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NON-CONFORMABLE FRACTIONAL LAPLACE TRANSFORM

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Abstract. In this paper we present an extension of Fractional Laplace Transform in the framework of the non-conformable local fractional derivative. Its main properties are studied and it is applied to the resolution of fractional differential equations.

1. Preliminaries

In mathematics, the Laplace transform is an integral transform n, it takes a function of a real variable t (often time) to a function of a complex variable s (complex frequency). Laplace transforms are usually restricted to functions of t with $t \geq 0$, consequently of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. The Laplace transform is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable s (often frequency) and yields a function of a real variable t (often time). Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications. So, for example, Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and convolution into multiplication.

Regarding the birth of the fractional calculus, all historians agree on the dating of the date and how it was produced. This fact took place after a publication of Leibniz where he introduced the notation of the differential calculus, in particular of

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the expression known today as $\frac{d^n y}{dx^n}$ that makes reference to the derivative of order n of the function and, with $n \in \mathbb{N}$. But did it make sense to extend the values of n to the set of rational, irrational, or complex numbers in that expression?

We know that the fractional derivative of a non-integer function can be conceived in two branches: global (classical) and local. The former are often defined by means of integral transforms, Fourier or Mellin, which means in particular that their nature is not local, has "memory", in the second case, they are defined locally by a certain incremental quotients. The first are associated with the emergence of the Fractional Calculation itself, with the pioneering works of Euler, Laplace, Lacroix, Fourier, Abel, Liouville,... until the establishment of the classical definitions of Riemann-Liouville and Caputo. Recent extensions and applications of these notions to various fields can be found in [2–4, 7, 13, 18, 21, 21]. There are some attempts to extend the classical notion of Laplace Transform to the non-integer case, we recommend consult [20].

Recently, in [8] Khalil et al. defined a new local fractional derivative called the conformable fractional derivative, based on the limit definition of the derivative. Namely, for a function $h : [0, \infty) \to \mathbb{R}$, the non-conformable fractional derivative of h of order α of h at t is defined by

$$D_\alpha(h)(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon t^{1-\alpha}\right) - h(t)}{\epsilon}, \quad \alpha \in (0, 1), t > 0.$$

In [1], Abdeljaward improve this new theory. For instance, definitions of left and right conformable derivatives and fractional integrals of higher order (i.e., of order $\alpha > 1$), Taylor power series, fractional integration by parts formulas and chain rule are provided by him.

Now, we give the definition of the non-conformable fractional derivative with its important properties which are useful in order to obtain our main results, which is explained in the following definition [5].

Definition 1.1. Given a function $h : [0, \infty) \to \mathbb{R}$. Then, the non-conformable fractional derivative $N_3^\alpha(h)(t)$ of order α of h at t is defined by

$$N_3^\alpha(h)(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon t^{-\alpha}\right) - h(t)}{\epsilon}, \quad \alpha \in (0, 1), t > 0.$$

If h is α-differentiable in some $(0, \alpha)$, $\alpha > 0$, $\lim_{t \to 0^+} h^{(\alpha)}(t)$ exist, then define

$$h^{(\alpha)}(0) = \lim_{t \to 0^+} h^{(\alpha)}(t).$$

Remark 1.1. Additionally, note that if h is differentiable, then

$$N_3^\alpha(h)(t) = t^{-\alpha} h'(t), \quad \text{where } h'(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon\right) - h(t)}{\epsilon}.$$

We can write $h^{(\alpha)}(t)$ for $D_\alpha(h)(t)$ or $\frac{d_\alpha}{d_\alpha t}(h(t))$ to denote the non-conformable fractional derivatives of h of order α at t. In addition, if the non-conformable fractional derivative N_3^α of h of order α exists, then we simply say h is N-differentiable.

In [5, 14], we can see that the chain rule is valid for non-conformable fractional derivatives.

Theorem 1.1. *Let $\alpha \in (0, 1]$, g a N-differentiable function at $t > 0$, f be differentiable in the range of $g(t)$. Then*

$$N_3^\alpha(f \circ g)(t) = f'(g(t))N_3^\alpha(g(t)).$$

Proof. We prove the result following a standard limit-approach. First case, if the function g is constant in a neighborhood of $a > 0$ then $N_3^\alpha(f \circ g)(t) = 0$. If g is not a constant in a neighborhood of $a > 0$ we can find and $\varepsilon_0 > 0$ such that $g(x_1) \neq g(x_2)$ for any $x_1, x_2 \in (a - t_0, a + t_0)$. Now, since g is continuous at a, for ε sufficiently small, we have

$$\begin{aligned}
N_3^\alpha(f \circ g)(a) =& \lim_{\varepsilon \to 0} \frac{fg((t + \varepsilon a^{-\alpha})) - f(g(a))}{\varepsilon} \\
=& \lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\
=& \lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\
=& \lim_{k \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon}.
\end{aligned}$$

Making $\varepsilon_1 = g(a + \varepsilon a^{-\alpha}) - g(a)$ in the first factor we have

$$\lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} = \lim_{\varepsilon_1 \to 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1},$$

and from here

$$\begin{aligned}
N_3^\alpha(f \circ g)(a) =& \lim_{\varepsilon_1 \to 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\
=& f'(g(a))N_3^\alpha g(a).
\end{aligned}$$

□

The following function will play an important role in our work.

Definition 1.2. Let $\alpha \in (0, 1)$ and c a real number. We define the fractional exponential in the following way

$$E_\alpha^{n_3}(c, t) = \exp\left(c \frac{t^{\alpha+1}}{\alpha + 1}\right).$$

Following the ideas presented in [5, 14] we can easily prove the next result.

Theorem 1.2. *Let $\alpha \in (0, 1]$ and h, g be α-differentiable at a point $t > 0$. Then*

(a) $N_3^\alpha(uf + vg) = uN_3^\alpha(h) + vN_3^\alpha(g)$ *for all* $u, v \in \mathbb{R}$*;*
(b) $N_3^\alpha(hg) = N_3^\alpha(g) + gN_3^\alpha(h)$*;*
(c) $N_3^\alpha\left(\frac{h}{g}\right) = \frac{hN_3^\alpha(g) - gN_3^\alpha(h)}{g^2}$*;*
(d) $N_3^\alpha(c) = 0$ *for all constant function* $h(t) = c$*;*

(e) $N_3^\alpha(1) = 0$;
(f) $N_3^\alpha\left(\frac{1}{1+\alpha}t^{1+\alpha}\right) = 1$;
(g) $N_3^\alpha\left(E_\alpha^{n_3}(c,t)\right) = cE_\alpha^{n_3}(c,t)$;
(h) $N_3^\alpha\left(\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)\right) = c\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)$;
(i) $N_3^\alpha\left(\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)\right) = -c\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)$.

Proof. (a) Let $H(t) = (af+bg)(t)$. Then $N_3^\alpha H(t) = \lim_{\varepsilon\to 0}\frac{H(t+\varepsilon t^{-\alpha})-H(t)}{\varepsilon}$ and from this we have the desired result.

(b) From definition we have

$$\begin{aligned}
N_3^\alpha(fg)(t) =& \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{-\alpha})g(t+\varepsilon t^{-\alpha}) - f(t)g(t)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{-\alpha})g(t+\varepsilon t^{-\alpha}) - f(t)g(t+\varepsilon t^{-\alpha}) + f(t)g(t+\varepsilon t^{-\alpha}) - f(t)g(t)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\frac{\left(f(t+\varepsilon t^{-\alpha}) - f(t)\right)g(t+\varepsilon t^{-\alpha})}{\varepsilon} + \lim_{\varepsilon\to 0}\frac{\left(g(t+\varepsilon t^{-\alpha}) - g(t)\right)f(t)}{\varepsilon}\\
=& fN_3^\alpha(g)(t) + gN_3^\alpha(f)(t).
\end{aligned}$$

(c) In a similar way to the previous one we have

$$N_3^\alpha\left(\frac{f}{g}\right)(t) = \lim_{\varepsilon\to 0}\frac{\frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})} - \frac{f(t)}{g(t)}}{\varepsilon}.$$

But

$$\begin{aligned}
\frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})} - \frac{f(t)}{g(t)} =& \frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})} - \frac{f(t)}{g(t)}\frac{g(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})}\\
=& \frac{f(t+\varepsilon t^{-\alpha})g(t) - f(t)g(t+\varepsilon t^{-\alpha})}{g(t)g(t+\varepsilon t^{-\alpha})}\\
=& \frac{f(t+\varepsilon t^{-\alpha})g(t) - f(t)g(t+\varepsilon t^{-\alpha}) - f(t)g(t) + f(t)g(t)}{g(t)g(t+\varepsilon t^{-\alpha})}\\
=& \frac{\left(f(t+\varepsilon t^{-\alpha}) - f(t)\right)g(t) - \left(g(t+\varepsilon t^{-\alpha}) - g(t)\right)f(t)}{g(t)g(t+\varepsilon t^{-\alpha})}.
\end{aligned}$$

From this last expression we obtain the expected result.

(d) Easily follows from definition.

(e) Is a particular case of the previous one.

(f) From Remark 1.1 we have

$$N_3^\alpha\left(\frac{1}{1+\alpha}t^{1+\alpha}\right) = t^{-\alpha}\frac{1}{1+\alpha}\left((1+\alpha)t^\alpha\right) = 1.$$

g) From Remark 1.1 and the chain rule we have

$$N_3^\alpha\left(E_\alpha^{n_3}(c,t)\right) = N_3^\alpha\left[\exp\left(c\frac{t^{\alpha+1}}{\alpha+1}\right)\right] = t^{-\alpha}\left[\exp\left(c\frac{t^{\alpha+1}}{\alpha+1}\right)\right]\left(c\frac{(\alpha+1)t^\alpha}{(\alpha+1)}\right)$$

$$=cE_{\alpha}^{n_3}(c,t).$$

To prove cases (h) and (i) it is sufficient to proceed as in the previous case, taking into account the Remark 1.1 and using the chain rule. □

Now, we give the definition of non-conformable fractional integral.

Definition 1.3. Let $\alpha \in (0,1]$ and $0 \leq u \leq v$. We say that a function $h : [u,v] \to \mathbb{R}$ is α-fractional integrable on $[u,v]$, if the integral

$$_{N_3}J_u^{\alpha}h(x) = \int_u^x \frac{h(t)}{t^{-\alpha}}dt$$

exists and is finite.

The following statement is analogous to the one known from the ordinary calculus (see [15]).

Theorem 1.3. *Let f be N-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

a) *if f is differentiable, then $_{N_3}J_{t_0}^{\alpha}\left(N_3^{\alpha}f(t)\right) = f(t) - f(t_0)$;*

b) $N_3^{\alpha}\left({}_{N_3}J_{t_0}^{\alpha}f(t)\right) = f(t)$.

Proof. a) From definition we have

$$_{N_3}J_{t_0}^{\alpha}\left(N_3^{\alpha}f(t)\right) = \int_{t_0}^{t}\frac{N_3^{\alpha}f(s)}{s^{-\alpha}}ds = \int_{t_0}^{t}\frac{f'(s)s^{-\alpha}}{s^{-\alpha}}ds = f(t) - f(t_0).$$

b) Analogously we have

$$N_3^{\alpha}\left({}_{N_3}J_{t_0}^{\alpha}f(t)\right) = t^{-\alpha}\frac{d}{dt}\left[\int_{t_0}^{t}\frac{f(s)}{s^{-\alpha}}ds\right] = f(t).$$ □

An important property, and necessary, in our work is that established in the following result.

Theorem 1.4 (Integration by parts)**.** *Let functions u, v be N-differentiable functions in (t_0,∞), with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

$$_{N_3}J_{t_0}^{\alpha}\left((uN_3^{\alpha}v)(t)\right) = [uv(t) - uv(t_0)] - {}_{N_3}J_{t_0}^{\alpha}\left((vN_3^{\alpha}u)(t)\right).$$

Proof. It is sufficient to use Theorem 1.2 and Theorem 1.3. □

In short time, many studies about theory and applications of the fractional differential equations which based on these new fractional derivative definitions [6,11,15,16,19].

In this paper we establish the first results to formalize a new version of a Laplace transform, in this case non-conformable, which will allow its application to a wide class of fractional differential equations. In the conformable case, there are some attempts that can be consulted in [6,9–12,19].

2. Results

Definition 2.1 (Exponential order)**.** A function f is said to be of generalized exponential order if there exist constants M and a such that $|f(t)| \leq ME_{\alpha}^{n_3}(a,t)$ for sufficiently large t.

We are now in a position to define the non-conformable fractional Laplace transform.

Definition 2.2. Let $\alpha \in (0,1)$ and c a real number. Let f be a real function defined for $t \geq 0$ and consider $s \in \mathbb{C}$, if the integral

$$_{N_3}J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) = \int_0^{+\infty} E_{\alpha}^{n_3}(-s,t)f(t)d_{\alpha}t = \int_0^{+\infty} \frac{E_{\alpha}^{n_3}(-s,t)f(t)}{t^{-\alpha}}dt$$

converge for the given value of s, you can define the function F given by the expression

$$F(s) =_{N_3} J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty), \tag{2.1}$$

and we will write $F = \mathcal{L}_N(f)$.

To the operator $\mathcal{L}_N$ we will call it the N-transformed of Laplace and we will say that F is the N-transformed of f. In turn, f is the N-inverse transform function of F and we will write it as $f = \mathcal{L}_N^{-1}\{F\}$, where $\mathcal{L}_N^{-1}$ is the N-transformed inverse Laplace operator.

As in the classic case, we must impose conditions to (2.1), so that the previous definition makes sense. If f satisfies the following two conditions:

(a) f is a piecewise continuous in the interval $(0,T]$ for any $T \in (0,+\infty)$;
(b) f is of generalized exponential order; that is, there are positive constants M and a, satisfying Definition 2.1 with $\mathrm{Re}\,(a-c) < 0$ and $|f(t)| \leq ME_{\alpha}^{n_3}(a,t)$ for all t and $\alpha \in (0,1]$.

Then the N-transformed of Laplace $F(s)$ of f exists for $s > a$. In effect, since f is of generalized exponential order, there exists constants $T > 0$, $K > 0$ and $a \in \mathbb{R}$ such that $|f(t)| \leq KE_{\alpha}^{n_3}(a,t)$ for all $t \geq T$ and $\alpha \in (0,1]$. Now we write $I =_{N_3} J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) =_{N_3} J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(T) +_{N_3} J_T^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) = I_1 + I_2$. Since f is a piecewise continuous, I_1 exists. For the second integral I_1, we note that for $t \geq T$ we have $|E_{\alpha}^{n_3}(-s,t)f(t)| \leq KE_{\alpha}^{n_3}(-(s-a),t)$. Thus,

$$_{N_3}J_T^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) \leq K_{N_3}J_T^{\alpha} E_{\alpha}^{n_3}(-(s-a),t)(+\infty) = \frac{K}{s-a}, \quad s > a.$$

Since the integral I_2 converges absolutely for $s > a$, I_2 converges for $s > a$. Thus, both I_1 and I_2 exist and hence I exists for $s > a$. Then we have that f is an N-transformable function.

Theorem 2.1. *Let $\alpha \in (0,1]$. So, we have*

(a) $\mathcal{L}_N(1) = \frac{1}{s}$*, from here we have* $\mathcal{L}_N(c) = cL_N(1)$ *for any* $c \in \mathbb{R}$*;*
(b) $\mathcal{L}_N(t^b) = \frac{(1+\alpha)^{\frac{b}{1+\alpha}}\Gamma\left(1+\frac{b}{1+\alpha}\right)}{s^{1+\frac{b}{1+\alpha}}}$*, where the gamma function* Γ *is defined by* $\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}dt$, $\Gamma(a,0) := \Gamma(a)$ *and* $b > -1$*;*

(c) $\mathcal{L}_N(E_\alpha^{n_3}(c,t)) = \frac{1}{s-c}$, *c any real number and* $s-c>0$*;*
(d) $\mathcal{L}_N(f(t)E_\alpha^{n_3}(c,t) = F(s-c)$, *with* $\mathcal{L}_N(f(t)) = F(s)$, *c any real number and* $s-c>0$*;*
(e) $\mathcal{L}_N(\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{c}{s^2+c^2}$*;*
(f) $\mathcal{L}_N(\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{s}{s^2+c^2}$*;*
(g) $\mathcal{L}_N(\sinh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{c}{s^2-c^2}$*;*
(h) $\mathcal{L}_N(\cosh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{s}{s^2-c^2}$*.*

Proof. (a) From definition directly.

(b) Through a change of variables we have

$$ {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)t^b(+\infty) = \frac{(1+\alpha)^{\frac{b}{1+\alpha}}}{s^{1+\frac{b}{1+\alpha}}}\,{}_{N_3}J_0^\alpha E_\alpha^{n_3}(-u)u^{\frac{b}{1+\alpha}}(+\infty), $$

where the desired result is obtained.

(c) Consider $f(t) = E_\alpha^{n_3}(c,t)$, with $c \in \mathbb{R}$. Then

$$ {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)E_\alpha^{n_3}(c,t)(+\infty) = {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-(s-c),t)(+\infty) = \frac{1}{s-c}. $$

(d) Suppose $\mathcal{L}_N f(t) = F(s)$ for $s>k$. So, we have

$$ \begin{aligned} {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)E_\alpha^{n_3}(c,t)f(t)(+\infty) =& {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-(s-c),t)f(t)(+\infty) \\ =& F(s-c), \quad s-c>k. \end{aligned} $$

(e) Using ${}_{N_3}J^\alpha E_\alpha^{n_3}(b,t)\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{E_\alpha^{n_3}(b,t)}{a^2+b^2}\left\{b\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) - a\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right)\right\}$ we obtain the expected result.

(f) Similar to previous one, using

$$ {}_{N_3}J^\alpha E_\alpha^{n_3}(b,t)\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{E_\alpha^{n_3}(b,t)}{a^2+b^2}\left\{b\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) + a\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right)\right\}. $$

(g) As $\mathcal{L}_N(\sinh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)) = \frac{1}{2}\left\{\mathcal{L}_N E_\alpha^{n_3}(c,t) - \mathcal{L}_N E_\alpha^{n_3}(-c,t)\right\}$ it is easy to get the required conclusion.

(h) From $\mathcal{L}_N(\cosh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)) = \frac{1}{2}\left\{\mathcal{L}_N E_\alpha^{n_3}(c,t) + \mathcal{L}_N E_\alpha^{n_3}(-c,t)\right\}$ it is obtained directly. □

Anallogously, the following propositions can be proved from the definition of N-transformed and the non-conformable integral.

Proposition 2.1. *If the functions f and g are transformable, then there is the transform of the sum and is equal to the sum of the transforms, that is*

$$ \mathcal{L}_N(f+g) = \mathcal{L}_N(f) + \mathcal{L}_N(g). $$

Proposition 2.2. *If the function f is transformable and λ is a real number, then there is the transform of product of λ by f and is equal to product of λ by the transform of f, that is*

$$\mathcal{L}_N(\lambda f) = \lambda \mathcal{L}_N(f).$$

Remark 2.1. Taking into account the two previous propositions, we say that $\mathcal{L}_N$ is a linear operator.

Proposition 2.3. *If f is a transformable function, then so is its N-derivative and you have*

$$\mathcal{L}_N(N_3^\alpha f) = s\mathcal{L}_N(f) - f(0). \tag{2.2}$$

Proof. Already $\mathcal{L}_N(N_3^\alpha f)$ exists, because f is of non-conformable exponential order and continuous. On an interval $[a,b]$ where $N_3^\alpha f$ is continuous, integrating by parts in (2.2), gives

$$\int_a^b \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt = f(b)E_\alpha^{N_3}(-s,b) - f(a)E_\alpha^{N_3}(-s,a) + s\int_a^b \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt.$$

On any interval $[0,K]$ there are finitely many intervals $[a,b]$ on each of which $N_3^\alpha f$ is continuous. Add above equality across these finitely many intervals $[a,b]$. The boundary values on adjacent intervals match and the integrals add to give

$$\int_0^K \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt = f(K)E_\alpha^{N_3}(-s,b) - f(0) + s\int_0^K \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt.$$

Taking the limit $K \to +\infty$ across this equality, we obtain the desired result. ☐

Analogously we have the following.

Proposition 2.4. *If the k consecutive derivatives $N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f)))$ are N-transformable, then we have*

$$\begin{aligned}&\mathcal{L}_N\left[N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f)))\right]\\ =&s^k\mathcal{L}_N(f) - s^{k-1}f(0) - s^{k-2}N_3^\alpha f(0) - s^{k-3}N_3^\alpha(N_3^\alpha f(0)) - \cdots - N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f(0)))).\end{aligned}$$

Proposition 2.5. *Let g be of non-conformable exponential order and continuous for $t \geq 0$. Then*

$$\mathcal{L}_N\left(\int_0^x \frac{g(x)}{x^{-\alpha}}dx\right) = \frac{1}{s}\mathcal{L}_N\left\{g(t)\right\}.$$

Proof. Let $f(t) = \left(\int_0^t \frac{g(x)}{x^{-\alpha}}dx\right)$. Then f is of exponential order and continuous then we have $\mathcal{L}_N\left(\int_0^t \frac{g(x)}{x^{-\alpha}}dx\right) = \mathcal{L}_N$ by definition and $\mathcal{L}_N = \frac{1}{s}\mathcal{L}\left(N_3^\alpha f(t)\right)$ because $f(0) = 0$. From here we reach the conclusion without difficulty. ☐

The following result establishes the relationship between the classic Laplace Transform and the N-transform defined above.

Theorem 2.2. *Let $\alpha \in (0,1)$ and f be a N-transformable function, then we have*

$$\mathcal{L}_N(f) = L\left[f\left(((1+\alpha)z)^{\frac{1}{1+\alpha}}\right)\right],$$

where $\mathcal{L}$ is the classical Laplace transform defined by $\mathcal{L}(g) = \int_0^{+\infty} e^{-st} g(t)dt$.

Proof. Simply make the change of the variables $z = \frac{t^{1+\alpha}}{1+\alpha}$. □

One of the most important results of the classic Laplace transform is the convolution product of two $\mathcal{L}$-transformable functions, we are already in a position to provide an analogous result for the N-transform defined in (2.1).

Theorem 2.3. *Let $\alpha \in (0,1]$ and $f, g : [0,+\infty] \to \mathbb{R}$ be real functions. If $F_\alpha(s) = L_N\left[f\left(t^{1+\alpha}\right)\right](s)$ and $G_\alpha(s) = \mathcal{L}_N\left[g\left(t\right)\right](s)$, then the next equality is satisfied*

$$\mathcal{L}_N(f * g)(s) = F_\alpha(s) G_\alpha(s),$$

where

$$(f * g)(t) = \int_0^t \left[f\left(t^{1+\alpha} - \tau^{1+\alpha}\right)\right] g(\tau) d_\alpha \tau.$$

Proof. It is sufficient to change the variables $u^{1+\alpha} = t^{1+\alpha} - \tau^{1+\alpha}$ and apply the properties of the $\mathcal{L}_N$ operator. □

2.1. **Existence of non-conformable Laplace transform.** In this subsection, the bounded and existence of non-conformable Laplace transform are presented.

Theorem 2.4. *Let f be piecewise continuous on $[0,\infty)$ and non-conformable exponentially bounded, then*

$$\lim_{s\to\infty} F_\alpha(s) = 0,$$

where $F_\alpha(s) = \mathcal{L}_\alpha[f(t)](s)$.

Proof. Since f is generalized order exponential, there exist t_0, M_1, c such that $|f(t)| \leq M_1 E_\alpha^{n_3}(c,t)$ for $t \geq t_0$. Also, f is piecewise continuous on $[0,t_0]$ and hence f is bounded, so there exists M_2 such that $|f(t)| \leq M_2$ for $t \in [0,t_0]$. Choosing $M = \max\{M_1, M_2\}$, we have $|f(t)| \leq M E_\alpha^{n_3}(c,t)$ for $t \geq 0$. Now, we have'

$$\begin{aligned}
\left|\int_0^\tau E_\alpha^{n_3}(-s,t) f(t) d_\alpha t\right| &\leq \int_0^\tau \left|E_\alpha^{n_3}(-s,t) f(t)\right| d_\alpha t \\
&\leq M \int_0^\tau E_\alpha^{n_3}(-s+c,t) d_\alpha t \\
&= \frac{M}{s-c} - \frac{E_\alpha^{n_3}(-s+c,t)}{s-c}.
\end{aligned}$$

This gives

$$\lim_{\tau\to\infty} \left|\int_0^\tau E_\alpha^{n_3}(-s,t) f(t) d_\alpha t\right| \leq \frac{M}{s-c}.$$

This completes the proof. □

3. Examples and Applications

Example 3.1. Consider the non-conformable differential equation:

$$N_3^\alpha x(t) = \lambda x(t), \quad x(0) = x_0, \quad \alpha \in (0,1]. \tag{3.1}$$

Clearly, if $\alpha = 1$ the equation above is just one of the simplest classical ordinary differential equations which is defined by the hypothesis that the rate of growth of a given function $x(t)$ is proportional to the current value (e.g. Maltius's population model), i.e., $x'(t) = \lambda x(t)$, $x(0) = x_0$ the exact solution of this is $x(t) = x_0 e^{\lambda t}$.

Applying the non-conformable Laplace Transform to both sides of equation (3.1), we get

$$\begin{aligned}\mathcal{L}_N \left(N_3^\alpha x(t)\right) =& \lambda \mathcal{L}_N \left(x(t)\right),\\ s\, X_\alpha(s) - x_0 =& \lambda X_\alpha(s).\end{aligned}$$

Simplifying this we get

$$X_\alpha(s) = \frac{x_0}{s+1}. \tag{3.2}$$

Taking the inverse non-conformable Laplace transform to (3.2), we get

$$x(t) = x_0 E_\alpha^{N_3}(-1,t) = -\frac{x_0}{\alpha+1} t^{\alpha+1}.$$

The solution of (3.1), obtained from non-conformable Laplace transformation method, are shown in Figure 1 for different values of α.

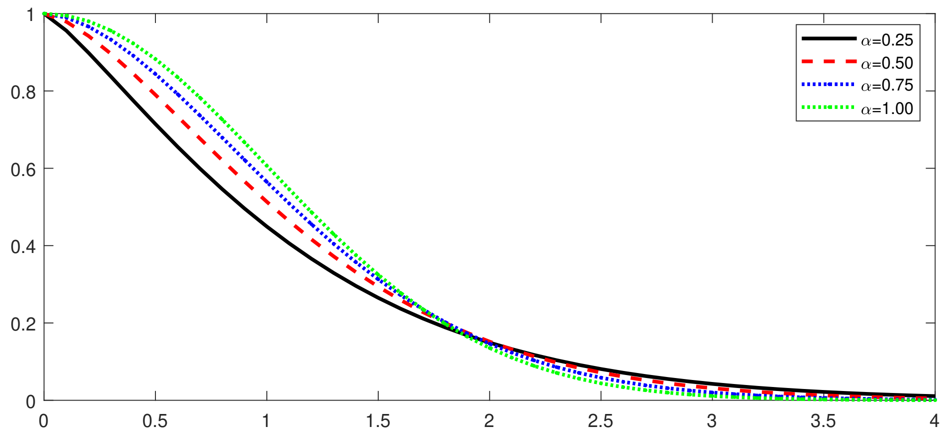

Figure 1. Non-conformable Laplace solution of (3.1) for different values of α.

Example 3.2. Consider the non-conformable fractional Bertalanffy-logistic differential equation

$$N_3^\alpha x(t) = x^{\frac{2}{3}}(t) - x(t), \quad x(0) = x_0, \quad \alpha \in (0,1). \tag{3.3}$$

The solution of the classic Bertalanffy-logistic differential equation $x'(t) = x^{\frac{2}{3}}(t) - x(t)$, $x(0) = x_0$ is $x(t) = \left[1 + \left(x_0^{\frac{2}{3}} - 1\right)e^{-\frac{t}{3}}\right]^3$. By using the change of variable $z = 3x^{\frac{1}{3}}$ in (3.3), we find

$$N_3^{\alpha} z(t) = 1 - \frac{2}{3} z(t), \quad z_0 = 3x_0^{\frac{1}{3}}. \tag{3.4}$$

Applying the non-conformable Laplace transform $\mathcal{L}$ to both sides of equation (3.4) we obtain

$$L_N(z(t)) = \frac{3}{s} + \frac{z_0 - 3}{s + \frac{1}{3}}.$$

Finally, applying the inverse Laplace transform we have the solution of (3.3) in the form $x(t) = \left[1 + \left(x_0^{\frac{2}{3}} - 1\right)e^{-\frac{t^{1+\alpha}}{3(1+\alpha)}}\right]^3$.

With $\alpha = 0.25, 0.50, 0.75, 1.00$, the non-conformable Laplace transformation solution of (3.3) are shown in Figures 2 and 3 for $x_0 = 2$ and $x_0 = 4$, respectively.

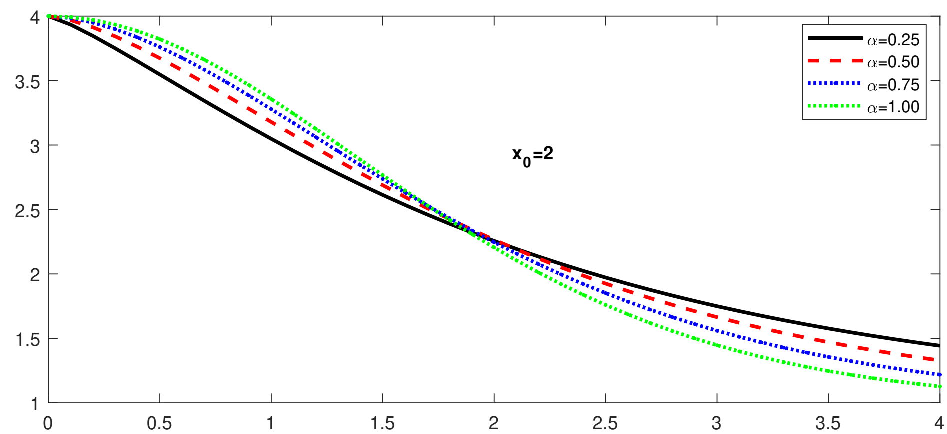

FIGURE 2. Non-conformable Laplace solution of (3.2) for $x_0 = 2$ and different values of α.

Example 3.3. Consider the non-conformable fractional differential equation

$$N_3^{\alpha}(N_3^{\alpha} x(t)) + cx(t) = 0, \quad \alpha \in (0, 1], \tag{3.5}$$

with the initial conditions $x(0) = x_0$, $N_3^{\alpha} x(0) = 0$. Clearly, if $\alpha = 1$ the previous differential equation approximates the characterization of small oscillations of a pendulum, i.e., $x''(t) + cx(t) = 0$, $x(0) = x_0$, $x'(0) = 0$, where $c = \frac{g}{L}$, with g the gravity acceleration and L the length of the pendulum rod. The exact solution to this problem is $x(t) = x_0 \cos \sqrt{c}t = x_0 \cos \sqrt{\frac{g}{L}}t$. Applying the non-conformable Laplace transform to

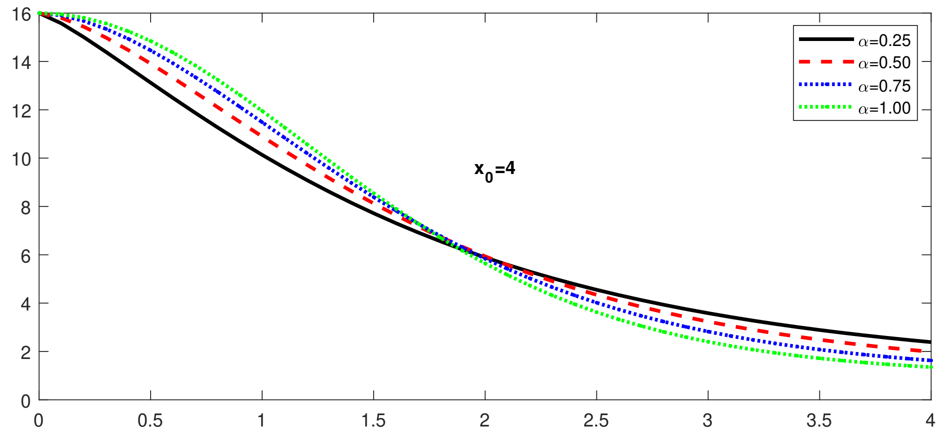

Figure 3. Non-conformable Laplace solution of (3.2) for $x_0 = 4$ and different values of α.

the both hand sides of (3.5), we get $(s^2+c)X(s)-sx_0 = 0$, thus $X(s) = \frac{sx_0}{(s^2+c)}$. Taking the inverse non-conformable Laplace transform we obtain $x(t) = x_0 \cos\left(\sqrt{\frac{g}{L}}\frac{t^{\alpha+1}}{\alpha+1}\right)$.

Example 3.4. Now consider the circuit consisting of a voltage source $v(t)$ in series with a resistor (R), a capacitor (C) and an inductor (L), as well as a switch that can be in the open or closed position. The circuit equation in the time domain is $Rx(t) + \frac{1}{c}\int_0^t x(u)du + v_C(0) + Lx'(t) = v(t)$, we assume that $x(0) = 0$ (i.e., the switch is open until $t = 0$, allowing the capacitor to maintain its initial condition $v_C(t)$ before that moment) and $v(t) = A$. The corresponding non-conformable fractional differential equation is

$$Rx(t) + \frac{1}{c}\,{}_N J_0^\alpha(x)(t) + v_C(0) + LN_3^\alpha x(t) = A, \quad \alpha \in (0,1].$$

Applying the non-conformable Laplace transform to both sides of above equation, we get $X(s) = \frac{A-v_C(0)}{L(s^2+\frac{R}{L}s+\frac{1}{LC})}$. The poles of the characteristic equation can be obtained as $s = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \left(\frac{R}{2C}\right)^2} = -\sigma \pm iw$ assuming the radicand is positive we have $X(s) = \frac{A-v_C(0)}{L((s+\sigma)^2+w^2)}$. After taking inverse N-transform and reorder you get

$$x(t) = \frac{A - v_C(0)}{wL}\, E_\alpha^{N_3}(-\sigma, t) \sin\left(w\frac{t^{\alpha+1}}{\alpha+1}\right).$$

4. Epilogue

The fundamental goal of this work has been to generalize the main theorems of the classical Laplace transform into the non-conformable Laplace transform. The goal has been achieved, whereby the non-conformable derivative definition has been used to construct some of these theorems and relations. We calculate the non-conformable

Laplace transform from some elementary functions and establish the non-conformable version of the transform of the successive derivative, the integral of a function and the convolution of the fractional functions. In addition, the bounded and the existence of the non-conformable Laplace transform are presented. The findings of this study indicate that the results obtained in the fractional case are adjusted to the results obtained in the ordinary case. Finally, we show the application of the N-transform to the resolution of fractional differential equations.

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