q^{2-\alpha}v(t)\right)=v(t).$$

Similar to last part, we obtain

$$x_v(0)+x_v'(0)+{}^cD_q^\beta x_v(0)=-c_{0v}-c_{1v}=\int_0^\eta x(s)ds$$

and

$$\begin{aligned} x_v(1)+x_v'(1)+{}^cD_q^\beta x_v(1)=&\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\ &+\left(\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\right)\\ &\times\left(\frac{1}{\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs\right)\\ &-\frac{\Gamma_q(2)a_1}{\Gamma_q(2-\beta)}-2c_{1v}-c_{0v}=\int_0^\nu x(s)ds.\end{aligned}$$

This finishes the proof. □

A solution of the inclusion problem (1.1) is an element $x\in AC^2([0,1],\mathbb{R})$ such that it satisfies the integral boundary conditions and there exists a function $v\in S_{F,x}$ such that $x(t)=I_q^\alpha v(t)-c_{0v}-c_{1v}t$ for all $t\in J$. Suppose that

$$\mathcal{X}=\left\{x\,|\,x,x',{}^cD_q^\beta x\in C(J,\mathbb{R})\text{ for all }\beta\in(0,1)\right\}, \tag{3.4}$$

endowed with the norm

$$\|x\|=\sup_{t\in J}|x(t)|+\sup_{t\in J}|x'(t)|+\sup_{t\in J}\left|{}^cD_q^\beta x(t)\right|. \tag{3.5}$$

Then, $(\mathcal{X},\|.\|)$ is a Banach space [24].

For investigation of the inclusion problem (1.1), we provide two different methods. In the first method which is used in Theorem 3.1, we showed a compact map F is upper semi-continuous and so by using fixed point theorem in Lemma 2.2, and in the second method which is presented in Theorem 3.2, by using fixed point theorem of

Covitz and Nadler, and consider three conditions, respectively, we found a solution for the inclusion problem (1.1).

Theorem 3.1. *Let* $F : J \times \mathbb{R}^3 \to P_{cp,cv}(\mathbb{R})$ *is a* L^1*-Caratheodory multifunction and there exist a bounded continuous increasing self map* ψ *define on* $[0,\infty)$ *and a continuous function* p *maps* J *into* $(0,\infty)$ *such that*

$$\begin{aligned}\left\|F\left(t,x(t),x'(t),{}^cD_q^\beta x(t)\right)\right\| &= \sup\left\{|v| \mid v \in F\left(t,x(t),x'(t),{}^cD_q^\beta x(t)\right)\right\}\\ &\leq p(t)\psi(\|x\|),\end{aligned}$$

for all $t \in J$ *and* $x \in \mathcal{X}$*. Then the inclusion problem* (1.1) *has at least one solution.*

Proof. First, define the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$N(x) = \left\{h \in \mathcal{X} \mid \text{exists } v \in S_{F,x} : h(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t,\ t \in J\right\}.$$

In the following, prove that the operator N has a fixed point.

Step I. We show that N maps bounded sets of $\mathcal{X}$ into bounded sets. Let $r > 0$ and $B_r = \{x \in \mathcal{X} \mid \|x\| \leq r\}$. Suppose that $x \in B_r$ and $h \in N(x)$. We can choose $v \in S_{F,x}$ such that $h(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t$ for almost all $t \in J$. Thus,

$$\begin{aligned}|h(t)| \leq &\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\frac{1}{\Gamma_q(\alpha)(1-\eta)}\int_0^\eta\int_0^s (s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s (s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\right|\int_0^1 (1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s (s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1 (1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\right|\int_0^1 (1-qs)^{(\alpha-2)}\left|v(s)\right|d_qs\\
&+\left|\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s (s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\right|\int_0^1 (1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s (s-qm)^{(\alpha-1)}|v(m)|d_qmds\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
&\leq\Lambda_1\left\|p\right\|_\infty\psi\left(\|x\|\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|{}^cD_q^\beta h(t)\right| \leq&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-\beta)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-1)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
\leq&\Lambda_2\left\|p\right\|_\infty\psi\left(\|x\|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|h'(t)| \leq&\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}|v(s)|d_qs\\
&+\left|\frac{(1-\nu)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\int_0^1(1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta-1)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
\leq&\Lambda_3\left\|p\right\|_\infty\psi\left(\|x\|\right),
\end{aligned}
$$

for all $t\in J$, where $\|p\|_\infty=\sup_{t\in J}|p(t)|$,

$$
\Lambda_1=\left[\frac{1}{\Gamma_q(\alpha+1)}+\frac{\eta^{\alpha+1}}{\Gamma_q(\alpha+2)(1-\eta)}+\left|\frac{(\eta^2-2)(\nu-1)\eta^{\alpha+1}}{2\gamma\Gamma_q(\alpha+2)}\right|\right. \tag{3.6}
$$

$$+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha+1)}\right|+\left|\frac{(\eta^2-2)(1-\eta)\nu^{\alpha+1}}{2\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta+1)}\right|$$
$$+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\right|+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)}\right|$$
$$+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\Bigg],$$

$$\Lambda_2=\Bigg[\frac{1}{\Gamma_q(\alpha-\beta+1)}+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right| \tag{3.7}$$
$$+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta)}\right|+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right|$$
$$+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\Bigg],$$

and

$$\Lambda_3=\Bigg[\frac{1}{\Gamma_q(\alpha)}+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)}\right|+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right| \tag{3.8}$$
$$+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\Bigg].$$

Hence,

$$\|h\|=\max_{t\in J}|h(t)|+\max_{t\in J}\left|{}^cD_q^{\beta}h(t)\right|+\max_{t\in J}|h'(t)|$$

is less than equal to $(\Lambda_1+\Lambda_2+\Lambda_3)\|p\|_\infty\psi(\|x\|)$.

Step II. We demonstrate that N maps bounded sets into equicontinuous subsets of $\mathcal{X}$. Let $x\in B_r$ and $t_1,t_2\in J$, with $t_1<t_2$. After that, for all $h\in N(x)$, we have

$$|h(t_2)-h(t_1)|=\Bigg|\frac{1}{\Gamma_q(\alpha)}\int_0^{t_2}(t_2-qs)^{(\alpha-1)}v(s)d_qs$$
$$-\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}v(s)d_qs$$
$$+\frac{(1-\nu)t_2}{\gamma\Gamma_q(\alpha)}\int_0^{\eta}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds$$
$$-\frac{(1-\nu)t_1}{\gamma\Gamma_q(\alpha)}\int_0^{\eta}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds$$
$$+\frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha)}\int_0^{1}(1-qs)^{(\alpha-1)}v(s)d_qs$$
$$-\left(\frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha)}\int_0^{1}(1-qs)^{(\alpha-1)}v(s)d_qs\right)$$

$$\times \left(\frac{(\eta-1)t_2}{\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} \upsilon(m) d_q m ds \right)$$
$$- \frac{(\eta-1)t_1}{\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} \upsilon(m) d_q m ds$$
$$+ \frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} \upsilon(s) d_q s$$
$$- \frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} \upsilon(s) d_q s$$
$$+ \frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \upsilon(s) d_q s$$
$$- \frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \upsilon(s) d_q s \Bigg|$$
$$\leq \|p\|_\infty \psi(\|x\|) \left[\left| \frac{t_2^\alpha - t_1^\alpha}{\Gamma_q(\alpha+1)} \right| + \left| \frac{(1-\nu)\eta^{\alpha+1}(t_2-t_1)}{\gamma\Gamma_q(\alpha+2)} \right| \right.$$
$$+ \left| \frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha+1)} \right| + \left| \frac{(\eta-1)\nu^{\alpha+1}(t_2-t_1)}{\gamma\Gamma_q(\alpha+2)} \right|$$
$$\left. + \left| \frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha-\beta+1)} \right| + \left| \frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha)} \right| \right],$$

$|h'(t_2) - h'(t_1)| \leq \|p\|_\infty \psi(\|x\|) \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\Gamma_q(\alpha)}$, and

$$\left| {}^c D_q^\beta h(t_2) - {}^c D_q^\beta h(t_1) \right| \leq \|p\|_\infty \psi(\|x\|) \left[\left| \frac{t_2^{\alpha-\beta} - t_1^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)} \right| \right.$$
$$+ \left| \frac{\left(t_2^{1-\beta} - t_1^{1-\beta}\right)(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)} \right|$$
$$+ \left| \frac{\left(t_2^{1-\beta} - t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta)} \right| + \left| \frac{\left(t_2^{1-\beta} - t_1^{1-\beta}\right)(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)} \right|$$
$$\left. + \left| \frac{\left(t_2^{1-\beta} - t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta)} \right| + \left| \frac{\left(t_2^{1-\beta} - t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)} \right| \right].$$

Hence,

$$\lim_{t_2 \to t_1} |h(t_2) - h(t_1)| = \lim_{t_2 \to t_1} |h'(t_2) - h'(t_1)| = \lim_{t_2 \to t_1} \left| {}^c D_q^\beta h(t_2) - {}^c D_q^\beta h(t_1) \right| = 0,$$

and so by using the Arzela-Ascoli theorem, N is completely continuous.

Step III. Now, we show that N has a closed graph. Let $x_n \to x_0$, $h_n \in N(x_n)$ for all n and $h_n \to h_0$. We prove that $h_0 \in N(x_0)$. For each n, choose $v_n \in S_{F,x_n}$ such that $h_n(t) = I_q^\alpha v_n(t) - c_{0v_n} - c_{1v_n} t$ for all $t \in J$. Consider the continuous linear operator

$$\begin{cases} \theta : L^1(J,\mathbb{R}) \to \mathcal{X}, \\ \theta(v)(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t. \end{cases}$$

It can be seen, by Lemma 2.1, $\theta o S_F$ is a closed graph operator. Since $x_n \to x_0$ and $h_n \in \theta(S_{F,x_n})$ for all n, there exists $v_0 \in S_{F,x_0}$ such that $h_0(t) = I_q^\alpha v_0(t) - c_{0v} - c_{1v_0}t$. Thus, N has a closed graph.

Step IV. In this level, we show that $N(x)$ is convex for all $x \in \mathcal{X}$. Let $h_1, h_2 \in N(x)$ and $0 \le w \le 1$. Choose $v_1, v_2 \in S_{F,x}$ such that $h_i(t) = I_q^\alpha v_i(t) - c_{0v_i} - c_{1v_i}t$, for almost all $t \in J$ and $i = 1, 2$. Then,

$$\begin{aligned}
&[wh_1 + (1-w)h_2](t) \\
&= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} [wv_1(s) + (1-w)v_2(s)] d_qs \\
&+ \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} \left[wv_1(m) + (1-w)v_2(m)\right] d_qm ds \\
&+ \frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} \left[wv_1(m) + (1-w)v_2(m)\right] d_qm ds \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs \\
&+ \frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} \left[wv_1(m) + (1-w)v_2(m)\right] d_qm ds \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs \\
&+ \frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} \left[wv_1(m) + (1-w)v_2(m)\right] d_qm ds \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs \\
&+ \frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} \left[wv_1(m) + (1-w)v_2(m)\right] d_qm ds \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \left[wv_1(s) + (1-w)v_2(s)\right] d_qs,
\end{aligned}$$

for $t \in J$. Since F has convex values, $S_{F,x}$ is convex and so $wh_1 + (1-w)h_2$ belongs to $N(x)$. If there exists $\lambda \in (0,1)$ such that $x \in \lambda N(x)$, then there exists $v \in S_{F,x}$

such that $x(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t$, for all $t \in J$. Choose $L > 0$ such that

$$\frac{L}{(\Lambda_1 + \Lambda_2 + \Lambda_3)\|p\|_\infty \psi(\|x\|)} > 1,$$

for all $x \in \mathcal{X}$. Thus, $\|x\| < L$. Now, put $U = \{x \in \mathcal{X} \mid \|x\| < L + 1\}$. Note that, there are no $x \in \partial U$ and $0 < \lambda < 1$ such that $x \in \lambda N(x)$ and the operator $N : \overline{U} \to P_{cp,cv}(\overline{U})$ is upper semi-continuous, because it is completely continuous. Therefore, by using Lemma 2.2, N has a fixed point in $\overline{U}$ which is a solution of the inclusion problem (1.1). This completes the proof. □

Here, by changing values of multifunction in the assumption Theorem 3.1, we provide another result about the existence of solutions for the problem (1.1).

Theorem 3.2. *Let $m \in C(J, \mathbb{R}^+)$ be such that $\|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) < 1$ and consider an integrable bounded multifunction $F : J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ such that the map $t \mapsto F(t, x, y, z)$ is measurable and*

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \leq m(t)\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\right), \tag{3.9}$$

for $t \in J$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Then the problem (1.1) has a solution.

Proof. Note that, the multivalued map $t \mapsto F\left(t, x(t), x'(t), {}^cD_q^B x(t)\right)$, for $x \in \mathcal{X}$, is measurable and closed valued. Hence, it has a measurable selection and so the set $S_{F,x}$ is nonempty. Now, consider the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ defined by

$$N(x) = \left\{h \in \mathcal{X} \mid \text{exists } v \in S_{F,x} : h(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t\right\},$$

for all $t \in J$.

Step I. We show that $N(x)$ is a closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(x)$ with $u_n \to u$. For each n, choose $v_n \in S_{F,x}$ such that $u_n(t) = I_q^\alpha v_n(t) - c_{0v_n} - c_{1v_n}t$ for $t \in J$. From being compacted values F, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1(J, \mathbb{R})$. Again the subsequence denote by $\{v_n\}_{n\geq 1}$. It is easy to check that $v \in S_{F,x}$ and $u_n(t) \to u(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t$ for all $t \in J$. This implies that $u \in N(x)$. Thus, the multifunction N has closed values.

Step II. In this level, we show that N is a contractive multifunction with constant $l := \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) < 1$. Let $x, y \in \mathcal{X}$ and $h_1 \in N(y)$. Choose $v_1 \in S_{F,y}$ such that $h_1(t) = I^\alpha v_1(t) - c_{0v_1} - c_{1v_1}t$ for almost all $t \in J$. Put

$$A_x = F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right),$$
$$A_y = F\left(t, y(t), y'(t), {}^cD_q^\beta y(t)\right).$$

By assumption, if

$$H_d\left(A_x, A_y\right) \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^cD_q^\beta x(t) - {}^cD_q^\beta y(t)\right|\right),$$

for all $t \in J$, then there exists $w \in F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right)$ such that

$$|v_1(t) - w| \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^cD_q^\beta x(t) - {}^cD_q^\beta y(t)\right|\right), \tag{3.10}$$

for almost all $t \in J$. For the multifunction $U : J \to 2^{\mathbb{R}}$, define $U(t)$ by the set of all $w \in \mathbb{R}$ where satisfies in (3.10) for $t \in J$. It is easy to check that the multifunction

$$U(\cdot) \cap F\left(\cdot, x(\cdot), x'(\cdot), {}^cD_q^\beta x(\cdot)\right),$$

is measurable. Therefore, we can choose $v_2 \in S_{F,x}$ such that

$$|v_1(t) - v_2(t)| \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^cD_q^\beta x(t) - {}^cD_q^\beta y(t)\right|\right),$$

for almost all $t \in J$. Now, define $h_2 \in N(x)$ by $h_2(t) = I_q^\alpha v(t) - c_{0v_2} - c_{1v_2}t$. Hence, we get

$$\begin{aligned}
|h_1(t) - h_2(t)| \leq & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} |v_1(s) - v_2(s)|\, d_qs \\
& + \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)|\, d_qmds \\
& + \left|\frac{(\eta^2 - 2)(\nu - 1)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)|\, d_qmds \\
& + \left|\frac{(\eta^2 - 2)(\eta - 1)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^1 (1 - qs)^{(alpha-1)} |v_1(s) - v_2(s)|\, d_qs \\
& + \left|\frac{(\eta^2 - 2)(1 - \eta)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^\nu \int_0^s (s - qm)^{\alpha-1} |v_1(m) - v_2(m)|\, d_qmds \\
& + \left|\frac{(\eta^2 - 2)(\eta - 1)}{2\gamma\Gamma_q(\alpha - \beta)}\right| \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |v(s)|\, d_qs \\
& + \left|\frac{(\eta^2 - 2)(\eta - 1)}{2\gamma\Gamma_q(\alpha - 1)}\right| \int_0^1 (1 - qs)^{(\alpha-2)} |v_1(s) - v_2(s)|\, d_qs \\
& + \left|\frac{(1 - \nu)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)|\, d_qmds \\
& + \left|\frac{(1 - \eta)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^1 (1 - qs)^{(\alpha-1)} |v_1(s) - v_2(s)|\, d_qs \\
& + \left|\frac{(\eta - 1)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^\nu \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)|\, d_qmds \\
& + \left|\frac{(1 - \eta)t}{\gamma\Gamma_q(\alpha - \beta)}\right| \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |v_1(s) - v_2(s)|\, d_qs \\
& + \left|\frac{(1 - \eta)t}{\gamma\Gamma_q(\alpha - 1)}\right| \int_0^1 (1 - qs)^{(\alpha-2)} |v_1(s) - v_2(s)|\, d_qs \\
\leq & \Lambda_1 \|m\|_\infty \|x - y\|,
\end{aligned}$$

$$|h_1'(t) - h_2'(t)| \leq \frac{1}{\Gamma_q(\alpha - 1)} \int_0^t (t - qs)^{(\alpha-2)} |v_1(s) - v_2(s)|\, d_qs$$

$$
\begin{aligned}
&+\left|\frac{(1-\nu)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma+q(\alpha)}\right|\int_0^1(1-qs)^{(\alpha-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(\eta-1)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1(1-qs)^{\alpha-2}\left|v_1(s)-v_2(s)\right|d_qs\\
&\leq\Lambda_3\|m\|_\infty\|x-y\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^cD^\beta h_1(t)-{}^cD^\beta h_2(t)\right|\\
\leq&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-\beta)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-1)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-2)}\left|v_1(s)-v_2(s)\right|d_qs\\
\leq&\Lambda_2\|m\|_\infty\|x-y\|.
\end{aligned}
$$

So,

$$\|h_1-h_2\|\leq(\Lambda_1+\Lambda_2+\Lambda_3)\,\|m\|_\infty\|x-y\|=l\|x-y\|.$$

This implies that the multifunction N is a contraction with closed values. Thus by using the result of Covitz and Nadler, N has a fixed point which is a solution for the inclusion problem (1.1). □

Here, we provide two examples for the results.

Example 3.1. Put $q=\frac{1}{3}$, $\alpha=\frac{5}{2}$, $\beta=\frac{1}{2}$, $\eta=\frac{1}{2}$, $\nu=\frac{1}{3}$, consider the fractional q-derivative inclusion

$$
{}^cD^{\frac{5}{2}}_{\frac{1}{3}}x(t)\in F\left(t,x(t),x'(t),{}^cD^{\frac{1}{2}}_{\frac{1}{3}}x(t)\right),\tag{3.11}
$$

with the boundary value conditions

$$\begin{cases} x(0) + x'(0) + {}^cD^{\frac{1}{2}}_{\frac{1}{3}} x(0) = \int_0^{\frac{1}{2}} x(s)ds, \\ x(1) + x'(1) + {}^cD^{\frac{1}{2}}_{\frac{1}{3}} x(1) = \int_0^{\frac{1}{3}} x(s)ds, \end{cases} \tag{3.12}$$

and consider the multifunction $F : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ defined by

$$F(t, x_1, x_2, x_3) = \left[\cos t + \frac{e^{-\sin^2 x_1}}{1 + e^{\cos^2 x_1}} + \sin x_2, 4 + t^2 + \frac{t+1}{2 + e^{|x_3|}}\right].$$

Note that, $\|F(t, x_1, x_2, x_3)\| = \sup\{|y| \mid y \in F(t, x_1, x_2, x_3)\} \leq 6$. If $p(t) = 1$ and $\psi(t) = 6$, then one can check that the assumptions of Theorem 3.1 hold and so the inclusion problem (3.11) has at least one solution.

Next example illustrates last result.

Example 3.2. Put $q = \frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$, $\alpha = \frac{7}{3}$, $\beta = \frac{1}{3}$, $\eta = \frac{1}{2}$, $\nu = \frac{1}{3}$, consider the inclusion problem

$${}^cD^{\frac{7}{3}}_{\frac{1}{2}} x(t) \in F\left(t, x(t), x'(t), {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(t)\right), \tag{3.13}$$

with the boundary value conditions

$$\begin{cases} x(0) + x'(0) + {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(0) = \int_0^{\frac{1}{2}} x(s)ds, \\ x(1) + x'(1) + {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(1) = \int_0^{\frac{1}{3}} x(s)ds, \end{cases} \tag{3.14}$$

and consider the multifunction $F : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ defined by

$$F(t, x_1, x_2, x_3) = \left[0, \frac{t \sin^2 x_1}{12(4 + 3t^2)} + \frac{(t+1)|x_2|}{100(2 + |x_2|)} + \frac{|x_3|}{100(1 + |x_3|)}\right].$$

It is easy to understand that

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \leq \left(\frac{t}{12(4 + 3t^2)} + \frac{t+1}{100} + \frac{1}{100}\right) \sum_{i=1}^{3} |x_i - y_i|,$$

for all $t \in J = [0, 1]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Thus, if

$$m(t) = \frac{t}{12(4 + 3t^2)} + \frac{t+1}{100} + \frac{1}{100},$$

for all $t \in J$, then

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \leq m(t) \sum_{i=1}^{3} |x_i - y_i|.$$

On the other side, we have three cases for q:

$q := \frac{1}{3}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(3.0182 + 2.0213 + 2.1289) \simeq 0.3643 < 1,$$

$q := \frac{1}{2}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(2.6576 + 1.8297 + 1.9831) \simeq 0.3289 < 1,$$

$q := \frac{2}{3}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(2.3812 + 1.6771 + 1.1.8681) \simeq 0.3012 < 1.$$

These values calculate by Algorithm 4, 5 and 6 which present in Table 5, 6 and 7. Consequently, the assumptions of Theorem 3.2 hold and then the inclusion problem (3.13) have at least one solution.

4. Computational Results

A simplified analysis can be performed to estimate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for calulated q-Gamma functiuon of order n in Algorithm 2.

Algorithm 1 The proposed method for calculated $(a-b)^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * \frac{a-b*a^k}{a-b*q^{\alpha+k}}$
  end for
  $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: n, $q \in (0,1)$, $x \in \mathbb{R}\backslash\{0,-1,2,\cdots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x

$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1 - q^{i+1})^{\alpha - 1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow \frac{x^{\alpha} * (1-q) * s}{\Gamma_q(x)}$

Output: $(I_q^{\alpha} f)(x)$

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with less values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third column of Table 2. Also, Table 3 is the same as Table 2, but x values increase in 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns.

Now, we investigate the computational complexity of Example 3.2 of Algorithm 4, 5 and 6. First, Table 4 shows the values of γ for $q \in (0,1)$, an approximate result is obtained with less than four decimal places indicated by underline. Furthermore, Tables 5, 6, 7 show valued calculations of Λ_1, Λ_2 and Λ_3 for $q = \frac{1}{3}$, $q = \frac{1}{2}$ and $q = \frac{2}{3}$, respectively.

Algorithm 4 The proposed method for calculated Λ_1

Input: n, $q \in (0,1)$, α, η, ν

for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu - 1)(\eta^2/2 - 1) + (\eta - 1)(\eta^2/2 - 2 - \Gamma_q(2)/(\Gamma_q(2) - \beta))$
  $\Lambda_{1_1} \leftarrow 1/\Gamma_q(\alpha+1) + \eta^{\alpha+1}/(\Gamma_q(\alpha+2)(1-\eta))$
  $\Lambda_{1_2} \leftarrow |((\eta^2 - 2)(\nu - 1)\eta^{\alpha+1})/(2\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_3} \leftarrow |((\eta^2 - 2)(\eta - 1))/(2\gamma\Gamma_q(\alpha+1))|$
  $\Lambda_{1_4} \leftarrow |((\eta^2 - 2)(1 - \eta)\nu^{\alpha+1})/(2\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_5} \leftarrow |((\eta^2 - 2)(\eta - 1))/(2\gamma\Gamma_q(\alpha - \beta + 1))|$
  $\Lambda_{1_6} \leftarrow |((\eta^2 - 2)(\eta - 1))/(2\gamma\Gamma_q(\alpha))| + |((1 - \nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_7} \leftarrow |(1 - \eta)/(\gamma\Gamma_q(\alpha+1))| + |((\eta - 1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_8} \leftarrow |(1 - \eta)/(\gamma\Gamma_q(\alpha - \beta + 1))| + |(1 - \eta)/(\gamma\Gamma_q(\alpha))|$
  $\Lambda_1 = \Lambda_{1_1} + \Lambda_{1_2} + \Lambda_{1_3} + \Lambda_{1_4} + \Lambda_{1_5} + \Lambda_{1_6} + \Lambda_{1_7} + \Lambda_{1_8}$
end for

Output: Λ_1

Algorithm 5 The proposed method for calculated Λ_2

Input: n, $q \in (0,1)$, α, η, ν

for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu - 1)(\eta^2/2 - 1) + (\eta - 1)(\eta^2/2 - 2 - \Gamma_q(2)/(\Gamma_q(2) - \beta))$
  $\Lambda_{2_1} \leftarrow 1/\Gamma_q(\alpha - \beta + 1)$
  $\Lambda_{2_2} \leftarrow |((1-\nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta))|$
  $\Lambda_{2_3} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta))|$
  $\Lambda_{2_4} \leftarrow |((\eta-1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta))|$
  $\Lambda_{2_5} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta))|$
  $\Lambda_{2_6} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta))|$
  $\Lambda_2 = \Lambda_{2_1} + \Lambda_{2_2} + \Lambda_{2_3} + \Lambda_{2_4} + \Lambda_{2_5} + \Lambda_{2_6}$
end for

Output: Λ_2

Algorithm 6 The proposed method for calculated Λ_3

Input: n, $q \in (0,1)$, α, η, ν

for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu - 1)(\eta^2/2 - 1) + (\eta - 1)(\eta^2/2 - 2 - \Gamma_q(2)/(\Gamma_q(2) - \beta))$
  $\Lambda_{3_1} \leftarrow 1/\Gamma_q(\alpha) + |((1-\nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{3_2} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha+1))|$
  $\Lambda_{3_3} \leftarrow |((\eta-1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{3_4} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha-\beta+1))| + |(1-\eta)/(\gamma\Gamma_q(\alpha))|$
  $\Lambda_3 = \Lambda_{3_1} + \Lambda_{3_2} + \Lambda_{3_3} + \Lambda_{3_4}$
end for

Output: Λ_3

All routines are written in “Matalab” software with the “Digits” 16 (Digits environment variable controls the number of digits in Matlab) and run on a PC with 2.90 GHz of Core 2 CPU and 4 GB of RAM.

TABLE 1. Some numerical results for calculation of $\Gamma_q(x)$, with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$, of Algorithm 2.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

TABLE 2. Some numerical results for calculation of $\Gamma_q(x)$, with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$, of Algorithm 2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

TABLE 3. Some numerical results for calculation of $\Gamma_q(x)$, with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$, of Algorithm 2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	<u>11.257095</u>	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	<u>49.065751</u>	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	<u>259.967394</u>
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

TABLE 4. Some numerical results for calculation of γ, with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	2.257197	2.226716	2.174059	11	2.270833	2.270788	2.268911
2	2.266232	2.248106	2.203418	12	2.270833	<u>2.270810</u>	2.269551
3	2.269293	2.259295	2.224501	13	2.270833	2.270822	2.269978
4	2.270319	2.265019	2.239296	14	2.270833	2.270828	2.270263
5	2.270662	2.267915	2.249509	15	2.270833	2.270830	2.270453
6	2.270776	2.269371	2.256481	16	2.270833	2.270832	2.270580
7	<u>2.270814</u>	2.270102	2.261204	17	2.270833	2.270833	2.270664
8	2.270827	2.270467	2.264386	18	2.270833	2.270833	<u>2.270721</u>
9	2.270831	2.270650	2.266523	19	2.270833	2.270833	2.270758
10	2.270833	2.270742	2.267954	20	2.270833	2.270833	2.270783

TABLE 5. Some numerical results for calculattion of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{1}{3}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	2.793328	1.846027	1.990304	6.629659
2	2.942153	1.961611	2.082118	6.985882
3	2.992794	2.001290	2.113262	7.107345
4	3.009790	2.014645	2.123703	7.148138
5	3.015468	2.019112	2.127190	7.161770
6	3.017362	2.020602	2.128353	7.166318
7	3.017993	2.021099	2.128741	7.167834
8	3.018204	2.021265	2.128870	7.168339
9	3.018274	2.021320	2.128913	$\underline{7.168508}$
10	3.018298	2.021339	2.128928	7.168564
11	3.018305	2.021345	2.128933	7.168583
12	3.018308	2.021347	2.128934	7.168589

TABLE 6. Some numerical results for calculation of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{1}{2}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	1.980443	1.311532	1.552811	4.844787
2	2.303542	1.554800	1.759966	5.618308
3	2.476635	1.688162	1.869507	6.034304
4	2.566137	1.757911	1.925802	6.249851
5	2.611636	1.793570	1.954335	6.359541
6	2.634573	1.811598	1.968699	6.414870
7	2.646088	1.820662	1.975905	6.442655
8	2.651858	1.825206	1.979514	6.456578
9	2.654746	1.827482	1.981320	6.463547
10	2.656191	1.828620	1.982223	6.467034
11	2.656913	1.829190	1.982675	6.468778
12	2.657274	1.829474	1.982901	6.469650
13	2.657455	1.829617	1.983014	6.470086
14	2.657545	1.829688	1.983070	6.470304
15	2.657591	1.829724	1.983098	6.470413
16	2.657613	1.829741	1.983113	6.470467
17	2.657624	1.829750	1.983120	6.470494
18	2.657630	1.829755	1.983123	$\underline{6.470508}$
19	2.657633	1.829757	1.983125	6.470515
20	2.657634	1.829758	1.983126	6.470518

TABLE 7. Some numerical results for calculation of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{2}{3}$ and $n = 1, 2, \ldots, 30$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	1.051016	0.687483	0.979592	2.718091
2	1.419580	0.948096	1.237258	3.604934
3	1.705375	1.157875	1.429740	4.292990
4	1.914775	1.315447	1.567753	4.797976
5	2.063077	1.428895	1.664216	5.156188
6	2.165905	1.508420	1.730549	5.404873
7	2.236244	1.563214	1.775683	5.575140
8	2.283940	1.600547	1.806181	5.690669
9	2.316097	1.625798	1.826697	5.768592
10	2.337695	1.642794	1.840456	5.820945
11	2.352165	1.654198	1.849665	5.856027
12	2.361843	1.661832	1.855820	5.879496
13	2.368310	1.666936	1.859931	5.895177
14	2.372627	1.670345	1.862675	5.905648
15	2.375508	1.672621	1.864506	5.912635
16	2.377430	1.674139	1.865727	5.917296
17	2.378712	1.675152	1.866541	5.920405
18	2.379567	1.675827	1.867084	5.922478
19	2.380137	1.676277	1.867446	5.923861
20	2.380517	1.676578	1.867688	5.924783
21	2.380770	1.676778	1.867849	5.925397
22	2.380939	1.676911	1.867956	5.925807
23	2.381052	1.677000	1.868028	5.926080
24	2.381127	1.677060	1.868075	5.926262
25	2.381177	1.677099	1.868107	5.926384
26	2.381211	1.677126	1.868128	5.926464
27	2.381233	1.677143	1.868142	5.926518
28	2.381248	1.677155	1.868152	5.926554
29	2.381258	1.677163	1.868158	5.926578
30	2.381264	1.677168	1.868162	5.926594

References

[1] T. Abdeljawad and J. Alzabut, *The q-fractional analogue for gronwall-type inequality*, Journal of Function Spaces and Applications **2013** (2013), 7 pages.

[2] T. Abdeljawad, J. Alzabut and D. Baleanu, *A generalized q-fractional gronwall inequality and its applications to non-linear delay q-fractional difference systems*, J. Inequal. Appl. **2016**(240) (2016), 13 pages.

[3] C. Adams, *The general theory of a class of linear partial q-difference equations*, Trans. Amer. Math. Soc. **26** (1924), 283–312.

[4] C. Adams, *Note on the existence of analytic solutions of non-homogeneous linear q-difference equations: ordinary and partial*, Annals of Mathematics **27** (1925), 73–83.

[5] C. Adams, *On the linear ordinary q-difference equation*, Trans. Amer. Math. Soc. Ser. B **30** (1929), 195–205.

[6] C. Adams, *Linear q-difference equations*, Bulletin of the American Mathematical Society **37 (6)** (1931), 361–400.

[7] R. Agarwal, *Certain fractional q-integrals and q-derivatives*, Math. Proc. Cambridge Philos. Soc. **66** (1969), 365–370.

[8] R. Agarwal and B. Ahmad, *Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions*, Comput. Math. Appl. **62** (2011), 1200–1214.

[9] R. Agarwal, D. Baleanu, V. Hedayati and S. Rezapour, *Two fractional derivative inclusion problems via integral boundary condition*, Appl. Math. Comput. **257** (2015), 205–212.

[10] R. Agarwal, M. Belmekki and M. Benchohra, *A survey on semilinear differential equations and inclusions invovling riemann-liouville fractional derivative*, Adv. Difference Equ. **2009** (2009), Article ID 981728, 47 pages.

[11] R. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973–1033.

[12] R. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2004.

[13] B. Ahmad, S. Etemad, M. Ettefagh and S. Rezapour, *On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **59(107)**(2) (2016), 119–134.

[14] B. Ahmad and S. Ntouyas, *Boundary value problem for fractional differential inclusions with four-point integral boundary conditions*, Surv. Math. Appl. **6** (2011), 175–193.

[15] B. Ahmad, S. Ntouyas and A. Alsaedi, *Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions*, Adv. Difference Equ. **2014** (2014), 18 pages.

[16] B. Ahmad, S. Ntouyas and A. Alsedi, *On fractional differential inclusions with with anti-periodic type integral boundary conditions*, Bound. Value Probl. **2013** (2013), 15 pages.

[17] B. Ahmad, S. Ntouyas and I. Purnaras, *Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations*, Adv. Difference Equ. **2012**(140) (2012), 15 pages.

[18] J. Alzabut and T. Abdeljawad, *Perron's theorem for q-delay difference equations*, Applied Mathematics and Information Sciences **5** (2011), 74–85.

[19] G. Anastassiou, *Principles of delta fractional calculus on time scales and inequalities*, Math. Comput. Model. **52** (2010), 556–566.

[20] M. Annaby and Z. Mansour, *q-Fractional Calculus and Equations*, Springer, Heidelberg, New York, 2012.

[21] J. Aubin and A. Ceuina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag, Berlin, 1984.

[22] D. Baleanu, H. Mohammadi and S. Rezapour, *The existence of solutions for a nonlinear mixed problem of singular fractional differential equations*, Adv. Difference Equ **2013**(359) (2013), 12 pages.
[23] M. Benchohra and N. Hamidi, *Fractional order differential inclusions on the half-line*, Surv. Math. Appl. **5** (2010), 99–111.
[24] V. Berinde and M. Pacurar, *The role of the Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform. **22** (2013), 143–150.
[25] M. Bragdi, A. Debbouche and D. Baleanu, *Existence of solutions for fractional differential inclusions with separated boundary conditions in banach space*, Adv. Math. Phys. (2013), Article ID 426061, 5 pages.
[26] R. Carmichael, *The general theory of linear q-difference equations*, Amer J. Math. **34** (1912), 147–168.
[27] H. Covitz and S. Nadler, *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.
[28] K. Deimling, *Multi-Valued Differential Equations*, Walter de Gruyter, Berlin, 1992.
[29] R. Ferreira, *Nontrivials solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ. **70** (2010), 1–101.
[30] R. Finkelstein and E. Marcus, *Transformation theory of the q-oscillator*, J. Math. Phys. **36** (1995), 2652–2672.
[31] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, Berlin, 2005.
[32] G. Han and J. Zeng, *On a q-sequence that generalizes the median genocchi numbers*, Annales des Sciences Mathématiques du Québec **23** (1999), 63–72.
[33] F. Jackson, *q-Difference equations*, Amer. J. Math. **32** (1910), 305–314.
[34] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, 2002.
[35] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, 1991.
[36] K. Lan and W. Lin, *Positive solutions of systems of caputo fractioal differential equations*, Communications in Applied Analysis **17** (2013), 61–86.
[37] A. Lasota and Z. Opial, *An application of the kakutani-ky fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci., SÃľr. Sci. Math. Astronom. Phys. **13** (1965), 781–786.
[38] X. Liu and Z. Liu, *Existence result for fractional differential inclusions with multivalued term depending on lower-order derivative*, Abstr. Appl. Anal. **2012** (2012), 24 pages.
[39] T. Mason, *On properties of the solution of linear q-difference equations with entire fucntion coefficients*, Amer. J. Math. **37** (1915), 439–444.
[40] K. Miller and B. Ross, *An introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[41] J. Nieto, A. Ouahab and P. Prakash, *Extremal solutions and relaxation problems for fractional differential inclusions*, Abstr. Appl. Anal. **2013** (2013), 9 pages.
[42] I. Podlubny, *Fractional Differential Equations*, Academic Press, San DIego, 1999.
[43] P. Rajković, S. Marinković and M. Stanković, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math. **1** (2007), 311–323.
[44] S. Rezapour and V. Hedayati, *On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions*, Kragujevac J. Math. **41**(1) (2017), 143–158.
[45] W. Trjitzinsky, *Analytic theory of linear q-difference equations*, Acta Math. **61** (1933), 1–38.
[46] Y. Zhao, H. Chen and Q. Zhang, *Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions*, Adv. Difference Equ. **2013**(48) (2013), 15 pages.
[47] H. Zhou, J. Alzabut and L. Yang, *On fractional langevin differential equations with anti-periodic boundary conditions*, The European Physical Journal Special Topics **226** (2017), 3577–3590.

[1]Department of Mathematics,
Faculty of Science,
Bu-Ali Sina University,
Hamedan, Iran
Email address: mesamei@gmail.com
Email address: mesamei@basu.ac.ir

[2]Department of Mathematics,
Faculty of Science,
Azarbaijan Shahid Madani University,
Tabriz, Iran
Email address: v.hedayati1367@gmail.com