

FINITE SUMS AND SOME PROPERTIES OF GENERALIZED FRAMES FOR OPERATORS IN HILBERT C^* -MODULES

MORTEZA MIRZAEI AZANDARYANI¹ AND FATEMEH ZAMANI MIRARKOULAEI²

ABSTRACT. Let K be an adjointable operator on a countably or finitely generated Hilbert C^* -module. In this paper, some new results concerning the construction of K - g -frames via finite sums and the action of adjointable operators are obtained. First, an equivalent condition for a sequence of adjointable operators to constitute a K - g -frame for the underlying Hilbert C^* -module is stated and then, using this equivalence, we get some properties of K - g -frames. Moreover, some sufficient conditions under which the sum of two sequences forms a K - g -frame are presented.

1. INTRODUCTION

Hilbert space frames as a generalization of orthonormal bases were first introduced by Duffin and Schaeffer in 1952 in [6] (see also [5]). A finite or countable subset of a Hilbert space \mathcal{H} like $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}}$ is called a frame for \mathcal{H} if there exist two positive numbers $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ such that

$$(1.1) \quad A_{\mathcal{F}} \|x\|^2 \leq \sum_{i \in \mathbb{I}} |\langle x, f_i \rangle|^2 \leq B_{\mathcal{F}} \|x\|^2,$$

for each x in \mathcal{H} . If \mathcal{F} satisfies the right-hand inequality, it is called a Bessel sequence.

A necessary and sufficient condition for a Bessel sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}}$ to be a frame is the existence of a Bessel sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}}$ such that

$$(1.2) \quad x = \sum_{i \in \mathbb{I}} \langle x, g_i \rangle f_i = \sum_{i \in \mathbb{I}} \langle x, f_i \rangle g_i,$$

for every x in \mathcal{H} . The Bessel sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}}$ is called a dual for $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}}$. In fact, (1.2) means that every x in \mathcal{H} can be represented in terms of the frame and

Key words and phrases. Hilbert C^* -modules, K - g -frames, Generalized frames, Finite sums.

2020 Mathematics Subject Classification. Primary: 42C15. Secondary: 47A05.

DOI

Received: July 16, 2025.

Accepted: December 26, 2025.

its dual. Hence, if a Bessel sequence is not a frame, i.e., it does not satisfy the first inequality in (1.1), then the stated reconstruction formulas in (1.2) would fail. But under certain circumstances, a Bessel sequence can be used for the reconstruction of every element in a proper subset of the underlying Hilbert space. One of the best choices for such subsets is the range of a bounded operator on the Hilbert space which is not necessarily surjective. Hence, the concept of K-frames was proposed in [9].

Definition 1.1. Let K be a bounded operator on a Hilbert space \mathcal{H} . A finite or countable subset $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}}$ in \mathcal{H} is called a K-frame for \mathcal{H} if there exist two positive numbers $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ such that

$$A_{\mathcal{F}} \|K^*x\|^2 \leq \sum_{i \in \mathbb{I}} |\langle x, f_i \rangle|^2 \leq B_{\mathcal{F}} \|x\|^2,$$

for each x in \mathcal{H} .

Obviously, a frame is a special case of a K-frame with $K = Id_{\mathcal{H}}$. As we can see, a K-frame (so a frame) is a sequence consisting of the elements in the underlying Hilbert space. But these elements can be considered as bounded operators using the inner product. This consideration caused the appearance of some important generalizations for frames such as g-frames and K-g-frames ([12, 21, 26]) that have been studied widely, mainly their finite sums have been studied, see [1, 2, 4, 9–11, 13, 14, 17, 20, 25].

In [8], Frank and Larson presented a general approach to the frame theory in Hilbert C^* -modules. Afterwards, g-frames, K-frames and K-g-frames in Hilbert C^* -modules were introduced in [15, 19, 23], respectively, and investigated in [18, 22, 24]. It is worth noting that due to the complexity of C^* -algebras and the absence of some useful properties of Hilbert spaces and the operators defined on them, the generalization of frame theory from Hilbert spaces to Hilbert C^* -modules is not trivial. As we know, a closed subspace of a Hilbert space is orthogonally complemented, so if \mathcal{H} and \mathcal{K} are Hilbert spaces and $T : \mathcal{H} \rightarrow \mathcal{K}$ is a bounded operator, then $\text{Rng}(T)$ ($\text{Rng}(T)$ denotes the range of T) is orthogonally complemented in \mathcal{K} . The conclusion does not hold for bounded operators on Hilbert C^* -modules, it is not valid even for adjointable ones. Indeed, there are abundant adjointable operators on Hilbert C^* -modules such that the closures of their ranges are not orthogonally complemented. Since a K-g-frame in a Hilbert C^* -module is a sequence of adjointable operators, the ranges of the operators and also the range of the operator K play an important role to study K-g-frames and their properties and also make the situation different from the Hilbert space setting. To see the importance of the ranges of adjointable operators in this paper, we refer to Theorems 3.1, 3.2 and the obtained results via them.

Since K-g-frames in Hilbert C^* -modules cover many of the frame generalizations, both in Hilbert spaces and Hilbert C^* -modules, the obtained results of this paper are presented for this concept. These results also hold for frames, g-frames and most of their generalizations in Hilbert spaces and Hilbert C^* -modules such as K-frames, K-g-frames, fusion frames and K-fusion frames.

In the present article, we mainly focus on some general ways to construct new K -g-frames in Hilbert C^* -modules via finite sums and the action of adjointable operators. Indeed, it is shown that, under some conditions, the sum of two sequences is a K -g-frame. In order to get more general results, two sequences of adjointable operators with some special properties will appear.

2. PRELIMINARIES

In this section, some basic concepts related to Hilbert C^* -modules, frames and g-frames in Hilbert C^* -modules are recalled. The main reference for Hilbert C^* -modules is the book [16] and the paper [8] is used for the presentation of some facts about frames in Hilbert C^* -modules (see also [3]). The paper [15] is considered for generalized frames or g-frames in Hilbert C^* -modules.

Suppose that \mathcal{A} is a unital C^* -algebra and \mathcal{H} is a left \mathcal{A} -module, such that the linear structures of \mathcal{A} and \mathcal{H} are compatible. \mathcal{H} is a pre-Hilbert \mathcal{A} -module if \mathcal{H} is equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ such that

- (i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{H}$;
- (ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathcal{A}$ and $x, y \in \mathcal{H}$;
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for all $x, y \in \mathcal{H}$;
- (iv) $\langle x, x \rangle \geq 0$, for each $x \in \mathcal{H}$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For every f in \mathcal{H} , we define $\|f\| := \|\langle f, f \rangle\|^{\frac{1}{2}}$. If \mathcal{H} is complete with $\|\cdot\|$, it is called a Hilbert \mathcal{A} -module or a Hilbert C^* -module over \mathcal{A} .

A Hilbert \mathcal{A} -module \mathcal{H} is finitely generated if there exists a finite set $\{x_1, \dots, x_n\} \subseteq \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an \mathcal{A} -linear combination $x = \sum_{i=1}^n a_i x_i$, $a_i \in \mathcal{A}$. A Hilbert \mathcal{A} -module \mathcal{H} is countably generated if there exists a countable set $\{x_i\}_{i \in \mathbb{I}} \subseteq \mathcal{H}$ such that \mathcal{H} equals the norm-closure of \mathcal{A} -linear hull of $\{x_i\}_{i \in \mathbb{I}}$.

We define $|x| := \langle x, x \rangle^{\frac{1}{2}}$, for each $x \in \mathcal{H}$. The center of \mathcal{A} is denoted by $\mathcal{Z}(\mathcal{A})$ and is defined by

$$\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba, \text{ for all } b \in \mathcal{A}\}.$$

We note that $\mathcal{Z}(\mathcal{A})$ is a commutative C^* -subalgebra of \mathcal{A} .

Let \mathcal{H} and \mathcal{K} be two Hilbert \mathcal{A} -modules. Then, a map $T: \mathcal{H} \rightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^*: \mathcal{K} \rightarrow \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

The notation $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ is used for the set of all adjointable operators from \mathcal{H} into \mathcal{K} and $\mathfrak{L}(\mathcal{H}, \mathcal{H})$ is abbreviated to $\mathfrak{L}(\mathcal{H})$. It is easy to see that every adjointable operator is bounded. We use the notations $\text{Rng}(T)$ and $\ker(T)$ for the range and the kernel of $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$, respectively.

We recall the following definition from [8].

Definition 2.1. Let \mathcal{H} be a Hilbert \mathcal{A} -module. A family $\{f_i\}_{i \in \mathbb{I}}$ in \mathcal{H} is a frame for \mathcal{H} , if there exist two positive constants A and B such that for all $f \in \mathcal{H}$,

$$(2.1) \quad A\langle f, f \rangle_{\mathcal{A}} \leq \sum_{i \in \mathbb{I}} \langle f, f_i \rangle_{\mathcal{A}} \langle f_i, f \rangle_{\mathcal{A}} \leq B\langle f, f \rangle_{\mathcal{A}}.$$

The numbers A and B are called the lower and upper bounds of the frame, respectively. If the sum in the middle of (2.1) is convergent in norm, the frame is called standard. The sequence $\{f_i\}_{i \in \mathbb{I}}$ is called a Bessel sequence with bound B if the right-hand inequality in (2.1) holds for every $f \in \mathcal{H}$.

Generalized frames or g-frames in Hilbert spaces were introduced in [21] and generalized to Hilbert C^* -modules in [15].

Let $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ be a sequence of Hilbert \mathcal{A} -modules. $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is called a g-frame for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ if there exist real constants $A_{\Lambda}, B_{\Lambda} > 0$ such that for each $x \in \mathcal{H}$,

$$A_{\Lambda} \langle x, x \rangle \leq \sum_{i \in \mathbb{I}} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B_{\Lambda} \langle x, x \rangle.$$

A_{Λ} and B_{Λ} are g-frame bounds of Λ . In this case, we call it an $(A_{\Lambda}, B_{\Lambda})$ -g-frame. The g-frame is standard if for each $x \in \mathcal{H}$, the sum converges in norm. If only the right-hand inequality is required, Λ is called a g-Bessel sequence. If $A_{\Lambda} = B_{\Lambda}$, the g-frame is called tight and if $A_{\Lambda} = B_{\Lambda} = 1$, the g-frame is called Parseval.

If $\{\mathcal{H}_i : i \in \mathbb{I}\}$ is a sequence of Hilbert \mathcal{A} -modules, then

$$\oplus_{i \in \mathbb{I}} \mathcal{H}_i = \left\{ x = \{x_i\}_{i \in \mathbb{I}} : x_i \in \mathcal{H}_i \text{ and } \sum_{i \in \mathbb{I}} \langle x_i, x_i \rangle \text{ is norm convergent in } \mathcal{A} \right\},$$

is a Hilbert \mathcal{A} -module with pointwise operations and \mathcal{A} -valued inner product

$$\langle x, y \rangle = \sum_{i \in \mathbb{I}} \langle x_i, y_i \rangle,$$

where $x = \{x_i\}_{i \in \mathbb{I}}$ and $y = \{y_i\}_{i \in \mathbb{I}}$.

For a standard g-Bessel sequence Λ , the operator $\mathcal{T}_{\Lambda} : \oplus_{i \in \mathbb{I}} \mathcal{H}_i \rightarrow \mathcal{H}$ which is defined by $\mathcal{T}_{\Lambda}(\{g_i\}_{i \in \mathbb{I}}) = \sum_{i \in \mathbb{I}} \Lambda_i^* g_i$ is called the synthesis operator of Λ . \mathcal{T}_{Λ} is adjointable and $\mathcal{T}_{\Lambda}^*(x) = \{\Lambda_i x\}_{i \in \mathbb{I}}$. The operator $\mathcal{S}_{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}$ which is defined by $\mathcal{S}_{\Lambda} x = \mathcal{T}_{\Lambda} \mathcal{T}_{\Lambda}^*(x) = \sum_{i \in \mathbb{I}} \Lambda_i^* \Lambda_i(x)$, is called the operator of Λ . If Λ is a standard $(A_{\Lambda}, B_{\Lambda})$ -g-frame, then $A_{\Lambda} \cdot \text{Id}_{\mathcal{H}} \leq \mathcal{S}_{\Lambda} \leq B_{\Lambda} \cdot \text{Id}_{\mathcal{H}}$. For more results about g-frames in Hilbert C^* -modules, see [15, 24].

Here, all C^* -algebras are unital and all Hilbert C^* -modules are finitely or countably generated (all C^* -algebras and Hilbert C^* -modules are supposed to be non-trivial, i.e., they are not $\{0\}$). All g-Bessel sequences are assumed to be standard.

3. SOME PROPERTIES OF STANDARD K-G-FRAMES

In this section, some properties of standard K-g-frames are obtained. First, we recall the definition of a K-g-frame from [23].

Definition 3.1. Let $K \in \mathfrak{L}(\mathcal{H})$ and let $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ be a sequence of operators. Then, Λ is called a K-g-frame for \mathcal{H} with respect to $\{\mathcal{H}_i: i \in \mathbb{I}\}$ if there exist real constants $A_\Lambda, B_\Lambda > 0$ such that for each $x \in \mathcal{H}$,

$$A_\Lambda \langle K^*(x), K^*(x) \rangle \leq \sum_{i \in \mathbb{I}} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B_\Lambda \langle x, x \rangle.$$

The K-g-frame is called standard if for each $x \in \mathcal{H}$, the sum converges in norm.

If $K = Id_{\mathcal{H}}$, the definition reduces to a g-frame.

We recall the following result from [7].

Theorem 3.1. Suppose that $\mathcal{H}_1, \mathcal{H}_2$ and \mathcal{H}_3 are three Hilbert C^* -modules, and let $T \in \mathfrak{L}(\mathcal{H}_1, \mathcal{H}_3)$ and $S \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_3)$. The following statements are equivalent.

- (i) There is some $\lambda > 0$ with $SS^* \leq \lambda TT^*$.
- (ii) There exists some $\mu > 0$ such that $\|S^*(x)\| \leq \mu \|T^*(x)\|$, for any $x \in \mathcal{H}_3$.

In [23, Theorem 3.8], using the assumption that $\overline{\text{Rng}(T)}$ is orthogonally complemented, where $T(f) = \{\Lambda_i f\}_{i \in \mathbb{I}}$, an equivalent condition for a sequence of adjointable operators to be a K-g-frame is obtained. Now, we use Theorem 3.1 to get the equivalent condition without taking into account $\overline{\text{Rng}(T)}$ is orthogonally complemented. We mention that this result will be used repeatedly in what follows.

Theorem 3.2. Let $K \in \mathfrak{L}(\mathcal{H})$. Then $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ is a standard K-g-frame if and only if there exist two positive numbers A_Λ and B_Λ such that

$$(3.1) \quad A_\Lambda \|K^* f\|^2 \leq \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\| \leq B_\Lambda \|f\|^2,$$

for every f in \mathcal{H} .

Proof. Observe that if (3.1) holds, then the operator $\mathcal{T}: \bigoplus_{i \in \mathbb{I}} \mathcal{H}_i \rightarrow \mathcal{H}$ defined by $\mathcal{T}(\{f_i\}_{i \in \mathbb{I}}) = \sum_{i \in \mathbb{I}} \Lambda_i^* f_i$ is adjointable with $\mathcal{T}^*(f) = \{\Lambda_i f\}_{i \in \mathbb{I}}$ and

$$A_\Lambda \|K^* f\|^2 \leq \|\mathcal{T}^*(f)\|^2 \leq B_\Lambda \|Id_{\mathcal{H}}(f)\|^2.$$

Now, Theorem 3.1 implies the existence of two positive numbers λ_1 and λ_2 with

$$\lambda_1 \langle K^* f, K^* f \rangle \leq \sum_{i \in \mathbb{I}} \langle \Lambda_i f, \Lambda_i f \rangle \leq \lambda_2 \langle f, f \rangle,$$

for each f in \mathcal{H} . Consequently, Λ is a K-g-frame. Trivially, if Λ is a standard K-g-frame, then (3.1) holds. \square

Example 3.1. Let $\{\mathcal{H}_i\}_{i=1}^{+\infty}$ be a sequence of Hilbert C^* -modules. Define the operator K on $\mathcal{H} := \bigoplus_{i=1}^{+\infty} \mathcal{H}_i$ by $K(\{f_i\}_{i=1}^{+\infty}) = \{g_i\}_{i=1}^{+\infty}$, where $g_i = f_i$, for $i = 2k$ and $g_i = 0$, for $i = 2k - 1$. It is easy to verify that K is a self-adjoint operator. Now, assume that P_i is the coordinate operator on \mathcal{H} . Then, for $\Lambda_i := P_i K$ and f in \mathcal{H} , we have

$$\left\| \sum_{i=1}^{+\infty} \langle \Lambda_i f, \Lambda_i f \rangle \right\| = \|Kf\|^2 = \|K^* f\|^2,$$

so $\Lambda := \{\Lambda_i\}_{i=1}^{+\infty}$ is a standard K - g -frame. But Λ is not a standard g -frame because if x is a non-zero element in \mathcal{H}_1 , then for $f := \{f_i\}_{i=1}^{+\infty}$, where $f_1 = x$ and $f_i = 0$, for $i > 1$, we get $\Lambda_i f = 0$, for every $i \in \mathbb{N}$ and $\|f\|^2 = \|x\|^2$, so there is no positive number to satisfy the lower bound condition of a standard g -frame.

Theorem 3.3. *Let $K \in \mathfrak{L}(\mathcal{H})$ and let $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ be a K - g -frame. Then, the following statements hold.*

- (i) *For every adjointable operator T on \mathcal{H} , Λ is a KT - g -frame.*
- (ii) *For every adjointable operator S on \mathcal{H} , which commutes with K , Λ is an SK - g -frame.*
- (iii) *If Λ is a \mathcal{U} - g -frame, for some adjointable operator \mathcal{U} , then Λ is a $(K + \mathcal{U})$ - g -frame.*

Proof. (i) It is obvious for $T = 0$, so let $T \neq 0$. Then, for each f in \mathcal{H} , we have

$$A_\Lambda \|(KT)^* f\|^2 \leq \|T^*\|^2 A_\Lambda \|K^* f\|^2 \leq \|T^*\|^2 \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\|.$$

Consequently,

$$\frac{A_\Lambda}{\|T^*\|^2} \|(KT)^* f\|^2 \leq \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\| \leq B_\Lambda \|f\|^2.$$

- (ii) The result follows from part (i) since $SK = KS$.
- (iii) Let $A_\mathcal{U}$ be a lower bound for Λ (as a \mathcal{U} - g -frame). For every f in \mathcal{H} , we get

$$\begin{aligned} \|(K + \mathcal{U})^* f\|^2 &\leq \|K^* f\|^2 + \|\mathcal{U}^* f\|^2 + 2 \|K^* f\| \cdot \|\mathcal{U}^* f\| \\ &\leq \left(\frac{1}{A_\Lambda} + \frac{1}{A_\mathcal{U}} + \frac{2}{\sqrt{A_\Lambda A_\mathcal{U}}} \right) \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\|. \end{aligned}$$

Thus,

$$\left(\frac{1}{A_\Lambda} + \frac{1}{A_\mathcal{U}} + \frac{2}{\sqrt{A_\Lambda A_\mathcal{U}}} \right)^{-1} \|(K + \mathcal{U})^* f\|^2 \leq \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\| \leq B_\Lambda \|f\|^2$$

and the result follows. \square

Corollary 3.1. *Let $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ be a g -Bessel sequence and assume that \mathfrak{K}_Λ is the set of all adjointable operators on \mathcal{H} , like K , for which Λ is a K - g -frame. The following statements hold.*

- (i) *If $\Lambda_i = 0$, for each $i \in \mathbb{I}$, then $\mathfrak{K}_\Lambda = \{0\}$.*
- (ii) *If $\Lambda_i \neq 0$, for some $i \in \mathbb{I}$, then $\{0\}$ is a proper subset of \mathfrak{K}_Λ .*
- (iii) *\mathfrak{K}_Λ is a right ideal of $\mathfrak{L}(\mathcal{H})$.*
- (iv) *\mathfrak{K}_Λ is equal to $\mathfrak{L}(\mathcal{H})$ if and only if Λ is a g -frame, i.e., $Id_\mathcal{H}$ belongs to \mathfrak{K}_Λ .*
- (v) *\mathfrak{K}_Λ is a $\mathfrak{Z}(\mathfrak{L}(\mathcal{H}))$ -module, where $\mathfrak{Z}(\mathfrak{L}(\mathcal{H}))$ is the center of the C^* -algebra $\mathfrak{L}(\mathcal{H})$.*

Proof. Trivially $K = 0$ belongs to \mathfrak{K}_Λ . For each $f \in \mathcal{H}$, we obtain that

$$\left\| \mathcal{S}_\Lambda^{\frac{1}{2}} f \right\|^2 = \left\| \sum_{i \in \mathbb{I}} |\Lambda_i f|^2 \right\| \leq B_\Lambda \|f\|^2,$$

so Λ is an $\mathcal{S}_\Lambda^{\frac{1}{2}}$ -g-frame. Thus, $\mathcal{S}_\Lambda^{\frac{1}{2}}$ belongs to \mathfrak{K}_Λ , consequently $\mathfrak{K}_\Lambda \neq \{0\}$, when $\Lambda_i \neq 0$, for at least one $i \in \mathbb{I}$. Now, the result follows from Theorem 3.3. \square

4. THE CONSTRUCTION OF K-G-FRAMES VIA FINITE SUMS

The purpose of this section is to present some conditions under which the sum of two sequences of adjointable operators gives a K-g-frame.

In [17], using the upper bounds of the synthesis operators, some conditions under which the sum of two g-Bessel sequences for a Hilbert space is a g-frame are derived. Now, we use the synthesis operators to obtain some conditions under which the sum of two standard g-Bessel sequences in a Hilbert C^* -module gives a K-g-frame. Of course, the approach is different from the Hilbert space case since some inequalities for the elements of Hilbert spaces are not valid in Hilbert C^* -modules, moreover, the adjointable operator K plays an important role here (the operator K is equal to the identity operator in the g-frame inequalities). Also, we use two sequences of adjointable operators ($\{T_i\}_{i \in \mathbb{I}}$, $\{S_i\}_{i \in \mathbb{I}}$) with some special properties to achieve more general results.

Theorem 4.1. *Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ and $\Gamma = \{\Gamma_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ are two g-Bessel sequences with $A \cdot KK^* \leq \mathcal{S}_\Lambda \leq B \cdot KK^*$ and $\mathcal{S}_\Gamma \leq C \cdot KK^*$, where A and B are two positive numbers and C is a non-negative number. Assume that $\{T_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ and $\{S_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ are two sequences of invertible operators with $\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-1}$, $\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-1} > 0$ and $\sup_{i \in \mathbb{I}} \|T_i\|$, $\sup_{i \in \mathbb{I}} \|S_i\| < +\infty$. Then, $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame if*

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} (\sup_{i \in \mathbb{I}} \|T_i\|) (\sup_{i \in \mathbb{I}} \|S_i\|)}{\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2}}.$$

Proof. First, we show that $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a standard g-Bessel sequence via Theorem 3.1 in [24] and then, using the assumptions, we obtain a lower bound and finally apply Theorem 3.2 to show that the sequence is a standard K-g-frame.

It follows from the inequality $(T_i^* T_i)^{-1} \leq \|(T_i^* T_i)^{-1}\| \cdot Id_{\mathcal{H}_i} = \|T_i^{-1}\|^2 \cdot Id_{\mathcal{H}_i}$ that $\|T_i^{-1}\|^{-2} \cdot Id_{\mathcal{H}_i} \leq T_i^* T_i$, for each $i \in \mathbb{I}$. Similarly, we get $\|S_i^{-1}\|^{-2} \cdot Id_{\mathcal{H}_i} \leq S_i^* S_i$. Now, the inequality

$$\left\| \sum_{i \in \mathbb{I}} |(T_i \Lambda_i + S_i \Gamma_i)x|^2 \right\| \leq \left\| \sum_{i \in \mathbb{I}} \langle T_i^* T_i \Lambda_i x, \Lambda_i x \rangle \right\| + \left\| \sum_{i \in \mathbb{I}} \langle S_i^* S_i \Gamma_i x, \Gamma_i x \rangle \right\|$$

$$\begin{aligned}
& + 2 \left\| \sum_{i \in \mathbb{I}} \langle T_i \Lambda_i x, T_i \Lambda_i x \rangle \right\|^{\frac{1}{2}} \left\| \sum_{i \in \mathbb{I}} \langle S_i \Gamma_i x, S_i \Gamma_i x \rangle \right\|^{\frac{1}{2}} \\
& \leq \left(\sqrt{B} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) + \sqrt{C} \left(\sup_{i \in \mathbb{I}} \|S_i\| \right) \right)^2 \|K^*\|^2 \|x\|^2
\end{aligned}$$

and Theorem 3.1 in [24] imply that $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a standard g -Bessel sequence. Also, for every $x \in \mathcal{H}$, we obtain that

$$\begin{aligned}
\left\| \sum_{i \in \mathbb{I}} |(T_i \Lambda_i + S_i \Gamma_i)x|^2 \right\| & \geq \left\| \sum_{i \in \mathbb{I}} \langle T_i^* T_i \Lambda_i x, \Lambda_i x \rangle + \sum_{i \in \mathbb{I}} \langle S_i^* S_i \Gamma_i x, \Gamma_i x \rangle \right\| \\
& \quad - \left\| \sum_{i \in \mathbb{I}} \langle T_i \Lambda_i x, S_i \Gamma_i x \rangle + \sum_{i \in \mathbb{I}} \langle S_i \Gamma_i x, T_i \Lambda_i x \rangle \right\| \\
& \geq \left\| \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) A \langle K^* x, K^* x \rangle + \inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} |\mathcal{J}_\Gamma^* x|^2 \right\| \\
& \quad - 2 \left\| \sum_{i \in \mathbb{I}} \langle T_i^* T_i \Lambda_i x, \Lambda_i x \rangle \right\|^{\frac{1}{2}} \left\| \sum_{i \in \mathbb{I}} \langle S_i^* S_i \Gamma_i x, \Gamma_i x \rangle \right\|^{\frac{1}{2}} \\
& \geq \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) A \|K^* x\|^2 - \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \|\mathcal{J}_\Gamma^* x\|^2 \right) \\
& \quad - 2\sqrt{BC} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right) \|K^* x\|^2.
\end{aligned}$$

Therefore, for every positive number R with

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{J}_\Gamma^* x\|^2}{\|K^* x\|^2} < R < \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right)}{\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2}},$$

we have

$$\left\| \sum_{i \in \mathbb{I}} \langle (T_i \Lambda_i + S_i \Gamma_i)x, (T_i \Lambda_i + S_i \Gamma_i)x \rangle \right\| \geq \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \right) (D - R) \|K^* x\|^2,$$

where

$$D := \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right)}{\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2}},$$

and the result follows from Theorem 3.2. \square

Corollary 4.1. *Suppose that $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ is a g -frame and $\Gamma = \{\Gamma_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ is a g -Bessel sequence. Let $\{T_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ and $\{S_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ be two sequences of invertible operators with $\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-1}, \inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-1} > 0$ and*

$\sup_{i \in \mathbb{I}} \|T_i\|, \sup_{i \in \mathbb{I}} \|S_i\| < +\infty$. Then, $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a g -frame if

$$\|\mathcal{J}_\Gamma\|^2 = \|\mathcal{J}_\Gamma^*\|^2 < \frac{A_\Lambda \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{B_\Lambda B_\Gamma} (\sup_{i \in \mathbb{I}} \|T_i\|) (\sup_{i \in \mathbb{I}} \|S_i\|)}{\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2}}.$$

Proof. The result follows from Theorem 4.1 by considering $K := Id_{\mathcal{H}}$, $A := A_\Lambda$, $B := B_\Lambda$ and $C := B_\Gamma$ since we have

$$\|\mathcal{J}_\Gamma\|^2 = \|\mathcal{J}_\Gamma^*\|^2 = \sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{J}_\Gamma^* x\|^2}{\|K^* x\|^2}.$$

□

Corollary 4.2. Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ and $\Gamma = \{\Gamma_i \in \mathfrak{L}(\mathcal{H}, \mathcal{H}_i)\}_{i \in \mathbb{I}}$ are two g -Bessel sequences with $A \cdot KK^* \leq \mathfrak{S}_\Lambda \leq B \cdot KK^*$ and $\mathfrak{S}_\Gamma \leq C \cdot KK^*$, where A and B are two positive numbers and C is non-negative.

(i) Let $\{a_i\}_{i \in \mathbb{I}}$ and $\{b_i\}_{i \in \mathbb{I}}$ be two sequences of invertible elements in the center of \mathcal{A} with $\inf_{i \in \mathbb{I}} \|a_i^{-1}\|^{-1}, \inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-1} > 0$ and $\sup_{i \in \mathbb{I}} \|a_i\|, \sup_{i \in \mathbb{I}} \|b_i\| < +\infty$, where

$$\|d_i\| = \sup\{\|d_i x\| : x \in \mathcal{H}_i, \|x\| \leq 1\},$$

for $d_i := a_i, a_i^{-1}, b_i, b_i^{-1}$. Then, $\{a_i \Lambda_i + b_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame if

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{J}_\Gamma^* x\|^2}{\|K^* x\|^2} < \frac{A \left(\inf_{i \in \mathbb{I}} \|a_i^{-1}\|^{-2} \right) - 2\sqrt{BC} (\sup_{i \in \mathbb{I}} \|a_i\|) (\sup_{i \in \mathbb{I}} \|b_i\|)}{\inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-2}}.$$

(ii) Assume that a and b are invertible elements in the center of \mathcal{A} such that

$$\|d\| = \sup\{\|dx\| : x \in \mathcal{H}_i, \|x\| \leq 1\},$$

for $d := a, a^{-1}, b, b^{-1}$ and $i \in \mathbb{I}$. Then, the sequence $\{a \Lambda_i + b \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame if

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{J}_\Gamma^* x\|^2}{\|K^* x\|^2} < \frac{A \|a^{-1}\|^{-2} - 2\sqrt{BC} \|a\| \cdot \|b\|}{\|b^{-1}\|^{-2}}.$$

(iii) If \mathcal{H}_i is non-trivial, i.e., $\mathcal{H}_i \neq \{0\}$, for each $i \in \mathbb{I}$ and

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{J}_\Gamma^* x\|^2}{\|K^* x\|^2} < A - 2\sqrt{BC},$$

then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.

Proof. (i) Since a_i and b_i are invertible and belong to the center of the C^* -algebra, for each $i \in \mathbb{I}$, the operators $T_i(x) = a_i x$ and $S_i(x) = b_i x$ are adjointable and invertible on \mathcal{H}_i with $\|T_i\| = \|a_i\|$, $\|S_i\| = \|b_i\|$ and $T_i^{-1}(x) = a_i^{-1} x$, $S_i^{-1}(x) = b_i^{-1} x$ with $\|T_i^{-1}\| = \|a_i^{-1}\|$ and $\|S_i^{-1}\| = \|b_i^{-1}\|$. Now, the result follows from Theorem 4.1.

(ii) The result follows from part (i) using $a_i := a$ and $b_i := b$, for each $i \in \mathbb{I}$.

(iii) The result follows from part (ii) using $a, b := 1_{\mathcal{A}}$. □

Proposition 4.1. *Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ is an (A_Λ, B_Λ) - K - g -frame and $\Gamma = \{\Gamma_i\}_{i \in \mathbb{I}}$ is a g -Bessel sequence.*

- (i) *If a and b are invertible elements in the center of the C^* -algebra \mathcal{A} such that $ab^* \mathcal{T}_\Gamma \mathcal{T}_\Lambda^* + ba^* \mathcal{T}_\Lambda \mathcal{T}_\Gamma^* \geq \alpha \cdot KK^*$, for some real number α and we have $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < \frac{A_\Lambda \|a^{-1}\|^{-2+\alpha}}{\|b^{-1}\|^{-2}}$, then $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*
- (ii) *If a and b are invertible elements in the center of \mathcal{A} such that $ab^* \mathcal{T}_\Gamma \mathcal{T}_\Lambda^* + ba^* \mathcal{T}_\Lambda \mathcal{T}_\Gamma^*$ is positive and $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < A_\Lambda \left(\frac{\|b^{-1}\|}{\|a^{-1}\|} \right)^2$, then $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*
- (iii) *If $\mathcal{T}_\Lambda \mathcal{T}_\Gamma^* + \mathcal{T}_\Gamma \mathcal{T}_\Lambda^* \geq \alpha \cdot KK^*$, for some real number α and we have the inequality $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < A_\Lambda + \alpha$, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*
- (iv) *If $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < A_\Lambda$ and $\mathcal{T}_\Lambda \mathcal{T}_\Gamma^* + \mathcal{T}_\Gamma \mathcal{T}_\Lambda^*$ is positive, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*
- (v) *If a and b are invertible elements in the center of \mathcal{A} such that $\mathcal{T}_\Lambda \mathcal{T}_\Gamma^* = 0$ and $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < A_\Lambda \left(\frac{\|b^{-1}\|}{\|a^{-1}\|} \right)^2$, then $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*
- (vi) *If $\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < A_\Lambda$ and $\mathcal{T}_\Lambda \mathcal{T}_\Gamma^* = 0$, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*

Proof. (i) It is easy to see that $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a g -Bessel sequence. Now, the result follows from the relation

$$\begin{aligned} & \left\| \sum_{i \in \mathbb{I}} |(a\Lambda_i + b\Gamma_i)x|^2 \right\| \\ &= \left\| a^* a \sum_{i \in \mathbb{I}} \langle \Lambda_i x, \Lambda_i x \rangle + b^* b \sum_{i \in \mathbb{I}} \langle \Gamma_i x, \Gamma_i x \rangle + \langle (ab^* \mathcal{T}_\Gamma \mathcal{T}_\Lambda^* + ba^* \mathcal{T}_\Lambda \mathcal{T}_\Gamma^*)x, x \rangle \right\| \\ &\geq \left(A_\Lambda \|a^{-1}\|^{-2} + \alpha \right) \|K^* x\|^2 - \|b^{-1}\|^{-2} \|\mathcal{T}_\Gamma^* x\|^2. \end{aligned}$$

(ii) It follows from part (i) using $\alpha = 0$.

(iii) The result is an immediate consequence of part (i) with $a, b := 1_A$.

(iv) We can obtain the result by considering $\alpha = 0$ in part (iii).

(v) The result is a consequence of part (ii).

(vi) The result follows from part (iv). □

The next proposition gives a condition (without the assumption of the invertibility for S_i 's) under which $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.

Proposition 4.2. *Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ and $\Gamma = \{\Gamma_i\}_{i \in \mathbb{I}}$ are two g -Bessel sequences with $A \cdot KK^* \leq \mathfrak{S}_\Lambda \leq B \cdot KK^*$ and $\mathfrak{S}_\Gamma \leq C \cdot KK^*$, where A and B are two positive numbers and C is a non-negative number. Assume that $\{T_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ is a sequence of invertible operators such that $\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-1} > 0$,*

$\sup_{i \in \mathbb{I}} \|T_i\| < +\infty$ and $\{S_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ is a sequence with $\sup_{i \in \mathbb{I}} \|S_i\| < +\infty$. Then, $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K -g-frame if

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} < \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} (\sup_{i \in \mathbb{I}} \|T_i\|) (\sup_{i \in \mathbb{I}} \|S_i\|)}{\sup_{i \in \mathbb{I}} \|S_i\|^2}.$$

Proof. The result can be obtained similar to the proof of Theorem 4.1 and using the following inequality

$$\begin{aligned} \left\| \sum_{i \in \mathbb{I}} |(T_i \Lambda_i + S_i \Gamma_i)x|^2 \right\| &\geq \left\| \sum_{i \in \mathbb{I}} \langle T_i^* T_i \Lambda_i x, \Lambda_i x \rangle \right\| - \left\| \sum_{i \in \mathbb{I}} \langle S_i^* S_i \Gamma_i x, \Gamma_i x \rangle \right\| \\ &\quad - \left\| \sum_{i \in \mathbb{I}} \langle T_i \Lambda_i x, S_i \Gamma_i x \rangle + \sum_{i \in \mathbb{I}} \langle S_i \Gamma_i x, T_i \Lambda_i x \rangle \right\| \\ &\geq A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) \|K^* x\|^2 - \sup_{i \in \mathbb{I}} \|S_i\|^2 \|\mathcal{T}_\Gamma^* x\|^2 \\ &\quad - 2\sqrt{BC} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right) \|K^* x\|^2. \end{aligned}$$

□

Remark 4.1. (i) We mention that the obtained inequality in Proposition 4.2 yields the other conditions for the cases that $\{b_i\}_{i \in \mathbb{I}}$ is a sequence of elements in the center of \mathcal{A} and b is an element in the center of the C^* -algebra (b_i s and b are not necessarily invertible). Indeed, in Corollary 4.2 and Proposition 4.1, $\inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-2}$ and $\|b^{-1}\|^{-2}$ are replaced by $\sup_{i \in \mathbb{I}} \|b_i\|^2$ and $\|b\|^2$, respectively.

(ii) In the case that the sequence $\{S_i\}_{i \in \mathbb{I}}$ consists of invertible operators, the inequality in Theorem 4.1 provides a better condition than the one stated in Proposition 4.2 since for each $i \in \mathbb{I}$, we have $\frac{1}{\|S_i\|^2} \leq \frac{1}{\|S_i^{-1}\|^{-2}}$. In fact, when the invertibility of every S_i is assumed, Proposition 4.2 will be a consequence of Theorem 4.1. The same conclusion holds for the sequence $\{b_i\}_{i \in \mathbb{I}}$ and the element b in the center of the C^* -algebra.

In Remark 4.1, we talked about "a better condition". It is worth noting that it is common in frame theory to find optimal values in inequalities, in particular, finding optimal lower and upper bounds is so important in applications, see [8].

Observe that if K^* is left-invertible, then a K -g-frame is a g-frame, so Corollary 4.1 gives a condition under which the stated sum is a g-frame. The precise condition, using a left inverse, is stated in the following example.

Example 4.1. With the assumptions of Theorem 4.1, we have the following.

(i) If K^* is left-invertible with a left inverse $(K^*)^\dagger$, then $\{T_i\Lambda_i + S_i\Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame if

$$\|\mathcal{T}_\Gamma\|^2 < \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} (\sup_{i \in \mathbb{I}} \|T_i\|) (\sup_{i \in \mathbb{I}} \|S_i\|)}{\|(K^*)^\dagger\|^2 \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \right)}$$

since

$$\sup_{x \in \mathcal{H} \setminus \ker K^*} \frac{\|\mathcal{T}_\Gamma^* x\|^2}{\|K^* x\|^2} \leq \left(\sup_{x \in \mathcal{H} \setminus \ker K^*} \left\| \mathcal{T}_\Gamma^* \frac{x}{\|x\|} \right\|^2 \right) \|(K^*)^\dagger\|^2 = \|\mathcal{T}_\Gamma\|^2 \|(K^*)^\dagger\|^2.$$

(ii) If K is co-isometric, then $\{T_i\Lambda_i + S_i\Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame if

$$\|\mathcal{T}_\Gamma\|^2 < \frac{A \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) - 2\sqrt{BC} (\sup_{i \in \mathbb{I}} \|T_i\|) (\sup_{i \in \mathbb{I}} \|S_i\|)}{\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2}}.$$

(iii) If K is the left-shift operator on $\ell^2(\mathbb{N})$, i.e., $K(\{\alpha_i\}_{i=1}^{+\infty}) = \{\alpha_i\}_{i=2}^{+\infty}$, then K is co-isometric.

(iv) Consider \mathcal{A} as a left Hilbert \mathcal{A} -module and let a and b be two elements in \mathcal{A} such that $a^*b = 1_{\mathcal{A}}$. Then, it is easy to see that the operator K defined on \mathcal{A} by $K(x) = xa$ is adjointable with $K^*(x) = xa^*$. Hence, L_b defined by $L_b(x) = xb$ is a left inverse of K^* with $\|L_b\| = \|b\|$.

(v) Let $\mathcal{A} := \mathcal{B}(\mathbb{C}^2)$ and let $a := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $b := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then, $a^*b = 1_{\mathcal{A}}$ with $\|b\| = \sqrt{\frac{3+\sqrt{5}}{2}}$.

Theorem 4.2. Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that Λ and Γ are two g-Bessel sequences such that $A_1 \cdot KK^* \leq \mathfrak{S}_\Lambda \leq B_1 \cdot KK^*$ and $A_2 \cdot KK^* \leq \mathfrak{S}_\Gamma \leq B_2 \cdot KK^*$, for positive numbers A_1, B_1 and non-negative numbers A_2, B_2 . Assume that $\{T_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ and $\{S_i \in \mathfrak{L}(\mathcal{H}_i)\}_{i \in \mathbb{I}}$ are two sequences of invertible operators such that $\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-1}, \inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-1} > 0$ and $\sup_{i \in \mathbb{I}} \|T_i\|, \sup_{i \in \mathbb{I}} \|S_i\| < +\infty$. If

$$\left(A_1 \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) + A_2 \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \right) \right) > 2\sqrt{B_1 B_2} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right),$$

then $\{T_i\Lambda_i + S_i\Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame.

Proof. Similar to the proof of Theorem 4.1, we first show that $\{T_i\Lambda_i + S_i\Gamma_i\}_{i \in \mathbb{I}}$ is a standard g-Bessel sequence and then, we obtain a lower bound to show that the sequence is a standard K-g-frame.

For each $x \in \mathcal{H}$, we get

$$\left\| \sum_{i \in \mathbb{I}} |(T_i\Lambda_i + S_i\Gamma_i)x|^2 \right\| \leq \left(\sqrt{B_1} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) + \sqrt{B_2} \left(\sup_{i \in \mathbb{I}} \|S_i\| \right) \right)^2 \|K^*\|^2 \|x\|^2.$$

On the other hand, we have

$$\begin{aligned} \left\| \sum_{i \in \mathbb{I}} |(T_i \Lambda_i + S_i \Gamma_i) x|^2 \right\| &\geq \left\| \sum_{i \in \mathbb{I}} \langle T_i^* T_i \Lambda_i x, \Lambda_i x \rangle + \sum_{i \in \mathbb{I}} \langle S_i^* S_i \Gamma_i x, \Gamma_i x \rangle \right\| \\ &\quad - \left\| \sum_{i \in \mathbb{I}} \langle T_i \Lambda_i x, S_i \Gamma_i x \rangle + \sum_{i \in \mathbb{I}} \langle S_i \Gamma_i x, T_i \Lambda_i x \rangle \right\| \\ &\geq \|K^* x\|^2 \left(A_1 \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) + A_2 \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \right) \right) \\ &\quad - 2\sqrt{B_1 B_2} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right) \|K^* x\|^2. \end{aligned}$$

Therefore, $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame. \square

Corollary 4.3. *Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ and $\Gamma = \{\Gamma_i\}_{i \in \mathbb{I}}$ are two g-Bessel sequences with $A_1 \cdot KK^* \leq \mathfrak{S}_\Lambda \leq B_1 \cdot KK^*$ and $A_2 \cdot KK^* \leq \mathfrak{S}_\Gamma \leq B_2 \cdot KK^*$, where A_1, B_1 are two positive numbers and A_2, B_2 are non-negative numbers.*

(i) *Assume that $\{a_i\}_{i \in \mathbb{I}}$ and $\{b_i\}_{i \in \mathbb{I}}$ are two sequences of invertible elements in the center of \mathcal{A} such that $\inf_{i \in \mathbb{I}} \|a_i^{-1}\|^{-1}, \inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-1} > 0$ and $\sup_{i \in \mathbb{I}} \|a_i\|, \sup_{i \in \mathbb{I}} \|b_i\| < +\infty$. If*

$$\left(A_1 \left(\inf_{i \in \mathbb{I}} \|a_i^{-1}\|^{-2} \right) + A_2 \left(\inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-2} \right) \right) > 2\sqrt{B_1 B_2} \left(\sup_{i \in \mathbb{I}} \|a_i\| \right) \left(\sup_{i \in \mathbb{I}} \|b_i\| \right),$$

then $\{a_i \Lambda_i + b_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame.

(ii) *Assume that a and b are invertible elements in the center of \mathcal{A} . Then, the sequence $\{a \Lambda_i + b \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame if*

$$\left(A_1 \|a^{-1}\|^{-2} + A_2 \|b^{-1}\|^{-2} \right) > 2\sqrt{B_1 B_2} \|a\| \cdot \|b\|.$$

(iii) *If $(A_1 + A_2) > 2\sqrt{B_1 B_2}$, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame.*

Proof. (i) The operators $T_i(x) = a_i x$ and $S_i(x) = b_i x$ are adjointable and invertible on \mathcal{H}_i with $\|T_i\| \leq \|a_i\|$, $\|S_i\| \leq \|b_i\|$ and $T_i^{-1}(x) = a_i^{-1} x$, $S_i^{-1}(x) = b_i^{-1} x$ with $\|T_i^{-1}\| \leq \|a_i^{-1}\|$ and $\|S_i^{-1}\| \leq \|b_i^{-1}\|$. Therefore,

$$\begin{aligned} A_1 \left(\inf_{i \in \mathbb{I}} \|T_i^{-1}\|^{-2} \right) + A_2 \left(\inf_{i \in \mathbb{I}} \|S_i^{-1}\|^{-2} \right) &\geq A_1 \left(\inf_{i \in \mathbb{I}} \|a_i^{-1}\|^{-2} \right) + A_2 \left(\inf_{i \in \mathbb{I}} \|b_i^{-1}\|^{-2} \right) \\ &> 2\sqrt{B_1 B_2} \left(\sup_{i \in \mathbb{I}} \|a_i\| \right) \left(\sup_{i \in \mathbb{I}} \|b_i\| \right) \\ &\geq 2\sqrt{B_1 B_2} \left(\sup_{i \in \mathbb{I}} \|T_i\| \right) \left(\sup_{i \in \mathbb{I}} \|S_i\| \right). \end{aligned}$$

Then, by Theorem 4.2, $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame which is equivalent to say that $\{a_i \Lambda_i + b_i \Gamma_i\}_{i \in \mathbb{I}}$ is a K-g-frame.

Parts (ii)-(iii) follow from (i). \square

Theorem 4.3. *Let $K \in \mathfrak{L}(\mathcal{H})$ and assume that \mathcal{U} and T belong to $\mathfrak{L}(\mathcal{H})$ such that they commute with K^* and there exist a positive number $A_{\mathcal{U}}$ and a real number A_T with $\langle \mathcal{U}(x), \mathcal{U}(x) \rangle \geq A_{\mathcal{U}} \langle x, x \rangle$ and $\langle T(x), T(x) \rangle \geq A_T \langle x, x \rangle$, for each $x \in \mathcal{H}$. If Λ and Γ are two g -Bessel sequences with $A_1 \cdot KK^* \leq \mathfrak{S}_{\Lambda} \leq B_1 \cdot KK^*$ and $A_2 \cdot KK^* \leq \mathfrak{S}_{\Gamma} \leq B_2 \cdot KK^*$, where $A_1, B_1 > 0$ and $A_2, B_2 \geq 0$, such that $A_1 A_{\mathcal{U}} + A_2 A_T > 2\sqrt{B_1 B_2} \|\mathcal{U}\| \cdot \|T\|$, then $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K - g -frame.*

Proof. For each x in \mathcal{H} , we get

$$\left\| \sum_{i \in \mathbb{I}} |(\Lambda_i \mathcal{U} + \Gamma_i T)x|^2 \right\| \leq \left(B_1 \|\mathcal{U}\|^2 + B_2 \|T\|^2 + 2\sqrt{B_1 B_2} \|\mathcal{U}\| \cdot \|T\| \right) \|K\|^2 \|x\|^2,$$

so $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a standard g -Bessel sequence. Moreover, we obtain that

$$\begin{aligned} & \left\| \sum_{i \in \mathbb{I}} |(\Lambda_i \mathcal{U} + \Gamma_i T)x|^2 \right\| \\ & \geq \left\| \sum_{i \in \mathbb{I}} |\Lambda_i \mathcal{U}(x)|^2 + \sum_{i \in \mathbb{I}} |\Gamma_i T(x)|^2 \right\| - \left\| \sum_{i \in \mathbb{I}} \langle \Lambda_i \mathcal{U}(x), \Gamma_i T(x) \rangle + \sum_{i \in \mathbb{I}} \langle \Gamma_i T(x), \Lambda_i \mathcal{U}(x) \rangle \right\| \\ & \geq \left(A_1 A_{\mathcal{U}} + A_2 A_T - 2\sqrt{B_1 B_2} \|\mathcal{U}\| \|T\| \right) \|K^* x\|^2. \end{aligned}$$

Therefore, $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K - g -frame. \square

The next result can be obtained using the proof of Theorem 4.3 and Proposition 4.1.

Proposition 4.3. *Let $K \in \mathfrak{L}(\mathcal{H})$. Suppose that $\Lambda = \{\Lambda_i\}_{i \in \mathbb{I}}$ is an (A_1, B_1) - K - g -frame and $\Gamma = \{\Gamma_i\}_{i \in \mathbb{I}}$ is a g -Bessel sequence with $A_2 \cdot KK^* \leq \mathfrak{S}_{\Gamma} \leq B_2 \cdot Id_{\mathcal{H}}$, where $A_2, B_2 \geq 0$, (for example Γ is an (A_2, B_2) - K - g -frame, i.e., $A_2, B_2 > 0$).*

(i) *Suppose that \mathcal{U} and T belong to $\mathfrak{L}(\mathcal{H})$ such that they commute with K^* and there exist a positive number $A_{\mathcal{U}}$ and a non-negative number A_T such that $\langle \mathcal{U}(x), \mathcal{U}(x) \rangle \geq A_{\mathcal{U}} \langle x, x \rangle$, $\langle T(x), T(x) \rangle \geq A_T \langle x, x \rangle$, for each $x \in \mathcal{H}$, and $T^* \mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* \mathcal{U} + \mathcal{U}^* \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^* T \geq \alpha \cdot KK^*$, for some $\alpha \in \mathbb{R}$. If $A_1 A_{\mathcal{U}} + A_2 A_T + \alpha > 0$, then $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K - g -frame. In the special case if $T^* \mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* \mathcal{U} + \mathcal{U}^* \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^* T$ is a positive operator, for example, when $\mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* = 0$, then the sequence $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K - g -frame (since $\alpha = 0$ and the inequality $A_1 A_{\mathcal{U}} + A_2 A_T > 0$ automatically holds).*

(ii) *If a and b are invertible elements in the center of \mathcal{A} such that $b^* a \mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* + a^* b \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^* \geq \alpha \cdot KK^*$, for some $\alpha \in \mathbb{R}$, and $A_1 \|a^{-1}\|^{-2} + A_2 \|b^{-1}\|^{-2} + \alpha > 0$, then $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame. In the special case if $b^* a \mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* + a^* b \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^*$ is a positive operator, for example, when $\mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* = 0$, then $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*

(iii) *If $\mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* + \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^* \geq \alpha \cdot KK^*$, for some $\alpha \in \mathbb{R}$, and $A_1 + A_2 + \alpha > 0$, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame. In the special case if $\mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* + \mathcal{T}_{\Lambda} \mathcal{T}_{\Gamma}^*$ is a positive operator, for example, when $\mathcal{T}_{\Gamma} \mathcal{T}_{\Lambda}^* = 0$, then $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$ is a K - g -frame.*

Remark 4.2. (i) When $\langle \mathcal{U}(x), \mathcal{U}(x) \rangle \geq A_{\mathcal{U}} \langle x, x \rangle$, for some positive number $A_{\mathcal{U}}$ and for all $x \in \mathcal{H}$, \mathcal{U} is called bounded below by the inner product. It follows from

Proposition 2.1 in [3] that it is equivalent to say that \mathcal{U} is bounded below by the norm, i.e., $\|\mathcal{U}x\| \geq A\|x\|$, for some positive number A and every x in \mathcal{H} . Hence, the conditions $\langle \mathcal{U}(x), \mathcal{U}(x) \rangle \geq A_{\mathcal{U}} \langle x, x \rangle$ and $\langle Tx, Tx \rangle \geq A_T \langle x, x \rangle$ in Theorem 4.3 and Proposition 4.3 can be changed using the concept of bounded below operators by the norm for the operators \mathcal{U} and T , when $A_{\mathcal{U}}$ and A_T are positive. Also, the conditions can be replaced by the surjectivity of the adjoint operators (see [3, Proposition 2.1]).

(ii) Note that Proposition 4.3 holds when Γ is a g-Bessel sequence (consider $A_2 = 0$). Now, let Λ be a K_1 -g-frame and Γ be a K_2 -g-frame. Then, under the assumptions of Proposition 4.3, $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K_i -g-frame, $i = 1, 2$. Now, Theorem 3.3 implies that $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ (so $\{a\Lambda_i + b\Gamma_i\}_{i \in \mathbb{I}}$ and $\{\Lambda_i + \Gamma_i\}_{i \in \mathbb{I}}$) is a $(K_1 + K_2)$ -g-frame. Hence, Corollary 3.3 in [1] is a special case of Proposition 4.3.

(iii) Note that in the proof of Theorem 4.3, the relation

$$\sum_{i \in \mathbb{I}} |\Lambda_i \mathcal{U}x|^2 \geq A_1 \langle K^* \mathcal{U}x, K^* \mathcal{U}x \rangle = A_1 \langle \mathcal{U}K^*x, \mathcal{U}K^*x \rangle \geq A_1 A_{\mathcal{U}} \langle K^*x, K^*x \rangle$$

is used. Hence, the result is also valid with the weaker condition $\langle \mathcal{U}y, \mathcal{U}y \rangle \geq A_{\mathcal{U}} \langle y, y \rangle$, $y \in \text{Rng}(K^*)$. The same conclusion holds for Γ and T . Thus, if there exist two positive numbers α_1 and α_2 with $(K\mathcal{U}^*)(K\mathcal{U}^*)^* \geq \alpha_1 \cdot KK^*$ and $(KT^*)(KT^*)^* \geq \alpha_2 \cdot KK^*$, then under the assumptions of Theorem 4.3 and Proposition 4.3, the sequence $\{\Lambda_i \mathcal{U} + \Gamma_i T\}_{i \in \mathbb{I}}$ is a K-g-frame. Also, Theorem 3.8 and Corollary 3.9 in [1] can be considered as special cases of Proposition 4.3.

It is mentioned that Remark 4.2 concerns bounded below operators and their properties. In fact, bounded below operators have a crucial role in frame theory, for instance a necessary and sufficient condition for a Bessel sequence (in a Hilbert space or a Hilbert C^* -module) to be a frame is that the adjoint of its synthesis operator is bounded below, see [8]. The bounded below operators or the operators with surjective adjoints are also helpful to study the sums of g-frames, see [1, 17].

Remark 4.3. We can consider a K-frame as a special case of a K-g-frame (where $\mathcal{H}_i = \mathcal{A}$, for all $i \in \mathbb{I}$). Therefore, the obtained results for K-g-frames yield some facts for K-frames. Especially, the sequences like $\{a_i f_i + b_i g_i\}_{i \in \mathbb{I}}$ and $\{a f_i + b g_i\}_{i \in \mathbb{I}}$ are focused where a, b, a_i 's and b_i 's are elements in the underlying C^* -algebra that do not necessarily belong to the center. Indeed, the operator $T_i: \mathcal{A} \rightarrow \mathcal{A}$ defined by $T_i(a) = a a_i^*$ is adjointable with $T_i^*(a) = a a_i$ and if $\{f_i\}_{i \in \mathbb{I}}$ is a Bessel sequence (resp. frame, K-frame), then $\{\Lambda_i\}_{i \in \mathbb{I}}$ is a g-Bessel sequence (resp. g-frame, K-g-frame), where $\Lambda_i(f) = \langle f, f_i \rangle$. Also, the operator $S_i: \mathcal{A} \rightarrow \mathcal{A}$ defined by $S_i(a) = a b_i^*$ is adjointable and for a Bessel sequence $\{g_i\}_{i \in \mathbb{I}}$ in \mathcal{H} and $\Gamma_i(f) = \langle f, g_i \rangle$, we have

$$\{(T_i \Lambda_i + S_i \Gamma_i) f\}_{i \in \mathbb{I}} = \{\langle f, a_i f_i + b_i g_i \rangle\}_{i \in \mathbb{I}}.$$

Thus, the mentioned results for the sequence $\{T_i \Lambda_i + S_i \Gamma_i\}_{i \in \mathbb{I}}$ imply some conditions under which the sequences like $\{a_i f_i + b_i g_i\}_{i \in \mathbb{I}}$ and $\{a f_i + b g_i\}_{i \in \mathbb{I}}$ are K-frames.

We mention that the sum of the Bessel sequences appeared in Remark 4.3 can be considered as a generalization of the stated ones in [20] to Hilbert C^* -modules.

5. CONCLUSION

The manuscript investigates the construction and characterization of K - g -frames in Hilbert C^* -modules. Indeed, the paper presents the following main contributions.

- In Theorem 3.2, using the majorization and range-inclusion result of [7], one stated assumption in [23] for characterizing K - g -frames in Hilbert C^* -modules is eliminated and then some important properties of K - g -frames are obtained via Theorem 3.2.

- Some previously established finite-sum construction results in Hilbert spaces are extended to Hilbert C^* -modules.

- Section 4 provides explicit conditions expressed in terms of operator norms and the elements in the center of the underlying C^* -algebra to make the results practically applicable.

Acknowledgements. The authors thank the referees for very valuable suggestions which improved the paper.

REFERENCES

- [1] F. Abtahi, Z. Kamali and Z. Keyshams, *On the sum of generalized frames in Hilbert spaces*, Mediterr. J. Math. **18** (2021), 1–21. <https://doi.org/10.1007/s00009-021-01811-8>
- [2] F. A. Neyshaburi and A. A. Arefijamaal, *Some constructions of K -frames and their duals*, Rocky Mountain J. Math. **47** (2017), 1749–1764. <https://doi.org/10.1216/RMJ-2017-47-6-1749>
- [3] L. Arambasic, *On frames for countably generated Hilbert C^* -modules*, Proc. Amer. Math. Soc. **135** (2007), 469–478. <https://doi.org/10.1090/S0002-9939-06-08498-X>
- [4] J. Baradaran and M. Zerehpoush, *On the sum of g -frames and their stability in Hilbert spaces*, Mediterr. J. Math. **20** (2023), Article ID 217. <https://doi.org/10.1007/s00009-023-02417-y>
- [5] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283. <https://doi.org/10.1063/1.527388>
- [6] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. <https://doi.org/10.2307/1990760>
- [7] X. Fang, M. Sal Moslehian and Q. Xu, *On majorization and range inclusion of operators on Hilbert C^* -modules*, Linear Multilinear Algebra **66** (2018), 2493–2500. <https://doi.org/10.1080/03081087.2017.1402859>
- [8] M. Frank and D. R. Larson, *Frames in Hilbert C^* -modules and C^* -algebras*, J. Operator Theory **48** (2002), 273–314.
- [9] L. Gavruta, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), 139–144. <https://doi.org/10.1016/j.acha.2011.07.006>
- [10] X. Guo, *Canonical dual K -Bessel sequences and dual K -Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. **444** (2016), 598–609. <https://doi.org/10.1016/j.jmaa.2016.06.055>
- [11] M. He, J. Leng, J. Yu and Y. Xu, *On the sum of K -frames in Hilbert spaces*, Mediterr. J. Math. **17** (2020), 1–19. <https://doi.org/10.1007/s00009-020-1487-7>
- [12] D. Hua and Y. Huang, *K - g -frames and stability of K - g -frames in Hilbert spaces*, J. Korean Math. Soc. **53** (2016), 1331–1345. <http://dx.doi.org/10.4134/JKMS.j150499>
- [13] Y. Huang and S. Shi, *New constructions of K - g -frames*, Results Math. **73** (2018), 1–13. <https://doi.org/10.1007/s00025-018-0924-4>
- [14] M. Jia and Y. C. Zhu, *Some results about the operator perturbation of a K -frame*, Results Math. **73** (2018), 1–11. <https://doi.org/10.1007/s00025-018-0902-x>

- [15] A. Khosravi and B. Khosravi, *Fusion frames and g -frames in Hilbert C^* -modules*, Int. J. Wavelets Multiresolut. Inf. Process. **6** (2008), 433–446. <https://doi.org/10.1142/S0219691308002458>
- [16] E. C. Lance, *Hilbert C^* -Modules- A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser. Vol. **210**, Cambridge University Press, Cambridge, England, 1995.
- [17] D. Li, J. Leng, T. Huang and G. Sun, *On sum and stability of g -frames in Hilbert spaces*, Linear Multilinear Algebra **66** (2018), 1578–1592. <https://doi.org/10.1080/03081087.2017.1364338>
- [18] M. Mahmoudieh, G. A. Tabadkan and A. A. Arefijamaal, *Sum of K -frames in Hilbert C^* -modules*, Filomat **34** (2020), 1771–1780. <https://doi.org/10.2298/FIL2006771M>
- [19] A. Najati, M. M. Saem and P. Gavruta, *Frames and operators in Hilbert C^* -modules*, Oper. Matrices **10** (2016), 73–81. <http://dx.doi.org/10.7153/oam-10-06>
- [20] S. Obeidat, S. Samarah, P. G. Casazza and J. C. Tremain, *Sums of Hilbert space frames*, J. Math. Anal. Appl. **351** (2009), 579–585. <https://doi.org/10.1016/j.jmaa.2008.10.040>
- [21] W. Sun, *G -frames and g -Riesz bases*, J. Math. Anal. Appl. **322** (2006), 437–452. <https://doi.org/10.1016/j.jmaa.2005.09.039>
- [22] Z. Q. Xiang, *New inequalities of K - g -frames in submodules*, Bull. Iran. Math. Soc. **48** (2022), 627–641. <https://doi.org/10.1007/s41980-021-00535-5>
- [23] Z. Q. Xiang and Y. M. Li, *G -frames for operators in Hilbert C^* -modules*, Turk. J. Math. **40** (2016), 453–469. <https://doi.org/10.3906/mat-1501-22>
- [24] X. Xiao and X. Zeng, *Some properties of g -frames in Hilbert C^* -modules*, J. Math. Anal. Appl. **363** (2010), 399–408. <https://doi.org/10.1016/j.jmaa.2009.08.043>
- [25] X. Xiao, Y. Zhu, and L. Gavruta, *Some properties of K -frames in Hilbert spaces*, Results Math. **63** (2013), 1243–1255. <https://doi.org/10.1007/s00025-012-0266-6>
- [26] Y. Zhou and Y. C. Zhu, *K - g -frames and dual g -frames for closed subspaces*, Acta Math. Sin. (Chin. Ser.) **56** (2013), 799–806.

¹DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF QOM,
QOM, IRAN
Email address: morteza_ma62@yahoo.com; m.mirzaee@qom.ac.ir
ORCID id: <https://orcid.org/0000-0003-2386-3311>

²DEPARTMENT OF MATHEMATICS,
KHARAZMI UNIVERSITY,
TEHRAN, IRAN
Email address: std_zamani243@khu.ac.ir
ORCID id: <https://orcid.org/0009-0003-2530-4190>