Kragujevac Journal of Mathematics
Volume 51(2) (2027), Pages 197–207.

ON STATISTICAL CONVERGENCE OF BOCHNER SUMMABLE FUNCTIONS AND KOROVKIN TYPE APPROXIMATION THEOREM

HEMANTA KALITA[1,*], BIPAN HAZARIKA[2], AND EKREM SAVAS[3]

Abstract. In this manuscript, we introduce the summability theory of Bochner integrable functions. Also, statistical convergence in Bochner spaces is considered. A criterion for statistical convergence is given. Several properties of Bochner summability, and statistical convergence in the sense of Bochner summable functions are discussed here. Finally, we prove Korovkin type approximation theorem via statistical Bochner summable functions in a Bochner space.

1. Introduction and Preliminaries

Summability theory deals with the generalisation of the concept of limit of a sequence or series, which is typically impacted by an auxiliary sequence of linear means of the given sequences or series. The original sequence or series may be divergent, but the linear mean sequence must converge. It is known that Zygmund [17] first proposed the concept of statistical convergence in his well-known work "Trigonometric series" in 1935. The notion was formally established by Fast [4] and Steinhaus. The notion of convergence of an infinite series was first resolved satisfactorily by the French mathematician A. L. Cauchy. One can see [5,14] and their references for recent trends of statistical convergence and their related works.

Bilalov and Nazarova [1] introduced the idea of the statistical convergence in Lebesgue spaces L^p. One can see [8–11] and references therein for several works

Key words and phrases. Summable functions, statistical convergence, Bochner space, Korovkin theorem.
2020 *Mathematics Subject Classification.* Primary: 11B05. Secondary: 26A05, 40A05, 40A30.
DOI
Received: January 21, 2025.
Accepted: July 16, 2025.

of statistical convergence. In the same works, the concept of statistical fundamentality in L^p was discussed and its equivalence to statistical convergence was proved. A. D. Gadjiev et al. [6] discussed the Korovkin type approximation theorem for a sequence of positive linear operators acting on $L^p([a,b])$. See also [14] for Korovkin-type approximation theorems with algebraic test functions for sequences of statistical Riemann and statistical Lebesgue integrable functions.

Bochner spaces, named after the mathematician Salomon Bochner, are a generalization of L^p spaces to functions lying in a Banach space which is not necessarily the space $\mathbb{R}$ or $\mathbb{C}$. Let (T, Σ, μ) be a measure space, where $T = [a,b]$. The Bochner space $L^p(T, \mathcal{X})$ or simply $L^p(\mathcal{X})$ consists of equivalence classes of all Bochner measurable functions with values in the Banach space $\mathcal{X}$ whose norm $\|f\|_{\mathcal{X}}$ belongs to the standard L^p space. A. Caushi et al. proposed Bochner integration in the context of statistical convergence in [3].

The work in [1] motivated us to study statistical convergence in Bochner spaces.

The structure of the manuscript is as follows. In Section 1, we recall several definitions and results that are useful for the following section. In Section 2, we discuss Bochner summability and several results related to the Bochner summability. In Section 3, we introduce statistical convergence of Bochner summable functions in Bochner spaces. Further, we establish a Korovkin type approximation theorem via statistical Bochner summable functions in Bochner spaces.

Let A be a subset of bounded natural number $\mathbb{N}$. The density of A is $\delta(A) = \lim_{n\to+\infty} \frac{n(A_n)}{n}$ where $A_n = \{k < n : k \in A\}$ and $n(A)$ denotes the cardinality of A. It is clear that density of a finite set is zero and $\delta(A^c) = 1 - \delta(A)$, where $A^c = \mathbb{N} \setminus A$.

Definition 1.1. A sequence of $x = (x_k)$ is statistically convergent to a vector L of normed space $\mathcal{X}$ if for each $\epsilon > 0$, $\lim_{n\to+\infty} \frac{1}{n} |\{k \le n : \|x_k - L\| \ge \epsilon\}| = 0$. We denote this by st-$\lim x_k = L$.

Definition 1.2. A sequence $x = (x_k)$ is a statistically Cauchy sequence if for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that

$$\lim_{n\to+\infty} \frac{1}{n} |\{k \le n : \|x_k - x_N\| \ge \epsilon\}| = 0.$$

Let (f_k) be a sequence of functions taking values in a normed space.

Definition 1.3. A sequence of functions $(f_k(x))$ is said to be statistically convergent to f if for every $\epsilon > 0$, $\lim_{n\to+\infty} \frac{1}{n} |\{k \le n : \|f_k(x) - f(x)\| \ge \epsilon\}| = 0$ for all x.

Recall statistical convergence in L^p as below.

Definition 1.4 ([1])**.** Let $(f_k) \subset L^p([a,b])$. A sequence (f_k) is said to be statistically convergent to some given function f in $L^p([a,b])$ if st-$\lim_{n\to+\infty} \int_a^b |f_k(x) - f(x)|^p dx = 0$, $1 \le p < +\infty$.

Recall the Bochner integral and the statistical Bochner integrable functions from [3, 13] as below.

Definition 1.5 ([13]). A function $f : T \to \mathcal{X}$ is called Bochner integrable if there exist a sequence of functions (f_k) such that

(a) $\sum_{k=1}^{+\infty} \int_a^b \|f_k(x)\| d\mu < +\infty$;
(b) $f(x) = \sum_{k=1}^{+\infty} f_k(x)$ at the points $x \in T$, where $\sum_{k=1}^{+\infty} \|f_k(x)\| < +\infty$.

Definition 1.6 ([3]). A function $f : T = [a, b] \to \mathcal{X}$ is called statistical Bochner integrable if there exists a st-Cauchy sequence of simple functions (f_k) such that

(a) it is statistically convergent a.e. by μ to the function f;
(b) st-$\lim_k \int_T \|f_k(t) - f(t)\| d\mu = 0$ a.e.

The limit st-$\lim_k \int_T f_k(t) d\mu$ is called statistical Bochner integral and denoted by $(B_s) \int_T f(t) d\mu$.

Theorem 1.1 ([3]). *If the function f is st-Bochner integrable, then the function $\|f\|$ is also st-Bochner integrable.*

Lemma 1.1 (Fatou Lemma [3]). *Let $(f_k(t))$ be the sequence of st-measurable functions statistically convergent a.e. to the function $f(t)$. If $\|f_k(t)\| \leq \|f_{k+1}(t)\|$, then st-$\lim_k \int_T \|f_k\| d\mu = (B_s) \int_T \|f(t)\| d\mu$.*

2. Bochner Summability

It is well known that the existence of the improper integral does not guarantee absolute integrability or simple summability, regardless of whether we are dealing with Riemann-integrable or Lebesgue-integrable functions. As an example, we can consider the nature of $\int_0^{+\infty} \frac{\sin x}{x} dx$. It is known that the Bochner integral is a generalized Lebesgue integral. In this section, we discuss the vector-valued summands of the Lebesgue integral. Several important results are discussed here. We begin the section with the following definition.

Definition 2.1. A vector-valued function f, which is integrable in the sense of Bochner on the interval $(1, +\infty)$ is said to be Bochner summable if there exists a sequence of simple functions (f_n), $f_n : (1, +\infty) \to \mathcal{X}$ such that $\lim_{n\to+\infty} f_n(x) = f(x)$ a.e. in $(1, +\infty)$ and

$$\lim_{n\to+\infty} \int_1^n \|f_n - f\|_{\mathcal{X}} \, d\mu = 0.$$

If for all $n \geq 1, f_n = L$, then we say that f is Bochner summable to L. It is clear from the above definition that all Bochner summable functions are Bochner integrable but all Bochner integrable functions may not be Bochner summable.

Example 2.1. Let $\mathcal{X}$ be an infinite dimensional Banach space. Let $\{I_k^n : n = 0, 1, \ldots; k = 1, 2, \ldots, 2^n\}$ be the dyadic intervals on $[0, 1]$. That is $I_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$. Let E_n be a 2^n-dimensional subspace of $\mathcal{X}$. Consider $T_n : l_2^{2^n} \to E_n$ such that $\{e_k^n : k = 1, 2, \ldots, 2^n\}$ be image under T_n of the standard unit vectors $\{u_k^n : k =$

$1, 2, \ldots, 2^n\}$ of $l_2^{2^n}$. Let us consider a sequence $f_n : T \to \mathfrak{X}$ by $f_n(t) = \sum_{k=1}^{2^n} e_k^n \chi_{I_k^n}(t)$. Then,

$$\|f_n\| = \sum_{k=1}^{2^n} \int_{I_k^n} \|e_k^n\| d\mu = 2^{-n} \sum_{k=1}^{2^n} \|e_k^n\|.$$

Since $1 \leq \|e_k^n\| \leq 2$, so $1 \leq \|f_n\| \leq 2$. Clearly, (f_n) is a sequence of Bochner summable respectively Bochner integrable. Let us construct with minor variation as below. Fix $0 < \beta < \frac{1}{2}$ and $\alpha_n = 2^{n\beta}$. Define $g_n : T \to \mathfrak{X}$ by $g_n = \alpha_n f_n$. Clearly, $\|g_n\| \to +\infty$ at the rate of α_n. Then, (g_n) is still Bochner integrable but not Bochner summable.

Theorem 2.1. *If $f : (1, +\infty) \to \mathfrak{X}$ is such that $f(t) = 0$ a.e. in $(1, +\infty)$, then f is Bochner summable to* 0.

Proof. Let $f_n : (1, +\infty) \to \mathfrak{X}$ be such that $f_n = 0$ for all $n \geq 1$. Then, there exists a sequence of simple functions such that $\lim_{n\to+\infty} f_n(x) = f(x)$ a.e. in $(1, +\infty)$. Now, since (f_n) is a L^p-Cauchy sequence so, we have

$$\lim_{n\to+\infty} \int_1^n \|f\|_{\mathfrak{X}}\, d\mu = 0.$$

So, f is Bochner summable to 0. □

Theorem 2.2. *Let $f, g : (1, +\infty) \to \mathfrak{X}$ be two Bochner summable functions and $\alpha \in \mathbb{R}$. Then, $f + g$ and αf are Bochner summable.*

Proof. Since $f, g : (1, +\infty) \to \mathfrak{X}$ are Bochner summable functions, there exist two sequences of simple functions (f_n) and (g_n) and two measurable subsets E, F in $(1, +\infty)$ such that $\lim_{n\to+\infty} f_n(x) = f(x)$ for all $x \in E$ and $\lim_{n\to+\infty} g_n(y) = g(y)$ for all $y \in F$, with $\mu(E^c) = 0$ and $\mu(F^c) = 0$. So, $\mu((E \cap F)^c) = 0$ and $\lim_{n\to+\infty}(f_n + g_n)(t) = (f + g)(t)$ for all $t \in E \cap F$. Now,

$$\int_1^n |(f_n + g_n) - (f + g)|_{\mathfrak{X}}\, d\mu \leq \int_1^n \|f_n - f\|_{\mathfrak{X}}\, d\mu + \int_1^n \|g_n - g\|_{\mathfrak{X}}\, d\mu \to 0,$$

as $n \to +\infty$. So, $f + g$ is Bochner summable. Similarly, we can show that αf is Bochner summable. □

Theorem 2.3. *If $f : (1, +\infty) \to \mathfrak{X}$ is Bochner summable and $g : (1, +\infty) \to \mathfrak{X}$ is such that $f = g$ a.e. in $(1, +\infty)$, then g is Bochner summable.*

Theorem 2.4. *Let $f : (1, +\infty) \to \mathfrak{X}$ be a countable valued measurable function with $f(t) = \sum_{i=1}^{+\infty} y_i \chi_{E_i}(t)$ where $E_i \subset (1, +\infty)$ are measurable and $E_m \cap E_n = \emptyset$ for $m \neq n$, $y_m \in \mathfrak{X}$ for all $m \geq 1$. If the series $\sum_{m\geq 1} \|y_m\|\, \mu(E_m)$ is convergent, then f is Bochner summable.*

Proof. Let us define a sequence $(f_n)_{n\in\mathbb{N}}$, by

$$f_n = \sum_{i=1}^{n} y_i \chi_{E_i}, \quad n \geq 1.$$

Then, (f_n) is a sequence of simple functions and $\lim_{n\to+\infty} f_n(t) = f(t)$ for all $t \in (1, +\infty)$. Now,

$$\begin{aligned}\int_1^n \|f_n - f\|_{\mathcal{X}}\, d\mu &= \int_1^n \left\| \sum_{m=n+1}^{+\infty} y_m \chi_{E_m} \right\|_{\mathcal{X}} d\mu \\ &\leq \int_{(1,+\infty)} \left\| \sum_{m=n+1}^{+\infty} y_m \chi_{E_m} \right\|_{\mathcal{X}} d\mu \\ &= \sum_{m=n+1}^{+\infty} \|y_m\|_{\mathcal{X}}\, \mu(E_m) \to 0, \quad \text{as } n \to +\infty.\end{aligned}$$

This shows that f is Bochner summable. □

Corollary 2.1. *A countable valued measurable function* $f : T \to \mathcal{X}$, *such that* $\|f(t)\|_{\mathcal{X}} \leq g(t)$ *a.e. in* T, *with* g *Bochner summable, is Bochner summable.*

Lemma 2.1. *A measurable function* $f : T \to \mathcal{X}$ *is Bochner summable if and only if* $\|f\|_{\mathcal{X}} : T \to \mathbb{R}$ *is Bochner summable.*

Proof. Let $f : T \to \mathcal{X}$ be Bochner summable. By definition $\|f\|_{\mathcal{X}}$ is Bochner summable.

Conversely, let $\|f\|_{\mathcal{X}}$ be Bochner summable. By definition,

$$\lim_{n\to+\infty} \|f_n - f\|_{\mathcal{X}} = \lim_{n\to+\infty} \int_T \|f_n - f\|_{\mathcal{X}} d\mu = 0.$$

Also, since f is measurable, for every $k \in \mathbb{N}$ there is a countable valued measurable sequence of functions f_k of the form $f_k(t) = \sum_{m=1}^{+\infty} y_{k,m} \chi_{E_{k,m}}(t),\ t \in T$, where $E_{k,m} \subset T,\ m \in \mathbb{N}$ is measurable, $E_{k,m} \cap E_{k,l} = \emptyset$ for $m \neq l,\ y_{k,m} \in \mathcal{X},\ m \in \mathbb{N}$ and f_k has the following property: there exists $J \subset T,\ \mu(J) = 0$ such that for every $k \in \mathbb{N}$ we have

$$\left\| f(t) - f_k(t) \right\|_{\mathcal{X}} < \frac{\epsilon}{\mu(T)}, \quad \text{for } t \in T \setminus J. \tag{2.1}$$

Hence,

$$\|f_k(t)\|_{\mathcal{X}} \leq \|f(t)\|_{\mathcal{X}} + \|f(t) - f_k(t)\|_{\mathcal{X}} < \|f(t)\|_{\mathcal{X}} + \frac{\epsilon}{\mu(T)}.$$

By Corollary 2.1, f_k is Bochner summable and

$$\int_T \|f_k\|_{\mathcal{X}} d\mu = \sum_{n=1}^{+\infty} \|y_{k,m}\|_{\mathcal{X}} \mu(E_{k,m}) < +\infty.$$

Let $r_k \in \mathbb{N}$ be such that $\sum_{n=r_k+1}^{+\infty} \|y_{k,m}\|_{\mathcal{X}} \mu(E_{k,m}) < \epsilon$. Since $\|f - f_k\|_{\mathcal{X}}$ is measurable and (2.1) holds, the function $\|f - f_k\|_{\mathcal{X}}$ is summable and $\int_T \|f - f_k\|_{\mathcal{X}} d\mu < \epsilon$. Let $g_k = \sum_{n=1}^{r_k} y_{k,m} \chi_{E_{k,m}}$. Then, g_k is a sequence of simple functions, $f_k = g_k + \sum_{n=r_k+1}^{+\infty} y_{k,m} \chi_{E_{k,m}}$ and

$$\begin{aligned}\int_T \|f - g_k\|_{\mathcal{X}} d\mu &\leq \int_T \|f - f_k\|_{\mathcal{X}} d\mu + \int_T \|f_k - g_k\|_{\mathcal{X}} d\mu \\ &< \frac{\epsilon}{2} + \sum_{n=r_k+1}^{+\infty} \|y_{k,m}\|_{\mathcal{X}} \mu(E_{k,m}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So, f is Bochner summable function. □

Lemma 2.2. *Let $f : T \to \mathcal{X}$ be a measurable function of the form $f = g + \sum_{n=1}^{+\infty} x_n \chi_{E_n}$ where $g : T \to \mathcal{X}$ is measurable and bounded, E_n are pairwise disjoint measurable subsets of T, $x_n \in \mathcal{X}$, $n \in \mathbb{N}$. Then, f is Bochner summable if and only if x_n and E_n, $n \in \mathbb{N}$, can be chosen such that the series $\sum_{n=1}^{+\infty} x_n \mu(E_n)$ converges absolutely in $\mathcal{X}$.*

Proof. The proof is analogous to [13, Proposition 1.4.5]. □

Theorem 2.5. *Let (x_k) be a sequence of elements belonging to $\mathcal{X}$ and let us define the function $f : T = [a, b] \to \mathcal{X}$ as follows*

$$f = \begin{cases} \sum\limits_{k=1}^{+\infty} \chi_{(\frac{1}{2^k}, \frac{1}{2^{k-1}}]} x_k, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $\left\| \frac{1}{2^k} x_k \right\|_{+\infty} < \beta$, $\beta > 1$, then f is Bochner summable if and only if $\sum_{k=1}^{+\infty} \frac{1}{2^k} x_k$ is absolutely convergent.

Proof. By Lemma 2.2 this happens if and only if the series $\sum_{k=1}^{+\infty} \frac{1}{2^k} x_k$ is absolutely convergent and the proof is completed. □

Remark 2.1. It is well known that every Bochner integral is Henstock-Kurzweil integral (see [13]). It is also well known that Kuelbs-Steadman spaces contain Henstock-Kurzweil integrable functions, and Lebesgue spaces are contained densely in Kuelbs-Steadman spaces (see [7]). It will be very interesting to find Henstock-Kurzweil summability in Kuelbs-Steadman spaces. One can also find how Bochner summability behaves in Kuelbs-Steadman spaces.

3. Statistical Bochner Summable Functions

It is clear from the definition of Bochner integral and statistical Bochner integral that every Bochner integral is a statistical Bochner integral. The following example shows that the statistical Bochner integral is not Bochner summable.

Example 3.1 ([3]). Let $(f_n) \subset \mathcal{X}$ be a sequence defined by

$$f_k(x) = \begin{cases} (k+1)(-x)^k, & \text{for } k \in [3^p, 3^p + p),\ p = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in \mathcal{X} \setminus [-1, 1]$, then $\lim_{n\to+\infty} \frac{1}{n} |\{k \leq n : f_k(x) \neq 0\}| = 0$. That is st-$\lim_n f_n = 0$. On the other hand the usual integral $\lim_{n\to+\infty} \int_1^n (k+1)(-x)^k d\mu = (-1)^k (n^{k+1} - 1)$ tends to $\pm\infty$. So, (f_n) is not Bochner summable although it is statistically Bochner integrable.

This is why we concentrate our study on statistical convergence of Bochner summable functions in Bochner spaces.

Definition 3.1. Let $f_n, f \in L^p\left((1,+\infty),\mathcal{X}\right)$. We say $f_n \to f$ is statistical convergent if

$$\text{st-}\lim_{n\to+\infty}\int_1^{+\infty}\|f_n(t)-f(t)\|_{\mathcal{X}}^p d\mu(t)=0,\quad 1\le p<+\infty. \tag{3.1}$$

If relation (3.1) is true, then there exists $K\equiv(n_k)_{k\in\mathbb{N}},\ n_1<n_2<n_3<\cdots,\ \delta(K)=1$, $\lim_{n\to+\infty}\int_1^{+\infty}\|f_{n_k}(t)-f(t)\|_{\mathcal{X}}^p d\mu(t)=0$. This shows that there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ of the sequence $(f_{n_k})_{n\in\mathbb{N}}$ such that $f_{k_n}(t)\to f(t)$ for almost every $t\in[1,+\infty)$. Let $\mathcal{K}=\{K\subset\mathbb{N}:\delta(K)=1\}$. Then, the following result holds.

Proposition 3.1. *Let $K_j\in\mathcal{K},\ j=1,2$. Then, $K_1\cap K_2\in\mathcal{K}$.*

The set of statistically convergent sequences in the sense of Bochner is denoted by [B-s] and we write [B-s]-lim $f(t)=L$.

Theorem 3.1. *Let f and g be vector-valued functions (in the sense of Bochner) in $(1,+\infty)$. Then, the following hold.*

(a) *If* [B-s]-lim $f(t)=L$ *and* $c\in\mathbb{R}$, *then* [B-s]-lim $cf(t)=cL$.
(b) *If* [B-s]-lim $f(t)=L_1$ *and* [B-s]-lim $g(t)=L_2$, *then* [B-s]-lim $f(t)+g(t)=L_1+L_2$.

Next we define a statistically fundamental sequence in the sense of Bochner as follows.

Definition 3.2. We say $(f_n)_{n\in\mathbb{N}}$ is statistically fundamental in $L^p(\mathcal{X})$ if for all $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N},\ \delta(\Delta_\epsilon)=0$, where

$$\Delta_\epsilon=\{n\in\mathbb{N}:\ \|f_n-f_{n_\epsilon}\|_{\mathcal{X}}^p\ge\epsilon\}\quad\text{and}\quad\|f\|_{L^p(\mathcal{X})}=\left(\int_1^{+\infty}\|f\|_{\mathcal{X}}^p d\mu(t)\right)^{\frac{1}{p}}.$$

Lemma 3.1. *Let $(f_n)_{n\in\mathbb{N}}\subset L^p(\mathcal{X})$ be a sequence. If st-$\lim_{n\to+\infty}f_n$ exists, then $(f_n)_{n\in\mathbb{N}}$ is st-fundamental in $L^p(\mathcal{X})$.*

Proof. Let $\epsilon>0$ be arbitrary, $n\in\mathbb{N}$, and $\Delta_\epsilon=\{n\in\mathbb{N}:\|f_n-f_{n_\epsilon}\|_{\mathcal{X}}^p\ge\epsilon\}$. Take $n_\epsilon\in\Delta_\epsilon^c$ such that $\|f_n-f_{n_\epsilon}\|_{\mathcal{X}}^p\ge\epsilon$. Then, we have

$$\{n:\|f_n-f\|_{\mathcal{X}}^p<\epsilon\}\subset\{n:\|f_n-f_{n_\epsilon}\|_{\mathcal{X}}^p<\epsilon\}.$$

So, it is clear that $\delta(\Delta_\epsilon^c)=1$. Consequently, $\delta(\Delta_\epsilon)=0$. □

Theorem 3.2. *Let $(f_n)_{n\in\mathbb{N}}\subset L^p(\mathcal{X})$ be a sequence in sense of Bochner integral such that st-$\lim_{n\to+\infty}f_n$ exists. Then, $(f_n)_{n\in\mathbb{N}}$ is st-fundamental if and only if there exists $(g_n)_{n\in\mathbb{N}}\subset L^p(\mathcal{X})$ such that $\lim_{n\to+\infty}g_n$ exists and $\{n:f_n=g_n\}\in\mathcal{K}$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be st-fundamental in $L^p(\mathcal{X})$. Let us consider $(M_n)_{n\in\mathbb{N}}$, a sequence of closed sets in $L^p(\mathcal{X})$ where diameter $d(M_n)\to0$ as $n\to+\infty$. Also, let $K_{(n)}=\{n:f_n\in M_n\}$. From the hypothesis $d(M_n)\to0$ as $n\to+\infty$, it follows that $\{f\}=\bigcap_n M_n$. Since $K_{(m)}\in\mathcal{K}$, there exists $(n_m)_{m\in\mathbb{N}}\subset\mathbb{N}:n_1<n_2<n_3<\cdots$ such that $\frac{1}{n}\left|\left\{k \in I_n : k \in K^c_{(m)}\right\}\right| < \frac{1}{m}$ for all $n > n_m$. Let $\mathbb{N}_0 = \{k \in \mathbb{N} : n_m < k \leq n_{m+1}$ and $k \in K^c_{(m)}\}$, and

$$g_k = \begin{cases} f, & \text{if } k \in \mathbb{N}_0 \text{ and } k \geq n_1, \\ f_k, & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$ be given. If $k \in \mathbb{N}_0$ and $k > n_1$, then $\|g_k - f\|^p_{\mathcal{X}} = 0 < \epsilon$. If $k \notin \mathbb{N}_0$, then $k \in K_{(m)}$ and $f_k \in M_m$. Then,

$$\|f_k - f\|^p_{\mathcal{X}} \leq \|f_k - f_{n_m}\|^p_{\mathcal{X}} + \|f_{n_m} - f\|^p_{\mathcal{X}} < \epsilon.$$

Consequently, $\lim_{k\to+\infty} g_k = f$. Let $f_k \neq g_k,\ k < n$. Next, if $k \in \mathbb{N}_0$, then $k \in K^c_{(m)}$ gives

$$\begin{aligned}\frac{1}{n}\|\{k \leq n :\ f_k \neq g_k\}\| &\leq \frac{1}{n}\left\|\left\{k \leq n :\ k \in K^c_{(m)}\right\}\right\| \\ &< \frac{1}{m} \to 0, \quad \text{as } n \to +\infty,\end{aligned}$$

which gives $m \to +\infty$. So,

$$\lim_{n\to+\infty} \frac{\|\{k \leq n : f_k \neq g_k\}\|}{n} = 0. \tag{3.2}$$

Consequently, $\{k \leq n : f_k \neq g_k\} \in \mathcal{K}$ and $\lim_{n\to+\infty} g_n = f$.

Conversely, suppose $(f_n)_{n\in\mathbb{N}} \subset L^p(\mathcal{X})$ is a sequence in sense of Bochner integral, and there exists $(g_n)_{n\in\mathbb{N}} \subset L^p(\mathcal{X})$ such that $\lim_{n\to+\infty} g_n$ exists and $\{n :\ f_n = g_n\} \in \mathcal{K}$. In order to prove (f_n) is fundamental in $L^p(\mathcal{X})$, it is enough to prove st-$\lim_{n\to+\infty} f_n$ exists and st-$\lim_{n\to+\infty} f_n = f$. Let $\epsilon > 0$. We have

$$\{k \leq n : \|f_k - f\| \geq \epsilon\} \subset \{k \leq n :\ f_k \neq g_k\} \bigcup \{k \leq n : \|g_k - f\| \geq \epsilon\}. \tag{3.3}$$

Since $\lim_{k\to+\infty} g_k = f$ in $L^p(\mathcal{X})$, so $\|g_k - f\| < \epsilon$ for all $k \geq n_\epsilon$. Next, $\|\{k \leq n : \|g_k - f\| \geq \epsilon\}\| < n_\epsilon$ implies $\frac{1}{n}\|\{k \leq n : \|g_k - f\| \geq \epsilon\}\| \to 0$ as $n \to +\infty$. From (3.2) and (3.3), we obtain

$$\frac{1}{n}\{k \leq n : \|f_k - f\| \geq \epsilon\} \leq \frac{1}{n}\{k \leq n : f_k \neq g_k\} + \frac{1}{n}\|\,\{k \leq n : \|g_k - f\| \geq \epsilon\}\,\| \to 0,$$

as $n \to +\infty$. So, st-$\lim_{n\to+\infty} f_n = f$. By Lemma 3.1, (f_n) is fundamental in $L^p(\mathcal{X})$. □

Corollary 3.1. *Let $(f_n)_{n\in\mathbb{N}} \subset L^p(\mathcal{X})$ and st-$\lim_{n\to+\infty} f_n = f$. Then, there exists $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N} : n_1 < n_2 < \cdots,\ \lim_{k\to+\infty} f_{n_k} = f$ and $\delta\left((n_k)_{k\in\mathbb{N}}\right) = 1$.*

Proposition 3.2. *Let $(t_k)_{k\in\mathbb{N}}$ be a sequence of numbers and $\sum_{k=1}^{+\infty} \chi_A(k) = +\infty$, where $A = \{k \in \mathbb{N} : t_k \neq 0\}$. Then, there exists $(f_n)_{n\in\mathbb{N}} \subset L^p(\mathcal{X})$ such that $\sum_{k=1}^{+\infty} t_k f_k(t) = +\infty$ for all $t \in (1, +\infty)$.*

Proof. The proof is similar to [1, Lemma 3.8]. □

Theorem 3.3. *If the function f is st-Bochner integrable in $L^p(\mathcal{X})$, then the function $\|f\|$ is also st-Bochner integrable in $L^p(\mathcal{X})$.*

Proof. The proof is analogous to Theorem 1.1. □

Next, we prove the Fatou lemma in $L^p(\mathcal{X})$ via statistically Bochner summable functions.

Theorem 3.4. *Let $(f_k(t))$ be the sequence of st-measurable functions statistically convergent a.e. to the function $f(t)$ in $L^p(\mathcal{X})$. If $\|f_k(t)\| \leq \|f_{k+1}(t)\|$, then*

$$st\text{-}\lim_k \int_I \|f_k\| d\mu = (B_s) \int_I \|f(t)\| d\mu.$$

Proof. The proof is similar to that of Lemma 1.1. □

The Korovkin approximation refers to a convergence statement in which a function is approximated by a certain sequence of functions. In practical applications, continuous functions can be approximated by polynomials. The Korovkin approximations provide a convergence for the entire approximation by analyzing the convergence of the process on a finite set of functions. This approximation is named in honor of Pavel Korovkin. Several applications of Korovkin's theorem can be found in [12,14–16] and the references therein. Finally, to formulate the classical Korovkin theorem in terms of statistical convergence in Bochner spaces with statistical Bochner summable functions, we consider $C_M(T, \mathcal{X})$ or simply $C_M(\mathcal{X})$ as the space of all functions f that are continuous and completely bounded at each point of $T = [a, b]$. That is $\|f(x)\| \leq M_f,\ -\infty < x < +\infty$, where M_f is a constant dependent on f. Let (A_n) be a sequence of positive linear operators, i.e., $A_n(f, x) \geq 0$ if $f(x) \geq 0$, which acts from $C_M(\mathcal{X})$ on the space $BD(\mathcal{X})$ of all bounded functions on T. It is known that $BD(\mathcal{X})$ is a Banach space with norm $\|f\| = \sup_{a \leq x \leq b} \|f(x)\|,\ f \in BD(\mathcal{X})$.

Theorem 3.5. *Let (A_n) be a sequence of positive linear operators $A_n : L^p(\mathcal{X}) \to L^p(\mathcal{X})$ and let $(\|A_n\|)$ be uniformly bounded. If $st\text{-}\lim_n \|A_n(t^\beta, x) - x^\beta\|_{\mathcal{X}} = 0,\ \beta = 0, 1, 2,$ then for any function $f \in L^p(\mathcal{X})$ we have $st\text{-}\lim_n \|A_n(f, x) - f(x)\|_{\mathcal{X}} = 0$.*

Proof. Let (A_n) be a sequence of positive linear operators $A_n : L^p(\mathcal{X}) \to L^p(\mathcal{X})$ and let $(\|A_n\|)$ be uniformly bounded. Let $\text{st-}\lim_n \|A_n(t^\beta, x) - x^\beta\|_{\mathcal{X}} = 0,\ \beta = 0, 1, 2$. Then, for a given $\epsilon > 0$ there exist $n_\beta(\epsilon),\ \beta = 0, 1, 2$ and subsets $\mathcal{K}_\beta,\ \beta = 0, 1, 2$ of density 1 such that

$$\|A_n(t^\beta, x) - x^\beta\|_{\mathcal{X}} < \epsilon, \quad \text{for all } n \in \mathcal{K}_\beta \text{ and } n > n_\beta,\ \beta = 0, 1, 2. \tag{3.4}$$

Since $\delta(\mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2) = 1$, (3.4) holds for $n \in \mathcal{K},\ \mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2$ and $n > \max\{n_0, n_1, n_2\}$. By our assumption there is a constant $M > 0$ such that $\|A_n\|_{L^p(\mathcal{X}) \to L^p(\mathcal{X})} \leq M,\ n = 1, 2, \ldots$ Since $C_M(\mathcal{X})$ is dense in $L^p(\mathcal{X})$, for a given $f \in L^p(\mathcal{X})$, there exists $g \in C_M(\mathcal{X})$ such that $\|f - g\|_{\mathcal{X}} < \epsilon$. Hence, we have

$$\begin{aligned} \|A_n(f, x) - f(x)\|_{\mathcal{X}} &\leq \|A_n(f - x, x)\|_{\mathcal{X}} + \|A_n(g, x) - g(x)\|_{\mathcal{X}} + \|f - g\|_{\mathcal{X}} \\ &< \epsilon(1 + M) + \|A_n(g, x) - g(x)\|_{\mathcal{X}}. \end{aligned}$$

By continuity of g we get $\|g(x)\| < \alpha$ for all $x \in T$ and for some constant α. Thus,

$$\|A_n(g,x) - g(x)\|_{\mathcal{X}} \leq \|A_n(\|g(t) - g(x)\|, x)\|_{\mathcal{X}} + \alpha\|A_n(1,x) - 1\|_{\mathcal{X}}.$$

Again, since $g \in C_M(\mathcal{X})$, for all $x \in T$, we have $\|g(t) - g(x)\| < \epsilon$. So,

$$\begin{aligned}\|A_n\Big(\|g(t) - g(x)\|, x\Big)\|_{\mathcal{X}} \leq & \epsilon\|A_n(1+x)\|_{\mathcal{X}} + \|A_n(t-x)^2, x\|_{\mathcal{X}} \\ \leq & \epsilon\Big(\|A_n(a,x) - 1\|_{\mathcal{X}} + 1\Big) + \|A_n(t^2,x) - x^2\|_{\mathcal{X}} \\ & + 2b\|A_n(t,x) - x\|_{\mathcal{X}} + b^2\|A_n(1,x) - 1\|_{\mathcal{X}} \\ < & \epsilon, \quad \text{for } n \in \mathcal{K} \text{ and } n > \max\{n_0, n_1, n_2\}.\end{aligned}$$

Hence, $\text{st-lim}_n \|A_n(f,x) - f(x)\|_{\mathcal{X}} = 0$. □

4. Conclusion

In this article, we introduce Bochner summability. Several properties of Bochner summability are discussed here. A necessary and sufficient condition for Bochner summability is demonstrated. We present the concept of statistical convergence for Bochner summable functions in Bochner spaces. In addition, a Korovkin-type approximation theorem is established by using statistical Bochner summable functions in Bochner spaces.

References

[1] B. T. Bilalov and T. Y. Nazarova, *Statistical convergence of functional sequences*, Rocky Mountain J. Math. **45**(5) (2015), 1413–1424. https://doi.org/10.1216/RMJ-2015-45-5-1413

[2] D. Borwein, *Linear functionals with strongly Cesáro summability*, J. Lond. Math. Soc. **40** (1965), 628–634. https://doi.org/10.1112/jlms/s1-40.1.628

[3] A. Caushi and A. Tato, *A statistical integral of Bohner type on Banach space*, Appl. Math. Sci. **6**(138) (2012), 6857–6870.

[4] H. Fast, *Sur la convergence statistique*, Colloquium Math. **2** (1951), 241–244.

[5] B. Das, P. Debnath and B. C. Tripathy, *On statistically convergent complex uncertain sequences*, Carpathian Math. Publ. **14**(1) (2022), 135–146. https://doi.org/10.15330/cmp.14.1.135-146

[6] A. D. Gadjieva and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**(1) (2002), 129–138. https://doi.org/10.1216/rmjm/1030539612

[7] T. L. Gill and W. W. Zachary, *Functional Analysis and Feynman Operator Calculus*, Springer, 2016.

[8] B. B. Jena, S. K. Paikray and M. Mursaleen, *On statistical Riemann-Stieltjes integrablility and deferred Cesaro summability*, Contemp. Math. **5** (2024), 1–16. https://doi.org/10.37256/cm.5320242466

[9] B. B. Jena and S. K. Paikray, *A new approach to statistical Riemann-Stieltjes integrals*, Miskolc Math. Notes **24**(2) (2023), 789–803. https://doi.org/10.18514/MMN.2023.3903

[10] B. B. Jena, S. K. Paikray and H. Dutta, *Statistically Riemann integrable and summable sequence of functions via deferred Cesáro mean*, Bull. Iranian Math. Soc. **48**(4) (2022), 1293–1309. https://doi.org/10.1007/s41980-021-00578-8

[11] B. B. Jena, S. K. Paikray and H. Dutta, *A new approach to Korvokin-type approximation via deferred Cesaro statistical measurable convergence*, Chaos, Solitons & Fractals **148** (2021), 1–9. https://doi.org/10.1016/j.chaos.2021.111016

[12] S. K. Paikray, P. Parida and S. A. Mohiuddine, *A certain class of relatively equi-statistical fuzzy approximation theorems*, Eur. J. Pure Appl. Math. **13** (2020), 1212–1230. https://doi.org/10.29020/nybg.ejpam.v13i5.3711
[13] S. Schwabik and G. Ye, *Topics in Banach Space Integration*, World Scientific, Singapore, 2005.
[14] H. M. Srivastava, B. B. Jena and S. K. Paikray, *Statistical Riemann and Lebesgue integrable sequence of functions with Korovkin type approximation theorems*, Axioms **10** (2021), 1–16. https://doi.org/10.3390/axioms10030229
[15] H. M. Srivastava, B. B. Jena and S. K. Paikray, *A new class of Korovkin-type theorems on double sequences*, Iran. J. Sci. (2025), 1–12. https://doi.org/10.1007/s40995-025-01793-2
[16] H. M. Srivastava, B. B. Jena, S. K. Paikray and U. K. Misra, *Korovkin-type theorems for positive linear operators based on the statistical derivative of deferred Cesáro summability*, Algorithms **18** (2025), 1–21. https://doi.org/10.3390/a18040218
[17] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, UK, 1979.

[1]Mathematics Division,
VIT Bhopal University,
Bhopal-Indore Highway, Kothrikalan, Sehore, Madhya Pradesh 466114, India
Email address: hemanta30kalita@gmail.com
ORCID iD: https://orcid.org/0000-0002-9798-6608

[2]Department of Mathematics,
Gauhati University,
Guwahati, Assam, India
Email address: bh_rgu@yahoo.co.in
ORCID iD: https://orcid.org/0000-0002-0644-0600

[3]Department of Mathematics,
Usak University,
Usak, Turkey
Email address: ekremsavas@yahoo.com
ORCID iD: https://orcid.org/0000-0003-2135-3094

*Corresponding Author